

On the comparison of results regarding the post-Newtonian approximate treatment of the dynamics of extended spinning compact binaries

S Hergt¹, J Steinhoff² and G Schäfer¹

¹ Theoretisch-Physikalisches Institut, Friedrich-Schiller-Universität, Max-Wien-Platz 1, 07743 Jena, Germany

² Centro Multidisciplinar de Astrofísica (CENTRA), Departamento de Física, Instituto Superior Técnico (IST), Universidade Técnica de Lisboa, Avenida Rovisco Pais 1, 1049-001 Lisboa, Portugal

E-mail: steven.hergt@uni-jena.de

Abstract. A brief review has been given of all the Hamiltonians, and effective potentials calculated hitherto covering the post-Newtonian (pN) dynamics of a two-body system. A method has been presented to compare (conservative) reduced Hamiltonians with non-reduced potentials directly at least up to the next-to-leading-pN order.

1. Post-Newtonian modelling and results of the two-body dynamics

The treatment of the two-body dynamics in general relativity through higher order post-Newtonian (pN) order has to incorporate both spin- and tidal-force-induced mass multipoles of the involved bodies [1]. In our analysis, we have focused on the spin-multipole degrees of freedom, and presented some illuminating results that will be useful for the extraction of binary-system parameters from the emitted gravitational waves. One easy way for the modelling of such systems is the making use of the Tulczyjew's singular stress-energy tensor [2] $T_{\mu\nu}$, with the Greek coordinate indices running from 0 to 3, of the following form:

$$\sqrt{-g}T^{\mu\nu}(x^\sigma) = \int d\tau \left[u^{(\mu} p^{\nu)} \delta_{(4)} + \left(u^{(\mu} S^{\nu)\alpha} \delta_{(4)} \right)_{||\alpha} + \frac{1}{3} R^{\mu}_{\alpha\beta\rho} J^{\nu)\rho\beta\alpha} \delta_{(4)} - \frac{2}{3} \left(J^{\mu\alpha\beta\nu} \delta_{(4)} \right)_{||(\alpha\beta)} + \dots \right], \quad u^\mu = \frac{dz^\mu}{d\tau}, \quad \delta_{(4)} = \delta(z^\sigma - x^\sigma), \quad (1)$$

with the body's 4-velocity u^μ , the 4-momentum p_μ , the antisymmetric spin tensor $S^{\mu\nu}$ modelling the pole-dipole structure, and Dixon's reduced quadrupole moment tensor $J^{\mu\alpha\beta\nu}$ modelling the first order finite size effects while possessing the same symmetries as the Riemann tensor $R^{\mu\alpha\beta\nu}$. $\delta_{(4)}$ is a 4-dimensional Dirac delta function, $\int d^4x \delta_{(4)} = 1$, and $||$ denotes the 4-dimensional covariant derivative. The tensor $J^{\mu\alpha\beta\nu}$ is decomposed into stress, flow and the symmetric trace-free mass quadrupole $Q_{\mu\nu}$. The latter is given by the ansatz with a vector f_μ to which, the spin is orthogonal; i.e., $S^{\mu\nu} f_\nu = 0$ (See [3], Eq. (5.11) therein), and

$$Q_{\mu\nu} = \frac{C_Q}{m} \left(S_{\mu\rho} S_\nu{}^\rho - \frac{1}{3} P_{\mu\nu} S^{\rho\sigma} S_{\rho\sigma} \right), \quad P^{\mu\nu} = g^{\mu\nu} - \frac{1}{f_\rho f^\rho} f^\mu f^\nu, \quad (2)$$



Table 1. Post-Newtonian Hamiltonians known to date.

Order	1.0	1.5	2.0	2.5	3.0	3.5	4.0	4.5
H^N								
PM	$+ H^{1pN}$		$+ H^{2pN}$	$+ H^{2.5pN}(t)[6]$	$+ H^{3pN}$	$+ H^{3.5pN}(t)[7, 8]$	$+ (H^{4pN})$	$+ \{H^{4.5pN}(t)\}$
SO		$+ H_{SO}^{LO}$		$+ H_{SO}^{NLO}$		$+ H_{SO}^{N^2LO}[9]$	$+ H_{SO}^{LO,R}(t)$	$+ (H_{SO}^{N^3LO})$
S_1^2			$+ H_{S_1^2}^{LO}$		$+ H_{S_1^2}^{NLO}$		$+ (H_{S_1^2}^{N^2LO})$	$+ \{H_{S_1^2}^{LO,R}(t)\}$
$S_1 S_2$			$+ H_{S_1 S_2}^{LO}$		$+ H_{S_1 S_2}^{NLO}$		$+ H_{S_1 S_2}^{N^2LO}[10]$	$+ H_{S_1 S_2}^{LO,R}(t)$
spin ³						$+ [H_{S_3}^{LO}]$		$+ (H_{S_3}^{NLO})$
spin ⁴							$+ [H_{S_4}^{LO}]$	

{.} Eqs. of motion known [.] For black holes only (.) Not known (yet)

and is parametrized by one constant C_Q in the leading order quadratic-in-spin level, in this way fully encoding the rotational deformation. For black holes, one has $C_Q = 1$ [4], while for neutron star models, C_Q depends on the equations of state [5] and varies between 4.3 and 7.4. The next step in performing explicit pN-calculations is complex in various ways. We have compared two prominent methods: One method aims at calculating a Hamiltonian. This is achieved by a 3+1 decomposition of Einstein's field equations and the energy-momentum tensor from Eq. (1) leading to constraints, which have to be fulfilled at all instants of time on the 3-dimensional hypersurfaces of constant coordinate time. We have then used the ADM formalism as outlined in [3] to find the canonical set of variables $(\hat{\mathbf{z}}_I, \hat{\mathbf{p}}_I, \hat{\mathbf{S}}_I)$ with the body label $I = 1, 2$ fulfilling their standard canonical Poisson bracket relations $\{\hat{z}_I^i, \hat{p}_{Jj}\} = \delta_{ij}\delta_{IJ}$ and $\{\hat{S}_{I(i)}, \hat{S}_{I(j)}\} = \epsilon_{ijk}\hat{S}_{I(k)}$ with i, j, k running from 1 to 3. Letters embraced by round brackets indicate the components in local Lorentz frames and are denoted by a, b, \dots from the beginning of the alphabet; so a can take the "values" $a \in \{(0), (i)\}$. The spin-tensor components S_{ab} are connected to the coordinate-frame components through a vierbein transformation $S_{ab} = e_{a\mu}e_{b\nu}S^{\mu\nu}$.

The ADM formalism also leads to a formula for calculation of the Hamiltonian in full reduced phase-space by imposing the ADMTT or transverse-traceless gauge to the 3-metric on the 3-dimensional hypersurfaces, and by choosing the canonical spin supplementary condition (SSC) to get canonical variables for the bodies. By expansion of the constraints in pN-powers of $\frac{v^2}{c^2} \sim \frac{Gm}{r^2c^2}$ (two-body system virial theorem, speed of light c is not put equal to one) one ends up with a perturbative scheme for the calculation of the various pN-Hamiltonians. As the spin is of pN order $1/c$ or $1/c^2$ depending on its strength, the formal labelling is such that we have called the first post-Newtonian spin-Hamiltonians not 1.5pN or 2.5pN according to formal counting rules but just as leading order (LO) ones and the higher corrections we have called next-to-leading (NLO), and next-to-next-to-leading order (N²LO).

In Table 1, we have given a list of all known pN Hamiltonians for the case of maximally rotating objects where $|\mathbf{S}| = \frac{Gm^2a}{c}$ with the dimensionless spin a . H^N is the Newtonian Hamiltonian, PM means "point-mass", i.e., without spin, H^{nPN} with $n \in \{1, 2, \dots\}$ are the conservative pure point-mass Hamiltonians, $H^{\frac{n}{2}PN}$ are the non-autonomous, radiation-reaction (dissipative) pure point-mass Hamiltonians, $S_1 S_2$ to spin(1)-

spin(2) coupling, and S_1^2 to spin-quadrupole coupling involving the constant C_Q . LO,R in the index indicates the leading-order reaction counterpart, so $H_{S_1^2}^{LO,R}$ is the reaction counterpart to the conservative part $H_{S_1^2}^{LO}$. Obviously, the reaction part is much higher in its pN order than the conservative part, but nevertheless they are important to cover the dynamics to 4.5pN order consistently; up until now the reaction field for point masses is known to 3.5pN order [7, 8]. One other method to arrive at pN equations of motion is the derivation of effective potentials, which are subtly related to Hamiltonians by a Legendre transformation. This derivation is most effectively achieved by sophisticated methods from Effective Field Theory (EFT) that uses full knowledge from quantum field theoretical calculations. Up until now, pN potentials have been calculated to 3pN order [11] for point masses and to NNLO for spin(1)-spin(2) coupling [12].

2. Comparison between Effective Field Theory potentials and ADM Hamiltonians

Effective potentials are part of a Lagrangian with the Newtonian kinetic energy T_N , given as:

$$L_{eff} = T_N - V_{eff} = \frac{m_1}{2}v_1^2 + \frac{m_2}{2}v_2^2 - V_{eff}. \quad (3)$$

The conservative effective potential V_{eff} for two interacting bodies is pN expanded up to next-to-leading order (NLO) spin effects in the following way:

$$V_{eff} = V_{PM} + V_{SO}^{LO} + V_{S_1^2}^{LO} + V_{S_2^2}^{LO} + V_{S_1S_2}^{LO} + V_{SO}^{NLO} + V_{S_1^2}^{NLO} + V_{S_2^2}^{NLO} + V_{S_1S_2}^{NLO}. \quad (4)$$

One key difference between EFT potentials and ADM Hamiltonians is that in most cases the potentials still depend on the $S^{(0)(i)}$ -components of the spin tensor, which have to be fixed by choosing an appropriate SSC.

For a direct comparison, a formal Legendre transformation of the non-reduced potentials is conducted yielding the effective Hamiltonian H_{eff} , which is followed by a reduction process in phase-space in order to arrive at a canonical set of variables (See [13] for details). This ‘canonicalization’ is most transparently accomplished by reducing the following effective action:

$$S_{eff} = \int dt L_{eff} = \int dt \left(p_{1i}\dot{z}_1^i + p_{2i}\dot{z}_2^i - \frac{1}{2}S_{1ab}\Omega_1^{ab} - \frac{1}{2}S_{2ab}\Omega_2^{ab} - H_{eff}(\mathbf{z}_I, \mathbf{p}_I, S_{Iab}) \right). \quad (5)$$

Here, we have defined the angular velocity tensor $\Omega^{ab} \equiv \Lambda_A^a \dot{\Lambda}^{Ab}$ rendering Ω^{ab} antisymmetric and $\Lambda_{A\mu}\Lambda_\nu^A = g_{\mu\nu}$, $\Lambda_{Aa}\Lambda_b^A = \eta_{ab}$ with $(A, B, \dots) \in \{[0], [i]\}$ being the body-fixed Lorentz-frame labels. The reduced action has to read as:

$$\hat{S}_{eff} = \int dt \left(\hat{p}_{1i}\dot{\hat{z}}_1^i + \hat{p}_{2i}\dot{\hat{z}}_2^i - \frac{1}{2}\hat{S}_{1(i)(j)}\hat{\Omega}_1^{(i)(j)} - \frac{1}{2}\hat{S}_{2(i)(j)}\hat{\Omega}_2^{(i)(j)} - H_{can}(\hat{\mathbf{z}}_I, \hat{\mathbf{p}}_I, \hat{\mathbf{S}}_I) \right) \quad (6)$$

with $\hat{\Omega}^{(i)(j)} = \hat{\Lambda}_{[k]}^{(i)}\dot{\hat{\Lambda}}^{[k](j)}$ given by a non-linear shift of $\Lambda^{[k](i)}$ to $\hat{\Lambda}^{[k](i)}$ so that $\hat{\Lambda}^{[k](i)}\hat{\Lambda}^{[k](j)} = \delta_{ij}$.

This reduction is achieved by inserting the covariant SSC $S_{ab}u^b = 0$ as well as its conjugate condition $\Lambda^{[i]a}u_a = 0$ into $\frac{1}{2}S_{ab}\Omega^{ab}$, and performing a pN approximate variable transformation of spin and position reading as:

$$\begin{aligned} z_1^i &= \hat{z}_1^i - \left[\frac{1}{2m_1^2}p_{1k}\hat{S}_{1(i)(k)} \left(1 - \frac{\mathbf{p}_1^2}{4m_1^2} \right) - G\frac{m_2}{m_1^2}\frac{p_{1k}\hat{S}_{1(i)(k)}}{\hat{r}_{12}} + \frac{3}{2}G\frac{p_{2k}\hat{S}_{1(i)(k)}}{m_1\hat{r}_{12}^2} \right. \\ &\quad \left. + \frac{G}{2}\frac{\hat{n}_{12}^k(\hat{\mathbf{n}}_{12} \cdot \mathbf{p}_2)\hat{S}_{1(i)(k)}}{m_1\hat{r}_{12}} + G\frac{m_2}{m_1^2}\frac{\hat{S}_{1(k)(l)}\hat{S}_{1(i)(l)}\hat{n}_{12}^k}{\hat{r}_{12}^2} + G\frac{\hat{n}_{12}^k\hat{S}_{1(i)(l)}\hat{S}_{2(k)(l)}}{m_1\hat{r}_{12}^2} \right], \end{aligned} \quad (7)$$

Table 2. Agreement between EFT potentials and ADM Hamiltonians.

V_{NLO}^{SO}	Levi[14]		H_{NLOADM}^{SO}	Damour/Jaranowski/Schäfer[15, 16]
V_{NLO}^{SO}	Porto[17]			
		\approx		
$V_{NLO}^{S_1 S_2}$	Porto/Rothstein[18, 19]		$H_{NLOADM}^{S_1 S_2}$	Steinhoff/Hergt/Schäfer[20, 16]
$V_{NLO}^{S_1^2}$	Porto/Rothstein[21, 22]		$H_{NLOADM}^{S_1^2}$	Hergt/Steinhoff/Schäfer[23]

$$\begin{aligned}
 S_{1(i)(j)} = & \hat{S}_{1(i)(j)} - \left[\frac{p_{1[i}\hat{S}_{1(j)](k)}p_{1k}}{m_1^2} \left(1 - \frac{\mathbf{p}_1^2}{4m_1^2} \right) - \frac{2Gm_2}{m_1^2\hat{r}_{12}} p_{1[i}\hat{S}_{1(j)](k)}p_{1k} \right. \\
 & + \frac{3G}{m_1\hat{r}_{12}} p_{1[i}\hat{S}_{1(j)](k)}p_{2k} + \frac{G}{m_1\hat{r}_{12}} p_{1[i}\hat{S}_{1(j)](k)}\hat{n}_{12}^k(\hat{\mathbf{n}}_{12} \cdot \mathbf{p}_2) \\
 & \left. + \frac{2Gm_2}{m_1^2\hat{r}_{12}^2} p_{1[i}\hat{S}_{1(j)](l)}\hat{S}_{1(k)(l)}\hat{n}_{12}^k + \frac{2G}{m_1\hat{r}_{12}^2} p_{1[i}\hat{S}_{1(j)](l)}\hat{S}_{2(k)(l)}\hat{n}_{12}^k \right]. \quad (8)
 \end{aligned}$$

Those formulas are valid to transform the potentials at least to NLO to their canonical Hamiltonian counterpart, which enabled us to obtain an overall agreement of all EFT NLO potentials with their corresponding ADM Hamiltonian as displayed in Table 2 up to canonical transformations indicated by \approx (See [13] for a thorough investigation).

Acknowledgments

This work is supported by the Deutsche Forschungsgemeinschaft (DFG) through SFB/TR7 “Gravitational Wave Astronomy,” project STE 2017/1-1, and GRK 1523, and by the FCT (Portugal) through PTDC project CTEAST/098034/2008.

References

- [1] Binington T and Poisson E 2009 *Phys. Rev. D* **80** 084018
- [2] Tulczyjew W M 1959 *Acta Phys. Pol.* **18** 393
- [3] Steinhoff J 2011 *Ann. Phys. (Berlin)* **523** 296
- [4] Thorne K S 1980 *Rev. Mod. Phys.* **52** 299
- [5] Laarakkers W G and Poisson E 1999 *Ap. J.* **512** 282
- [6] Schäfer G 1985 *Ann. Phys. (N.Y.)* **161** 81
- [7] Jaranowski P and Schäfer G 1997 *Phys. Rev. D* **55** 4712
- [8] Königsdörffer C, Faye G and Schäfer G 2003 *Phys. Rev. D* **68** 044004
- [9] Hartung J and Steinhoff J 2011 *Ann. Phys. (Berlin)* **523** 783
- [10] Hartung J and Steinhoff J 2011 *Ann. Phys. (Berlin)* **523** 919
- [11] Foffa S and Sturani R 2011 *Phys. Rev. D* **84** 044031
- [12] Levi M 2011 [gr-qc/1107.4322]
- [13] Hergt S, Steinhoff J and Schäfer G 2012 *Ann. Phys. (N.Y.)* **327** 1494
- [14] Levi M 2010 *Phys. Rev. D* **82** 104004
- [15] Damour T, Jaranowski P and Schäfer G 2008 *Phys. Rev. D* **77** 064032
- [16] Steinhoff J, Schäfer G and Hergt S 2008 *Phys. Rev. D* **77** 104018
- [17] Porto R A 2010 *Class. Quant. Grav.* **27** 205001

- [18] Porto R A and Rothstein I Z 2008 *Phys. Rev. D* **78** 044012
- [19] Porto R A and Rothstein I Z 2010 *Phys. Rev. D* **81** 029904(E)
- [20] Steinhoff J, Hergt S and Schäfer G 2008 *Phys. Rev. D* **77** 081501(R)
- [21] Porto R A and Rothstein I Z 2008 *Phys. Rev. D* **78** 044013
- [22] Porto R A and Rothstein I Z 2010 *Phys. Rev. D* **81** 029905(E)
- [23] Hergt S, Steinhoff J and Schäfer G 2010 *Class. Quant. Grav.* **27** 135007