

Localizing the energy and momentum of linear gravity

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Abstract. Although there is little doubt that gravitational waves exist and carry energy as they propagate, it has been notoriously difficult to explain where in spacetime this energy resides. We have summarized a new approach to the localization of gravitational energy-momentum, valid within the linear approximation to general relativity. Built around a local description of the *exchange* of energy-momentum between matter and linear gravity, the framework defines a *unique* symmetric gravitational energy-momentum tensor, free of second derivatives, and motivates a natural gauge-fixing programme, which renders the description unambiguous. Once the gauge has been fixed according to this programme, the gravitational energy-momentum tensor obeys the dominant energy condition: gravitational energy-density is never negative, and gravitational energy-flux is never spacelike.

1. Introduction

Half a century ago, a simple argument established that gravitational waves carry energy and can exchange this energy with matter. Often attributed to Feynman [1] (certainly popularized by Bondi [2]), the argument asked us to imagine a gravitational detector comprising a rigid rod along which two “sticky beads” are threaded. A passing gravitational wave then acts to alter the proper distance between the beads, and this motion, opposed by friction, heats the detector, and thus, mediates a transfer of energy from gravity to matter. Despite the simplicity of this idea, even after fifty years, it has not been possible to explain *where* in spacetime this gravitational energy resides, and it is generally accepted that attempts to do so are “looking for the right answer to the wrong question” [3].

The elusiveness of the “right answer”, and the wrongness of the question, are often identified as arising from gravity’s gauge freedom, the consequence of which is a one-to-many mapping between physical spacetime and whatever localization of gravitational energy-momentum might be proposed. However, there is no reason *a priori* that gauge dependence should preclude the construction of a physically unambiguous tensor, provided we are prepared to remove the gauge freedom in some well-defined way. Unfortunately, no previous approach has supplied instructions of this nature, and more importantly, neither the construction of these energy-momentum objects, nor their key properties, appear to favour one gauge over another; thus, it appears impossible to justify any of these seemingly arbitrary choices as *natural*.

In spite of this difficulty, one aspect of this enduring problem stands opposed to conventional wisdom, and motivates further consideration: when gravity and matter interact, the *exchange* of energy can be localised! To see this, we need look no further than the sticky bead detector: here,



the energy exchange is certainly localized in so far as it takes place only within the confines of the detector. Furthermore, we can imagine a very small detector, much smaller than a wavelength of the incident gravitational radiation, and observe that at each instant, a well-defined power is developed in the detector as heat; thus, at least in this case, the rate of energy exchange is associated with a particular point in spacetime. One might hope, therefore, that consistency with this phenomenon would be enough to localize the energy and momentum of the gravitational field outside the detector, or even when no detector is present.

Based on these ideas, a framework has been developed to localize the energy and momentum of gravity, within the linear approximation to general relativity [4]; what follows is a summary of this work.¹

2. Derivation

We define the gravitational field h_{ab} on a *flat* background spacetime $(\check{\mathcal{M}}, \check{g}_{ab})$ by a diffeomorphism $\phi : \mathcal{M} \rightarrow \check{\mathcal{M}}$ that maps the physical spacetime (\mathcal{M}, g_{ab}) onto the background²:

$$\phi^* g_{ab} = \check{g}_{ab} + h_{ab}. \quad (1)$$

The physical spacetime is assumed to be “nearly flat”, and ϕ is chosen such that h_{ab} is small everywhere, so that the linear approximation to the Einstein’s field equations is valid:

$$\hat{G}_{ab}{}^{cd} h_{cd} = \kappa \check{T}_{ab} + O(h^2), \quad (2)$$

where $\check{T}_{ab} \equiv \phi^* T_{ab} \sim O(h)$ is the matter energy-momentum tensor T_{ab} mapped onto the background, and

$$\hat{G}_{ab}{}^{cd} h_{cd} \equiv \check{\nabla}_c \check{\nabla}_{(a} h_{b)}{}^c - \frac{1}{2} \check{\nabla}^2 h_{ab} - \frac{1}{2} \check{\nabla}_a \check{\nabla}_b h + \frac{1}{2} \check{g}_{ab} \left(\check{\nabla}^2 h - \check{\nabla}_c \check{\nabla}^c h \right) \quad (3)$$

is the linearized Einstein tensor $G_{ab}^{(1)}$.

The gravitational energy-momentum tensor τ_{ab} is defined by seeking a symmetric tensor, quadratic in $\check{\nabla}_c h_{ab}$, which solves:

$$\check{\nabla}_a j_\mu{}^a + \phi^* (\nabla_a J_\mu{}^a) = 0, \quad (4)$$

neglecting terms $O(h^3)$. In the above equation, $J_\mu{}^a \equiv T_b{}^a e_\mu{}^b$ are the (1 energy, 3 momentum) current-densities of matter, associated with the (1 timelike, 3 spacelike) vector fields $e_\mu{}^a \equiv (\phi^{-1})^* \check{e}_\mu{}^a$, the images of the Lorentzian coordinate basis $\check{e}_\mu{}^a \equiv (\partial/\partial x^\mu)^a$ that generate the translational symmetries of the background; the $j_\mu{}^a \equiv \tau_b{}^a \check{e}_\mu{}^b = \tau_\mu{}^a$ constitute the energy-momentum current-densities of the gravitational field. Consequently, Eq. (4) indicates that the extent to which material energy-momentum fails to be conserved at a point in the physical spacetime is exactly equal and opposite to the extent to which gravitational energy-momentum fails to be conserved at the corresponding point in the background. Interactions between matter and gravity can then be understood in terms of a *local exchange* of energy and momentum between the two.

It is not possible to construct a τ_{ab} to solve Eq. (4) for all gravitational fields, so a condition must be placed on h_{ab} in order to proceed. Of all possible symmetric tensors τ_{ab} , quadratic in

¹ We have worked in units where $c = 1$, write $\kappa \equiv 8\pi G$, and use the sign conventions of Wald [5]: $\eta_{\mu\nu} \equiv \text{diag}(-1, 1, 1, 1)$, $[\nabla_c, \nabla_d]v^a \equiv 2\nabla_{[c}\nabla_{d]}v^a \equiv R^a{}_{bcd}v^b$, and $R_{ab} \equiv R^c{}_{acb}$; Roman letters (except i, j, k, l) are used as abstract tensor indices and Greek letters as numerical indices running from 0 to 3. The indices i, j, k, l are reserved for spatial components, and run from 1 to 3.

² As usual, fields defined on \mathcal{M} have their indices raised and lowered with g_{ab} , and those on $\check{\mathcal{M}}$ with \check{g}_{ab} . Lorentzian coordinates $\{x^\mu\}$ are commonly deployed in $\check{\mathcal{M}}$, for which $\check{g}_{\mu\nu} = \eta_{\mu\nu}$.

$\check{\nabla}_c h_{ab}$, and all (non-trivial, linear and Lorentz invariant) field conditions, only one combination solves the Eq. (4) as:

$$\kappa \bar{\tau}_{ab} = \frac{1}{4} \check{\nabla}_a h_{cd} \check{\nabla}_b \bar{h}^{cd}, \quad \text{and} \quad (5)$$

$$\check{\nabla}^a \bar{h}_{ab} = 0, \quad (6)$$

where the overbars signify trace-reversal. Because Eq. (6) is simply the equation of *harmonic gauge*, which can always be satisfied through a choice of ϕ , the field condition does not restrict the physical applicability of our approach in any respect. In fact, the only effect of the field condition is to vastly reduce the gauge freedom in our description of gravitational energy-momentum given in Eq. (5). Essentially, Eq. (6) indicates that ϕ is to be chosen such that it maps Lorentzian coordinates $\{x^\mu\}$ of the background onto harmonic coordinates $y^\mu(p) \equiv x^\mu(\phi(p))$ of the physical spacetime. This ensures that the energy-momentum currents J_μ^a are defined by the generators of a harmonic coordinate system; these represent the approximate translational symmetries of the physical spacetime (present due to its small curvature), and give a sensible replacement for Killing vectors in the absence of an exact symmetry.

3. Infinitesimal probe

A small amount of gauge freedom remains after the harmonic condition in Eq. (6) has been enforced; this ambiguity is extinguished by considering the local exchange of energy-momentum between gravity and an *infinitesimal detector*, as discussed in the introduction. The detector takes the form of a matter “point-source”, with an energy-momentum tensor:

$$\begin{aligned} \check{T}_{00} &= M\delta(\vec{x}) + \frac{1}{2} I_{ij} \partial_i \partial_j \delta(\vec{x}), \\ \check{T}_{0i} &= \frac{1}{2} (\dot{I}_{ij} - L_{ij}) \partial_j \delta(\vec{x}), \quad \text{and} \\ \check{T}_{ij} &= \frac{1}{2} \ddot{I}_{ij} \delta(\vec{x}), \end{aligned} \quad (7)$$

derived by shrinking a compact source down to a point³. The exchange is rendered completely gauge invariant by the *monopole-free micro-average*: The incoming wave is split into a sum of Heaviside step-functions, and the energy-momentum delivered by each is integrated over a vanishingly small 4-volume centred on the probe⁴. The result:

$$\langle \partial^\mu \tau_{\mu\nu} \rangle_{\int}^M = -\frac{1}{4} \delta(\vec{x}) \ddot{I}_{ij} \partial_\nu h_{ij}^{\text{TT}} \quad (8)$$

is equal to the bare (i.e., not micro-averaged) energy-momentum delivered by the incident field in transverse-traceless gauge. This motivates the programme of fixing the final piece of gauge freedom by insisting that the incident h_{ab} be transverse-traceless; consequently, τ_{ab} represents the gauge-invariant gravitational energy-momentum that is accessible to an infinitesimal probe at rest in the transverse-traceless frame.

4. Local positivity

The use of transverse-traceless gauge is further motivated by the *positivity property* of τ_{ab} : Whenever the gravitational field is transverse-traceless, τ_{ab} describes positive energy-density

³ M , I_{ij} and L_{ij} are the mass, moment of inertia, and angular momentum of the detector respectively. Overdots indicate differentiation with respect to $t \equiv x^0$, and the three spatial coordinates are abbreviated $\vec{x} = (x^1, x^2, x^3)$.

⁴ Details are to be found in Sections IV B, and IV C of [4].

and causal energy-flux, for all observers. To state this more rigorously: If, at some point $p \in \check{\mathcal{M}}$, the gravitational field h_{ab} obeys the transverse-traceless conditions:

$$\check{\nabla}^a h_{ab} = 0, \quad h = 0, \quad \text{and} \quad u^a h_{ab} = 0, \quad (9)$$

for some timelike vector u^a , then τ_{ab} satisfies the following inequalities:

$$v^a \tau_{ab} v^b \geq 0, \quad \text{and} \quad (10a)$$

$$v^a \tau_{ac} \tau^c_b v^b \leq 0, \quad (10b)$$

at p , for any timelike vector v^a . This is a remarkable and exceptionally valuable property that has not been seen in any previous localization of gravitational energy-momentum. It extends the *dominant energy condition* to include the energy-momentum of the gravitational field, and ensures that τ_{ab} provides a description of gravitational energy-momentum that makes intuitive sense on local scales.

5. Closing remarks

By considering the local exchange of energy-momentum between matter and linear gravity in Eq. (4), a unique symmetric gravitational energy-momentum tensor τ_{ab} , free of second derivatives, was derived in Eq. (5); the harmonic gauge condition given in Eq. (6) arose as a natural consequence of this derivation, and extinguished the vast majority of the description's gauge freedom. The last trace of gauge freedom was then fixed by insisting that the gravitational field be transverse-traceless. This programme, not only guarantees an agreement with the gauge-invariant energy-momentum transferred onto a infinitesimal probe given in Eq. (8), but also ensures that τ_{ab} encodes positive energy-density in Eq. (10a) and causal energy flux given in Eq. (10b).

References

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