

Gravity in five-dimensional warped product spacetimes with time-dependent warp factor

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Abstract. We have considered gravity in a 5-dimensional warped product spacetime with a time-dependent warp factor and time-dependent extra dimension. The braneworld has been described by a spatially flat FRW-type metric. The five-dimensional field equations have been constructed and solved, and the status of the energy conditions have been examined. In the high energy regime, the bulk is assumed to be sourced by a scalar field. The effective cosmological constant of the 4-dimensional universe is a variable quantity monitored by the time-dependent warp factor, and leads to a geometric interpretation of dynamical dark energy.

1. Introduction

From the higher-dimensional theories of gravity [1], it appears that extra-dimensions play crucial role in a dynamical spacetime. The simplest example is available in the framework of a five-dimensional (5D) spacetime with a single warped extra dimension, in which the 4D cosmological constant arises from the curvature related to the expansion of a (3+1) spacetime.

In the RS model with a single extra dimension [2], matter fields are localized on a 4D hypersurface in a constant curvature 5D bulk with a constant extra-dimensional scale factor. The exponential warp factor is a function of the extra coordinate, and reflects the confining role of the negative bulk cosmological constant to localize gravity on the 3-brane through the curvature of the bulk [3]. However, the process of localization of gravity may include some time-dependence during a particular phase of evolution [4] and may be related to a time-dependent warp factor, which in turn may be related to a time-varying bulk cosmological constant. Hence, we have considered RS-type braneworlds in a 5D warped product spacetime having a non-compact fifth dimension and an exponential warp factor, which depends both on time as well as on the extra-dimensional coordinate. The braneworld is chosen to be spatially flat FRW-type, as it can be embedded in any constant curvature bulk [5]. A stabilized bulk [6] with constant curvature, and characterized by a negative cosmological constant removes the appearance of non-conventional cosmologies [7], and the ordinary FRW equations are recovered at low energies [8]. In the high energy regime, gravity along with scalar fields and particles are able to access the extra dimensions. We, therefore, have considered a bulk sourced by a scalar field minimally coupled to gravity in the high energy regime [9]. The interaction between the bulk and the brane introduces a new term for the change in the extrinsic curvature of the brane into the EFE on the brane [5]. This leads to a geometric interpretation of dynamical dark energy: The effective cosmological constant of the 4D universe is a variable quantity monitored by the time-dependent



warp factor. The gravitational force law is sensitive to the background cosmological expansion at early times and is related to the time-dependent extrinsic curvature of the brane [10]. The zero-mode graviton fluctuation is not guaranteed to be localized on the brane and the effective 4D Newton's constant may not be finite. However, the effect of this time-dependence dies down at later times.

The five-dimensional action is of the form [11]:

$$S = \int d^5x \sqrt{g} \left[\frac{1}{2\bar{\kappa}^2} (\bar{R} - 2\bar{\Lambda}) + \frac{1}{2} g^{AB} \nabla_A \psi \nabla_B \psi - V(\psi) \right] + \int d^4x \sqrt{-g} L_m, \quad (1)$$

and supplemented with the brane curvature term:

$$-M_P^2 \int d^4x \sqrt{-g} R. \quad (2)$$

The 5D metric is given by:

$$dS^2 = e^{2f(t,y)} \left(dt^2 - R^2(t)(dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2) \right) - A^2(t,y) dy^2. \quad (3)$$

As the graviton crosses the brane, it produces a 'bending effect', which varies with time, and is expressed in terms of the extrinsic curvature of the brane, representing the tangential components of the local variation of the normal unit vector at that point [12]. This bending gives rise to an observable result in the form of a smooth scalar function represented by the warping function, which is, therefore, a function of both t and y . The 4D effective action has terms depending on the time-dependent warp factor, and thus, the 4D Newton's constant is time-dependent [13].

2. Five-dimensional field equations

The non-vanishing components of the five-dimensional Einstein tensor are:

$$\bar{G}_t^t = \frac{3}{e^{2f}} \left(\frac{\dot{R}^2}{R^2} + \frac{2\dot{R}\dot{f}}{R} + \dot{f}^2 + \frac{\dot{f}\dot{A}}{A} + \frac{\dot{R}\dot{A}}{R\dot{A}} \right) - \frac{3}{A^2} \left(2f'^2 + f'' - \frac{f'A'}{A} \right), \quad (4)$$

$$\bar{G}_y^t = -\frac{3}{e^{2f}} \left((\dot{f})' - \frac{\dot{A}}{A} f' \right), \quad \bar{G}_t^y = \frac{3}{A^2} \left((\dot{f})' - \frac{\dot{A}}{A} f' \right), \quad (5)$$

$$\bar{G}_y^y = \frac{3}{e^{2f}} \left(\frac{\ddot{R}}{R} + \frac{\dot{R}^2}{R^2} + \frac{3\dot{R}\dot{f}}{R} + \dot{f}^2 + \ddot{f} \right) - \frac{6f'^2}{A^2}, \quad \text{and} \quad (6)$$

$$\bar{G}_j^i = \frac{1}{e^{2f}} \left(\frac{2\ddot{R}}{R} + \frac{\dot{R}^2}{R^2} + \frac{4\dot{R}\dot{f}}{R} + \dot{f}^2 + 2\ddot{f} + \frac{\dot{f}\dot{A}}{A} + \frac{2\dot{R}\dot{A}}{R\dot{A}} + \frac{\ddot{A}}{A} \right) - \frac{3}{A^2} \left(2f'^2 + f'' - \frac{f'A'}{A} \right). \quad (7)$$

The bulk stress-energy tensors are $\bar{T}_t^t = \rho_B$, $\bar{T}_j^i = -P_B$, $\bar{T}_y^y = -P_y$, and $\bar{T}_y^t = -Q_B$. The conservation of the energy-momentum tensor $T_{b;a}^a = 0$ leads to the two equations:

$$\dot{\rho}_B + 3(\rho_B + P_B)(\dot{f} + \frac{\dot{R}}{R}) + (\rho_B + P_y)\frac{\dot{A}}{A} = 0, \quad \text{and} \quad (8)$$

$$P_y' + (\rho_B - 3P_B + 4P_y)f' = 0. \quad (9)$$

3. Results

For an isotropic bulk, $P_B = P_y$, and we have $(\rho_B + P_B) \geq |2Q_B|$, where Q_B may be positive or negative, but $(\rho_B + P_B)$ must be non-negative so as not to violate the WEC. The condition $(\rho_B + P_B) > 0$, corresponds to the *high energy condition* for which $|Q_B| \neq 0$. At low energies, $|Q_B| = 0$ and hence, $(\rho_B + P_B) = 0$. Assuming positive flux for Q_B , we can write $\rho_B = -2Q_B \left(3\dot{f} + \frac{3\dot{R}}{R} + \frac{\dot{A}}{A} \right)$, which represents the equation of continuity for this isotropic bulk.

3.1. Low energy regime

To prevent matter or energy flowing out of the brane along the fifth dimension, we require that $\bar{T}_y^t = 0$, which implies that $\bar{G}_y^t = 0$ and hence, we obtain:

$$\dot{f}' = \frac{\dot{A}}{A} f'. \quad (10)$$

Assuming that A , f and their first order derivatives are continuous, Eq. (10) can be easily integrated to give the result:

$$A(t, y) = \chi(y) f'(t, y). \quad (11)$$

Thus, the extra-dimensional scale factor at a given y and t , depends on the way the warping function varies along the extra dimension at that instant at the specific location. The localization of gravity at different locations in the bulk will be different at different times. Since there are six unknowns (f , R , χ , ρ_B , P_B and P_y), but only three independent field equations as they include the conservation equations, we need to impose additional constraints to solve the field equations.

3.1.1. Conformally flat bulk: The non-zero components of the Weyl tensor for the case in Eq. (11) are equated to zero and solved for $f(t, y)$. This leads us to the condition [14]:

$$f(t, y) = F_1(t) F_2(y), \quad (12)$$

where F_1 and F_2 are arbitrary functions such that:

$$\frac{dF_2(y)}{dy} = 0, \quad (13)$$

which, therefore, represents the unwarped case. Thus, we cannot have a conformally flat bulk with a generalized warp factor and extra-dimensional scale factor.

3.1.2. A stabilized bulk: For this condition, we must have $\dot{A} = 0$, so that $A = A(y)$. Re-scaling $A(y) \rightarrow 1$, we find that for $T_y^t = 0$, we need $\dot{f}' = 0$. Thus, either $f = f(y)$ or $f = f(t)$.

Case (i): $f = f(y)$: This corresponds to the usual RS scenario. For $f'' = 0$, we have $f = \text{const} \times y$, which represents a RS 3-brane embedded into 5D AdS. For a constant curvature bulk, the bulk Weyl tensor vanishes and the bulk is characterized by a negative cosmological constant with $\rho_B = -P_B$. The observed universe will have $q = -1$.

Case (ii): $f = f(t)$: This is the trivial case. We can absorb $f(t)$ inside the 4D metric.

3.1.3. Specific choice $f(t, y) = \frac{1}{2}(\ln \tau(t) + \ln \Gamma(y))$: In this case, the validity of the strong energy condition in the bulk is governed by the nature of warping, as well as by the effect of the extra dimension. For an isotropic bulk with $\tau(t) = t^{-1/2}$ and $R(t) = t$, the weak energy condition is found to be valid.

3.2. High energy regime

Here, the energy scale is in the GeV range. The bulk energy-momentum tensor for a non-self-interacting massless scalar field source [15] is given by:

$$\bar{T}_{IJ}^{scalar} = -\partial_I \phi \partial_J \phi + \frac{1}{2} g_{IJ} \partial_K \phi \partial^K \phi. \quad (14)$$

The components of the bulk energy-momentum tensor for the given metric are:

$$\bar{T}_t^t = \frac{1}{2} \dot{\phi}^2 e^{-2f} + \frac{\phi'^2}{2A^2} = -\bar{T}_y^y, \quad \bar{T}_i^i = \frac{\phi'^2}{2A^2} - \frac{1}{2} \dot{\phi}^2 e^{-2f}, \quad \text{and} \quad \bar{T}_y^t = \dot{\phi} \phi' e^{-2f}. \quad (15)$$

Assuming that $\phi(t, y) = \phi_1(t) + \phi_2(y)$, we find that $\dot{\phi}_1 = k \frac{\dot{A}}{A}$ and $\phi'_2 = \frac{3}{k} f'$, where k is a scalar. Consequently, the solution for the bulk scalar field turns out to be:

$$\phi(t, y) = k \ln A + \frac{3}{k} f. \quad (16)$$

The effective Newton's constant and the cosmological constant on the brane will depend on the scalar field in the bulk.

4. Four-dimensional cosmology

The extrinsic curvature of the brane is governed by the time-dependent warp factor and the field equations for the effective matter has an extra term arising out of the junction conditions. It is geometrical in origin, which modifies the dynamics of the gravitational field compared to Einstein gravity. Consequently, the effective cosmological constant of the 4D universe is a variable quantity monitored by time-dependent warp factor. For an isotropic bulk with $f(t, y) = \frac{1}{2}(\ln \tau(t) + \ln \Gamma(y))$, the universe is initially decelerated, but makes a transition to an accelerated phase at later times.

5. Conclusions

The time-dependent warp factor monitors the localization of gravity and the evolution of the 4-dimensional universe up to the radiation-dominated phase, leading to a geometric interpretation of a dynamical dark energy. At high energies, the effective Newton's constant and the effective cosmological constant on the brane depend on the scalar fields in the bulk.

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