Quantum Gaudin model and classical KP hierarchy

A. Zabrodin

Institute of Biochemical Physics, 4 Kosygina, 119334, Moscow, Russia; ITEP, 25 B. Cheremushkinskaya, 117218, Moscow, Russia; National Research University Higher School of Economics, 20 Myasnitskaya Ulitsa, Moscow 101000, Russia

Abstract. This short note is a review of the intriguing connection between the quantum Gaudin model and the classical KP hierarchy recently established in [1]. We construct the generating function of integrals of motion for the quantum Gaudin model with twisted boundary conditions (the master T-operator) and show that it satisfies the bilinear identity and Hirota equations for the classical KP hierarchy. This implies that zeros of eigenvalues of the master T-operator in the spectral parameter have the same dynamics as the Calogero-Moser system of particles.

1. Introduction

In [1], a remarkable correspondence between the quantum Gaudin model and the classical Kadomtsev-Petviashvili (KP) hierarchy was established. It is a limiting case of the correspondence between quantum spin chains with the Yangian $Y(gl(N))$ symmetry algebra and the classical modified KP (mKP) hierarchy based on the construction of the master T-operator [2, 3]. The master T-operator was introduced in [2] (in a preliminary form, it previously appeared in [4]). It is a special generating function for commuting integrals of motion in the quantum model. In the Gaudin model case, any eigenvalue of the master T-operator appears to be the tau-function of the KP hierarchy, with polynomial dependence on the spectral parameter. Taking into account the well known story about dynamics of poles of rational solutions to soliton equations [5, 6, 7], this implies a link between the quantum Gaudin model [8] and the classical Calogero-Moser (CM) system of particles [9]. This link was also observed earlier in [10] using different arguments.

In this paper we present the results of [1] in a short compressed form. All proofs and technical details are omitted. Here we outline the results reviewed in the paper. Most of them can be obtained by a limiting procedure from the corresponding results for quantum spin chains proved in [2].

Using the matrix derivative operation, we construct commuting integrals of motion for the $gl(N)$ Gaudin model, with twisted boundary conditions and vector representations at the marked points in the quantum space, corresponding to arbitrary representations in the auxiliary space. The master T-operator is their generating function. It depends on an infinite number of auxiliary "time variables" $\mathbf{t} = \{t_1, t_2, t_3, \ldots\}$, where t_1 can be identified with the spectral parameter x. The master T-operator satisfies the bilinear identity for the classical KP hierarchy and hence any of its eigenvalues is a KP tau-function [13, 14]. This is a development of earlier


Content from this work may be used under the terms of the Creative Commons Attribution 3.0 licence. Any further distribution of this work must maintain attribution to the author(s) and the title of the work, journal citation and DOI.
Published under licence by IOP Publishing Ltd

studies [15, 16, 17] clarifying the role of the Hirota bilinear equations [18] in quantum integrable models.

Moreover, all eigenvalues of the master T-operator are *polynomial* tau-functions in $x = t_1$. Therefore, according to [6, 7], the dynamics of their roots in t_i with $i > 1$ is given by equations of motion of the classical CM system of particles. The marked points x_i in the Gaudin model (the inhomogeneities at the sites in the spin chain language) should be identified with initial coordinates of the CM particles while eigenvalues of the Gaudin Hamiltonians are proportional to their initial momenta. Eigenvalues of the Lax matrix for the CM model coincide with eigenvalues of the twist matrix (with certain multiplicities). Therefore, with fixed integrals of motion in the CM model determined by invariants of the twist matrix, there are finite number of solutions for the values of initial momenta which correspond to different eigenstates of the Gaudin model. In other words, the eigenstates of the Gaudin Hamiltonians are in one-to-one correspondence with (a finite number of) intersection points of two Lagrangian submanifolds in the phase space of the classical CM model. This "quantum-classical correspondence" was also discussed in [11, 12] in the context of supersymmetric gauge theories and branes.

In distinction to paper [1], here we consider the Gaudin model with a formal Planck's constant $\hbar$. In this case the master T-operator is the tau-function of the $\hbar$-dependent version of the KP hierarchy [19] and the coupling constant of the CM model becomes proportional to $\hbar^2$.

2. The quantum Gaudin model

Let $\mathsf{e}^{\hbar}_{ab}$ be generators of the "$\hbar$-dependent version" of the universal enveloping algebra $U(gl(N))$ with the commutation relations $[\mathsf{e}^{\hbar}_{ab}, \mathsf{e}^{\hbar}_{a'b'}] = \hbar(\delta_{a'b}\mathsf{e}^{\hbar}_{ab'} - \delta_{ab'}\mathsf{e}^{\hbar}_{a'b})$. The parameter $\hbar$ will play the role of the Planck's constant. Let π_λ be the finite dimensional irreducible representation of $U(gl(N))$ with the highest weight λ. We identify λ with the Young diagram $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$ with $\ell = \ell(\lambda)$ non-zero rows, where $\lambda_i \in \mathbb{Z}_+$, $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_\ell > 0$. For example, $\pi_{(1)}$ is the N-dimensional vector representation corresponding to the 1-box diagram $\lambda = (1)$. We have $\pi_{(1)}(\mathsf{e}^{\hbar}_{ab}) = \hbar e_{ab}$, where e_{ab} is the standard basis in the space of $N \times N$ matrices: the matrix e_{ab} has only one non-zero element (equal to 1) at the place ab: $(e_{ab})_{a'b'} = \delta_{aa'}\delta_{bb'}$. Note that $I = \sum_a e_{aa}$ is the unity matrix and $P = \sum_{ab} e_{ab} \otimes e_{ba}$ is the permutation operator acting in the space $\mathbb{C}^N \otimes \mathbb{C}^N$.

In the tensor product $U(gl(N))^{\otimes n}$ the generators $\mathsf{e}^{\hbar}_{ab}$ can be realized as $\mathsf{e}^{\hbar\,(i)}_{ab} := \mathrm{id}^{\otimes(i-1)} \otimes \mathsf{e}^{\hbar}_{ab} \otimes \mathrm{id}^{\otimes(n-i)}$. Clearly, they commute for any $i \neq j$ and any a, b because act non-trivially in different spaces. Similarly, for any matrix $g \in \mathrm{End}(\mathbb{C}^N)$ we define $g^{(i)}$ acting in the tensor product $\mathcal{V} = (\mathbb{C}^N)^{\otimes n}$: $g^{(i)} = I^{\otimes(i-1)} \otimes g \otimes I^{\otimes(n-i)} \in \mathrm{End}(\mathcal{V})$. In this notation, $P_{ij} := \sum_{a,b} e^{(i)}_{ab} e^{(j)}_{ba}$ is the permutation operator of the i-th and j-th tensor factors in $\mathcal{V} = \mathbb{C}^N \otimes \ldots \otimes \mathbb{C}^N$.

Fix n distinct numbers $x_i \in \mathbb{C}$ and a diagonal $N \times N$ matrix $g_0 = \mathrm{diag}\,(k_1, \ldots, k_N)$. (We assume that the k_i's are all distinct and non-zero.) We will call g_0 the twist matrix. The commuting Gaudin Hamiltonians are

$$H_i = \sum_{a=1}^{N} k_a \mathsf{e}^{\hbar\,(i)}_{aa} + \sum_{j\neq i} \sum_{a,b=1}^{N} \frac{\mathsf{e}^{\hbar\,(i)}_{ab}\mathsf{e}^{\hbar\,(j)}_{ba}}{x_i - x_j}\,, \quad i = 1, \ldots, n. \tag{1}$$

The Hamiltonians of the quantum Gaudin model [8] with the Hilbert space $\mathcal{V} = (\mathbb{C}^N)^{\otimes n}$ are restrictions of the operators (1) to the N-dimensional vector representation:

$$H_i = \hbar \sum_{a=1}^{N} k_a e^{(i)}_{aa} + \hbar^2 \sum_{j\neq i} \sum_{a,b=1}^{N} \frac{e^{(i)}_{ab} e^{(j)}_{ba}}{x_i - x_j} = \hbar g_0^{(i)} + \hbar^2 \sum_{j\neq i} \frac{P_{ij}}{x_i - x_j}\,, \quad i = 1, \ldots, n. \tag{2}$$

It is easy to check that the operators

$$M_a = \sum_{l=1}^{n} e_{aa}^{(l)} \tag{3}$$

commute with the Gaudin Hamiltonians: $[H_i, M_a] = 0$. Therefore, common eigenstates of the Hamiltonians can be classified according to eigenvalues of the operators M_a. Let

$$\mathcal{V} = \bigotimes_{i=1}^{n} V_i \;=\; \bigoplus_{m_1,\ldots,m_N} \mathcal{V}(\{m_a\})$$

be the decomposition of the Hilbert space of the Gaudin model $\mathcal{V}$ into the direct sum of eigenspaces for the operators M_a with the eigenvalues $m_a \in \mathbb{Z}_{\geq 0}$, $a = 1, \ldots, N$. Then the eigenstates of H_i's are in the spaces $\mathcal{V}(\{m_a\})$. Clearly, $\sum_a M_a = nI^{\otimes n}$, and hence $\sum_{a=1}^{N} m_a = n$. Note also that $\sum_{i=1}^{n} H_i = \sum_{a=1}^{N} k_a M_a$.

A more general family of commuting Gaudin Hamiltonians was discussed in [20, 21, 22]. In [1], it was shown that the $gl(N)$ Gaudin model with vector representations at the sites admits a very simple construction of the higher commuting Hamiltonians, which is a version of the one suggested in [23] for the spin chains of the XXX-type. The main technical tool is the matrix derivative.

Let g be an element of the Lie algebra $gl(N)$ and f be any function of g, with values in $\mathrm{End}(L)$, where L is the space of any representation of $gl(N)$. The matrix derivative is defined as follows:

$$\mathsf{d} f(g) = \frac{\partial}{\partial \varepsilon} \sum_{ab} e_{ab} \otimes f(g + \varepsilon \mathsf{e}_{ba})\Big|_{\varepsilon=0}. \tag{4}$$

The right hand side belongs to $\mathrm{End}\,(\mathbb{C}^N \otimes L)$. For example:

$$\mathsf{d}\,(\operatorname{tr} g)^k = k(\operatorname{tr} g)^{k-1} I, \quad \mathsf{d} \operatorname{tr} g^k = k g^{k-1}, \quad \mathsf{d} g^k = P \sum_{i=0}^{k-1} g^i \otimes g^{k-i-1} \quad \text{for} \quad k \in \mathbb{Z}_{\geq 0}$$

When the number of tensor factors is more than two another notation is more convenient. Let $V_i \cong \mathbb{C}^N$ be copies of $\mathbb{C}^N$ and $\mathcal{V} = V_1 \otimes \ldots \otimes V_n$ as before. Then, applying the matrix derivatives to a scalar function f several times, we can embed the result into $\mathrm{End}(\mathcal{V})$ according to the formulas

$$\mathsf{d}_i f(g) = \frac{\partial}{\partial \varepsilon} \sum_{ab} e_{ab}^{(i)} f(g + \varepsilon \mathsf{e}_{ba})\Big|_{\varepsilon=0},$$

$$\mathsf{d}_{i_2}\mathsf{d}_{i_1} f(g) = \frac{\partial}{\partial \varepsilon_2}\frac{\partial}{\partial \varepsilon_1} \sum_{a_2 b_2} \sum_{a_1 b_1} e_{a_2 b_2}^{(i_2)} e_{a_1 b_1}^{(i_1)} f\left(g + \varepsilon_1 \mathsf{e}_{b_1 a_1} + \varepsilon_2 \mathsf{e}_{b_2 a_2}\right)\Big|_{\varepsilon_1=\varepsilon_2=0}$$

and so on. The lower indices of d show in which tensor factors the resulting operator acts non-trivially.

Let $\chi_\lambda(g) = \operatorname{tr}_{\pi_\lambda} g$ be the character of the representation π_λ calculated for a matrix g. It is given in terms of the Schur polynomials $s_\lambda(\mathbf{y})$ of the variables $\mathbf{y} = \{y_1, y_2, \ldots\}$, $y_k = \frac{1}{k} \operatorname{tr} g^k$:

$$\chi_\lambda(g) = s_\lambda(\mathbf{y}) = \det_{i,j=1,\ldots,\ell(\lambda)} h_{\lambda_i - i + j}(\mathbf{y}) \tag{5}$$

(the Jacobi-Trudi formula), with the complete symmetric polynomials $h_k(\mathbf{y}) = s_{(k)}(\mathbf{y})$ defined by

$$\exp\Big(\xi(\mathbf{y},z)\Big) = \sum_{k=0}^{\infty} h_k(\mathbf{y}) z^k, \tag{6}$$

where $\xi(\mathbf{y},z) := \sum_{k\geq 1} y_k z^k$. For example, $\chi_{(1)}(g) = \operatorname{tr} g$. It is convenient to set $h_k = 0$ at $k < 0$. Let $p_1, \ldots, p_N$ be eigenvalues of g realized as an element of End $(\mathbb{C}^N)$. Then $y_k = \frac{1}{k}\,(p_1^k + \ldots + p_N^k)$ and $\chi_\lambda(g) = \dfrac{\det_{1\leq i,j\leq N}\big(p_j^{\lambda_i+N-i}\big)}{\det_{1\leq i,j\leq N}\big(p_j^{N-i}\big)}$ (see [24]). This formula implies that $\chi_\emptyset(g) = s_\emptyset(\mathbf{y}) = 1$. The characters form a special class of scalar functions on the space of $N\times N$ matrices which is of primary importance for us.

Now we are ready to construct the family of commuting operators for the Gaudin model:

$$\mathsf{T}^G_\lambda(x) = \Big(1 + \frac{\hbar \mathsf{d}_n}{x - x_n}\Big) \ldots \Big(1 + \frac{\hbar \mathsf{d}_1}{x - x_1}\Big) \chi_\lambda(g_0) \tag{7}$$

By the analogy with spin chains, we will call them Gaudin transfer-matrices. The first few are: $\mathsf{T}^G_\emptyset(x) = 1$, $\mathsf{T}^G_{(1)}(x) = \operatorname{tr} g_0 + \sum_i \dfrac{\hbar}{x - x_i}$ and

$$\mathsf{T}^G_{(1^2)}(x) = \chi_{(1^2)}(g_0) + \operatorname{tr} g_0 \sum_i \frac{\hbar}{x - x_i} + \sum_{i<j} \frac{\hbar^2}{(x - x_i)(x - x_j)} - \sum_i \frac{H_i}{x - x_i}, \tag{8}$$

From the last formula we see that the Gaudin Hamiltonians H_i belong to this family. For one-column diagrams, this construction agrees with Talalaev's prescription [22]. At $n = 0$ the transfer matrix is just the character: $\mathsf{T}^{G\,(n=0)}_\lambda(x) = \chi_\lambda(g_0)$.

In what follows, the normalization such that any Gaudin transfer matrix is a polynomial in x will be more convenient. In the polynomial normalization, the Gaudin transfer matrices are introduced by the formula

$$T^G_\lambda(x) = (x - x_n + \hbar \mathsf{d}_n) \ldots (x - x_1 + \hbar \mathsf{d}_1) \chi_\lambda(g_0). \tag{9}$$

All these operators commute for any x and λ.

3. The master T-operator and the KP hierarchy

Let $\mathbf{t} = \{t_1, t_2, \ldots\}$ be an infinite set of "time parameters". The master T-operator for the Gaudin model is the following generating function for the transfer-matrices $T^G_\lambda(x)$:

$$T^G(x, \mathbf{t}) = \sum_\lambda T^G_\lambda(x) s_\lambda(\mathbf{t}/\hbar). \tag{10}$$

These operators commute for different values of the parameters: $[T^G(x,\mathbf{t}),\, T^G(x',\mathbf{t}')] = 0$. Since $\sum_\lambda \chi_\lambda(g_0) s_\lambda(\mathbf{t}/\hbar) = \exp\Big(\frac{1}{\hbar}\sum_{k\geq 1} t_k \operatorname{tr} g_0^k\Big)$ (the Cauchy-Littlewoodn identity, see, e.g., [24]), we can define the master T-operator more explicitly as

$$T^G(x, \mathbf{t}) = (x - x_n + \hbar \mathsf{d}_n) \ldots (x - x_1 + \hbar \mathsf{d}_1) \exp\Big(\frac{1}{\hbar} \sum_{k\geq 1} t_k \operatorname{tr} g_0^k\Big). \tag{11}$$

Note that because $e^{-t \operatorname{tr} g} \mathsf{d} e^{t \operatorname{tr} g} = t$, the role of the variable t_1 is to shift $x \to x + t_1$, so that $e^{\frac{1}{\hbar} x \operatorname{tr} g_0} T^G(x, \mathbf{t})$ depends on x, t_1 only through their sum $x + t_1$.

The master T-operator contains the complete information about the spectrum of all transfer-matrices. They can be restored from it according to the formula

$$T^G_\lambda(x) = s_\lambda(\hbar\tilde\partial)T^G(x,\mathbf{t})\Big|_{\mathbf{t}=0}, \tag{12}$$

where $\tilde\partial = \{\partial_{t_1}, \frac{1}{2}\,\partial_{t_2}, \frac{1}{3}\,\partial_{t_3}, \ldots\}$. In particular,

$$T^G_{(1)}(x) = \hbar\partial_{t_1}T^G(x,\mathbf{t})\Big|_{\mathbf{t}=0}, \quad T^G_{(1^2)}(x) = \frac{1}{2}\,(\hbar^2\partial^2_{t_1} - \hbar\partial_{t_2})T^G(x,\mathbf{t})\Big|_{\mathbf{t}=0}. \tag{13}$$

For any $z\in\mathbb{C}$ we put $\mathbf{t}\pm\hbar[z^{-1}] := \Big\{t_1 \pm \hbar z^{-1},\, t_2 \pm \frac{\hbar}{2}\,z^{-2},\, t_3 \pm \frac{\hbar}{3}\,z^{-3},\, \ldots\Big\}$. As we shall see below, $T^G(x,\mathbf{t}\pm\hbar[z^{-1}])$ regarded as functions of z with fixed $\mathbf{t}$ play an important role. Here we only note that equation (12) implies that $T^G(x,0\pm\hbar[z^{-1}])$ is the generating series for T-operators corresponding to the one-row and one-column diagrams:

$$T^G(x,\hbar[z^{-1}]) = \sum_{s\geq 0} z^{-s}T^G_{(s)}(x), \qquad T^G(x,-\hbar[z^{-1}]) = \sum_{a=0}^{N}(-z)^{-a}T^G_{(1^a)}(x). \tag{14}$$

The following theorem is the main result of [1].

Theorem 3.1 *The master T-operator (11) satisfies the bilinear identity for the $\hbar$-dependent KP hierarchy [13, 14, 19]:*

$$\oint_\infty e^{\frac{1}{\hbar}\xi(\mathbf{t}-\mathbf{t}',z)}\,T^G\left(x,\mathbf{t}-\hbar[z^{-1}]\right)T^G\left(x,\mathbf{t}'+\hbar[z^{-1}]\right)dz = 0 \quad \textit{for all } \mathbf{t},\mathbf{t}'. \tag{15}$$

The integration contour is chosen in such a way that it encircles all singularities coming from the T^G's and none of those coming from $e^{\frac{1}{\hbar}\xi(\mathbf{t}-\mathbf{t}',z)}$.

This means that each eigenvalue of the master T-operator is a tau-function of the KP hierarchy. The general bilinear identity (15) implies many bilinear funtional relations for the master T-operator (the Hirota equations). Some of them are written explicitly in [1]. Equation (10) is the expansion of the tau-function in Schur polynomials [13, 25, 26].

As soon as the Gaudin model is linked to the KP hierarchy, it is tempting to ask about the role of the other standard ingredients of the KP theory. We will be mostly interested in the Baker-Akhiezer (BA) function. According to the general scheme, the BA function corresponding to the tau-function (11) is given by the formula

$$\psi(x,\mathbf{t};z) = e^{\frac{1}{\hbar}(xz+\xi(\mathbf{t},z))}(T^G(x,\mathbf{t}))^{-1}\,T^G\big(x,\mathbf{t}-\hbar[z^{-1}]\big). \tag{16}$$

and satisfies the differential equation

$$\hbar\partial_{t_2}\psi(x,\mathbf{t};z) = \hbar^2\partial_x^2\psi(x,\mathbf{t};z) + 2u(x,\mathbf{t})\psi(x,\mathbf{t};z), \tag{17}$$

where $u(x,\mathbf{t}) = \hbar^2\partial_x^2\log T^G(x,\mathbf{t})$. Using the definition (11), we have:

$$\psi(x,\mathbf{t};z) = z^{-N}e^{\frac{1}{\hbar}(xz+\xi(\mathbf{t},z))}(T^G(x,\mathbf{t}))^{-1}\prod_{i=1}^{\overleftarrow{n}}(x-x_i+\hbar\mathsf{d}_i)\left[\det\,(zI-g_0)\exp\Big(\frac{1}{\hbar}\sum_{k\geq 1}t_k\,\mathrm{tr}\,h^k\Big)\right] \tag{18}$$

(here and below $\prod_{i=1}^{\leftarrow n} A_i = A_n \dots A_2 A_1$). From this formula we see that $e^{-\frac{1}{\hbar}(xz+\xi(\mathbf{t},z))}\psi(x,\mathbf{t};z)$ is a polynomial in z^{-1} of degree N. Regarded as a function of x, it is a rational function with n zeros and n poles which are simple in general position. From (11), (18) it follows that

$$\lim_{x\to\infty}\left(e^{-\frac{1}{\hbar}(xz+\xi(\mathbf{t},z))}\psi(x,\mathbf{t};z)\right) = z^{-N}\det(zI-g_0). \tag{19}$$

We also note the formula for the *stationary* BA function $\psi(x,z) := \psi(x,0;z)$, which directly follows from (18):

$$\psi(x,z) = z^{-N}e^{\frac{1}{\hbar}xz}\prod_{i=1}^{\leftarrow n}\left(1+\frac{\hbar \mathsf{d}_i}{x-x_i}\right)\det(zI-g_0). \tag{20}$$

4. The link to the classical Calogero-Moser model

Integrability of the Gaudin model implies that eigenvalues of the master T-operator in the polynomial normalization are polynomials in the spectral parameter x of degree n:

$$T^G(x,\mathbf{t}) = e^{\frac{1}{\hbar}t_1\operatorname{tr} g_0+\frac{1}{\hbar}t_2\operatorname{tr} g_0^2+\dots}\prod_{k=1}^{n}(x+t_1-x_k(t_2,t_3,\dots)). \tag{21}$$

The roots of each eigenvalue have their own dynamics in the times t_k. This dynamics is known [6, 7] to be given by the rational CM model [9]. The parameters x_i of the Gaudin model play the role of coordinates of the CM particles at $t_i = 0$: $x_j = x_j(0)$. In particular, we have $T^G(x,0) = T^G_\emptyset(x) = \prod_{k=1}^n(x-x_k)$.

Using (13), we easily obtain the formula for $\mathsf{T}^G_{(1)}(x) = T^G_{(1)}(x)/T^G_\emptyset(x)$ given in (8). For $\mathsf{T}^G_{(1^2)}(x)$ the second equation in (13) yields

$$\mathsf{T}^G_{(1^2)}(x) = \chi_{(1^2)}(g_0) + \operatorname{tr} g_0\sum_i\frac{\hbar}{x-x_i} + \sum_{i<j}\frac{\hbar^2}{(x-x_i)(x-x_j)} + \frac{1}{2}\sum_i\frac{\hbar\dot{x}_i}{x-x_i},$$

where $\dot{x}_i = \partial_{t_2}x_i(t_2)\Big|_{t_2=0}$. Comparing with the third equation in (8), we conclude that the initial velocities are proportional to eigenvalues of the Gaudin Hamiltonians:

$$\hbar\dot{x}_i = -2H_i. \tag{22}$$

This unexpected connection between the quantum Gaudin model and the classical CM model was observed in [10] using different arguments.

Following [6], one can derive equations of motion for the t_k-dynamics of the x_i's performing the pole expansion of the linear problem (17) for the BA function ψ. It is convenient to denote $t_2 = t$ and put all othet times equal to 0 because they are irrelevant in this derivation. The pole ansatz for the BA function is

$$\psi = e^{\frac{1}{\hbar}(xz+tz^2)}\left(c_0(z)+\sum_{i=1}^{n}\frac{c_i(z,t)}{x-x_i(t)}\right), \tag{23}$$

where $c_0(z) = \det(I-z^{-1}g_0)$. One should substitute it into the linear problem (17) with $u = -\sum_{i=1}^{n}\frac{\hbar^2}{(x-x_i)^2}$ and cancel all the poles at the points x_i. This yields an overdetermined

system of equations for the coefficients c_i. Their compatibility implies the the Lax representation for the CM model:

$$\dot{Y} = [T,\, Y], \tag{24}$$

where the $n\times n$ matrices Y, T are given by

$$Y_{ik} = -p_i\delta_{ik} - \hbar\,\frac{1-\delta_{ik}}{x_i - x_k}\,, \qquad p_i := \frac{1}{2}\,\dot{x}_i, \tag{25}$$

$$T_{ik} = -\delta_{ik}\sum_{j\neq i}\frac{2\hbar}{(x_i-x_j)^2} + \frac{2\hbar(1-\delta_{ik})}{(x_i-x_k)^2}. \tag{26}$$

The equations of motion are:

$$\ddot{x}_i = -8\sum_{j\neq i}\frac{\hbar^2}{(x_i-x_j)^3}. \tag{27}$$

Set $X = X(\mathbf{t}) = \mathrm{diag}(x_1(\mathbf{t}),\dots,x_n(\mathbf{t}))$. For the function ψ itself we then have:

$$\psi = \det(I - z^{-1}g_0)\, e^{\frac{1}{\hbar}(xz+tz^2+t_3z^3+\dots)}\Big(1 - \hbar 1^{\mathsf{t}}(xI-X)^{-1}(zI-Y)^{-1}1\Big), \tag{28}$$

where $1 = (1,1,\dots,1)^{\mathsf{t}}$ is the n-component vector. As is well known (and easy to check), the matrices X, Y satisfy the commutation relation

$$[X,\, Y] = \hbar(I - 1\otimes 1^{\mathsf{t}}) \tag{29}$$

(here $1\otimes 1^{\mathsf{t}}$ is the $n\times n$ matrix of rank 1 with all entries equal to 1).

The matrix Y is the Lax matrix for the CM model. As is seen from (24), the time evolution preserves its spectrum, i.e., the coefficients $\mathcal{J}_k$ of the characteristic polynomial

$$\det(zI - Y(t)) = \sum_{k=0}^{n}\mathcal{J}_k z^{n-k} \tag{30}$$

are integrals of motion. The highest integral, $\mathcal{J}_n$, was found explicitly in [27], where a recurrence procedure for finding all other integrals of motion was also suggested. In fact this procedure is equivalent to the following explicit expression for the characteristic polynomial:

$$\det\big(zI - Y(t)\big) = \exp\Big(\sum_{i<j}\frac{\hbar^2\partial_{p_i}\partial_{p_j}}{(x_i-x_j)^2}\Big)\prod_{l=1}^{n}(z+p_l). \tag{31}$$

Note that this expression is well-defined because the sum obtained after expansion of the exponential function in the r.h.s. contains a finite number of non-zero terms.

One can see that eigenvalues of the Lax matrix Y are the same as eigenvalues of the twist matrix g_0 (with appropriate multiplicities). Indeed, let us compare expansions of (20) and (28) at large x. From (20) we have:

$$\psi(x,z) = \det(I - z^{-1}g_0)\, e^{\frac{1}{\hbar}xz}\left(1 - \frac{\hbar}{x}\sum_i\sum_a\frac{e^{(i)}_{aa}}{z-k_a} + O(x^{-2})\right).$$

Using the commutation relation (29) it is easy to check that for any $k\geq 0$ it holds $1^{\mathsf{t}}Y^k 1 = \mathrm{tr}\, Y^k$. Taking this into account, we can expand (28) at $\mathbf{t}=0$:

$$\psi(x,z) = \det(I - z^{-1}g_0)\, e^{\frac{1}{\hbar}xz}\left(1 - \frac{\hbar}{x}\,\mathrm{tr}\,\frac{1}{z-Y_0} + O(x^{-2})\right),$$

where $Y_0 := Y(0)$. Therefore, we conclude that $\mathrm{tr}\,\dfrac{1}{zI-Y_0} = \sum\limits_i \sum\limits_a \dfrac{e^{(i)}_{aa}}{z-k_a}$ and, since $\mathrm{tr}\,(zI-Y_0)^{-1} = \partial_z \log\det(zI-Y_0)$, we have

$$\det(zI-Y_0) = \prod_{a=1}^{N}(z-k_a)^{\sum_{i=1}^n e^{(i)}_{aa}} = \prod_{a=1}^{N}(z-k_a)^{M_a}, \tag{32}$$

where M_a is the operator (3). Hence we see that the M_a is the "operator multiplicity" of the eigenvalue k_a. In the sector $\mathcal{V}(\{m_a\})$ the multiplicity becomes equal to m_a. This argument allows one to prove the following important statement:

Theorem 4.1 *The eigenvalues of the Lax matrix Y of the CM model are numbers from the set $\{k_1, k_2, \ldots, k_N\}$ (the eigenvalues of the twist matrix) with multiplicities $m_a \geq 0$ such that $m_1 + \ldots + m_N = n$, with the m_a's being eigenvalues of the operators M_a.*

The hamiltonian form of equations of motion (27) is $\begin{pmatrix} \dot{x}_i \\ \dot{p}_i \end{pmatrix} = \begin{pmatrix} \partial_{p_i}\mathcal{H}_2 \\ -\partial_{x_i}\mathcal{H}_2 \end{pmatrix}$ with the Hamiltonian

$$\mathcal{H}_2 = \mathrm{tr}\,Y^2 = \sum_i p_i^2 - \sum_{i<j}\frac{2\hbar^2}{(x_i-x_j)^2}\,. \tag{33}$$

This result can be extended to the whole hierarchy [7]:

$$\begin{pmatrix} \partial_{t_k} x_i \\ \partial_{t_k} p_i \end{pmatrix} = \begin{pmatrix} \partial_{p_i}\mathcal{H}_k \\ -\partial_{x_i}\mathcal{H}_k \end{pmatrix}, \qquad \mathcal{H}_k = \mathrm{tr}\,Y^k. \tag{34}$$

The $\mathcal{H}_k$'s are higher integrals of motion for the CM model. They are known to be in involution [27, 28, 29]. This agrees with commutativity of the KP flows. The integrals $\mathcal{H}_k$ are connected with the integrals $\mathcal{J}_k$ introduces in (30) by Newton's formula [24] $\sum\limits_{k=0}^{n}\mathcal{J}_{n-k}\mathcal{H}_k = 0$ (we have set $\mathcal{H}_0 = \mathrm{tr}Y^0 = n$).

The results of [7] imply an explicit determinant representation of the tau-function. It is easy to adopt it for the master T-operator $T^G(x,\mathbf{t})$ (21). Let $X_0 = X(0)$ be the diagonal matrix $X_0 = \mathrm{diag}(x_1, x_2, \ldots, x_n)$, where $x_i = x_i(0)$ and Y_0 be the Lax matrix (25) at $\mathbf{t} = 0$, with the diagonal elements being proportional to the Gaudin Hamiltonians $H_i = -\hbar p_i(0)$:

$$Y_0 = \begin{pmatrix} \dfrac{H_1}{\hbar} & \dfrac{\hbar}{x_2-x_1} & \dfrac{\hbar}{x_3-x_1} & \cdots & \dfrac{\hbar}{x_n-x_1} \\ \dfrac{\hbar}{x_1-x_2} & \dfrac{H_2}{\hbar} & \dfrac{\hbar}{x_3-x_2} & \cdots & \dfrac{\hbar}{x_n-x_2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \dfrac{\hbar}{x_1-x_n} & \dfrac{\hbar}{x_2-x_n} & \dfrac{\hbar}{x_3-x_n} & \cdots & \dfrac{H_n}{\hbar} \end{pmatrix}. \tag{35}$$

Theorem 4.2 *The master T-operator for the Gaudin model is given by*

$$T^G(x,\mathbf{t}) = e^{\frac{1}{\hbar}\sum_{k\geq 1} t_k \mathrm{tr}\,h^k}\det\left(xI - X_0 + \sum_{k\geq 1} k t_k Y_0^{k-1}\right). \tag{36}$$

It follows from the above arguments that eigenvalues of the Gaudin Hamiltonians H_i, $i = 1, \dots, n$ (2), can be found in the framework of the classical CM system with n particles. Namely, the spectrum of H_i's in the space $\mathcal{V}(\{m_a\})$ is determined by the conditions

$$\operatorname{tr} Y_0^j = \sum_{a=1}^{N} m_a k_a^j \quad \text{for all} \quad j \geq 1, \tag{37}$$

i.e., given the initial coordinates x_i and the action variables $\mathcal{H}_j = \operatorname{tr} Y_0^j$ one has to find possible values of the initial momenta $p_i = -H_i/\hbar$. Taking into account equations (30) and (31), we can represent the equations for H_i in the form of the equality

$$\exp\Bigl(\hbar^4 \sum_{i<j} x_{ij}^{-2} \partial_{H_i} \partial_{H_j}\Bigr) \prod_{l=1}^{n} (z - \hbar^{-1} H_l) = \prod_{a=1}^{N} (z - k_a)^{m_a}\,, \qquad x_{ij} \equiv x_i - x_j \tag{38}$$

which has to be satisfied identically in z. This identity is equivalent to n algebraic equations for n quantities $H_1, \dots, H_n$.

We see that the eigenstates of the Gaudin Hamiltonians correspond to the intersection points of two Lagrangian submanifolds: one obtained by fixing the x_i's and the other obtained by fixing the $\mathcal{H}_i$'s, with values of the latter being determined by eigenvalues of the twist matrix. This purely classical prescription appears to be equivalent to the Bethe ansatz solution and solves the spectral problem for the quantum Gaudin Hamiltonians.

Example. Consider the vector $\mathsf{v}_a \in \mathbb{C}^N$ with components $(\mathsf{v}_a)_b = \delta_{ab}$. Since $P_{ij}(\mathsf{v}_a)^{\otimes n} = (\mathsf{v}_a)^{\otimes n}$, the vector $(\mathsf{v}_a)^{\otimes n}$ is an eigenstate for the Gaudin Hamiltonians H_i with the eigenvalues $k_a + \sum_{j \neq i} \dfrac{\hbar}{x_i - x_j}$. It is also an eigenvector for the operators M_b with eigenvalues $m_b = n\delta_{ab}$. The matrix (35) in this case is the $n \times n$ Jordan block with the only eigenvector $\mathbf{1}$ with eigenvalue k_a and $\operatorname{tr} Y_0^j = n k_a^j$.

Acknowledgments

Discussions with A.Alexandrov, A.Gorsky, V.Kazakov, S.Khoroshkin, I.Krichever, S.Leurent, M.Olshanetsky, A.Orlov, T.Takebe, Z.Tsuboi, and A.Zotov are gratefully acknowledged. Some of these results were reported at the International conference "Physics and Mathematics of Nonlinear Phenomena" (22-29 June 2013, Gallipoli, Italy). The author thanks the organizers and especially professor B.Konopelchenko for the invitation and support. This work was supported in part by RFBR grant 12-01-00525, by joint RFBR grants 12-02-91052-CNRS, 12-02-92108-JSPS and by Ministry of Science and Education of Russian Federation under contract 8207 and by grant NSh-3349.2012.2 for support of leading scientific schools.

References

[1] A. Alexandrov, S. Leurent, Z. Tsuboi and A. Zabrodin, *The master T-operator for the Gaudin model and the KP hierarchy* [arXiv:1306.1111].

[2] A. Alexandrov, V. Kazakov, S. Leurent, Z. Tsuboi and A. Zabrodin, *Classical tau-function for quantum spin chains*, JHEP **1309** (2013) 064 [arXiv:1112.3310].

[3] A. Zabrodin, *The master T-operator for vertex models with trigonometric R-matrices as classical tau-function*, Teor. Mat. Fys. **171:1** (2013) 59-76 (Theor. Math. Phys. **174** (2013) 52-67) [arXiv:1205.4152]; A. Zabrodin, *Hirota equation and Bethe ansatz in integrable models*, Suuri-kagaku Journal (in Japanese), Number 596 (2013) 7-12 [arXiv:1211.4428].

[4] V. Kazakov, S. Leurent and Z. Tsuboi, *Baxter's Q-operators and operatorial Bäcklund flow for quantum (super)-spin chains*, Commun. Math. Phys. **311** (2012) 787-814 [arXiv:1010.4022].

[5] H. Airault, H. P. McKean, and J. Moser, *Rational and elliptic solutions of the Korteweg-De Vries equation and a related many-body problem*, Commun. Pure Appl. Math., **30** (1977) 95-148.

[6] I.M. Krichever, *Rational solutions of the Kadomtsev-Petviashvili equation and integrable systems of N particles on a line*, Funct. Anal. Appl. **12:1** (1978) 59-61;
I.M. Krichever, *Rational solutions of the Zakharov-Shabat equations and completely integrable systems of N particles on a line*, J. Sov. Math., **21:3** (1983) 335-345.
[7] T. Shiota, *Calogero-Moser hierarchy and KP hierarchy*, J. Math. Phys. **35** (1994) 5844-5849.
[8] M. Gaudin, *Diagonalisation d'une classe d'hamiltoniens de spin*, J. de Phys. **37** (1976), no. 10 1087-1098.
[9] F. Calogero, *Solution of the one-dimensional N-body problems with quadratic and/or inversely quadratic pair potentials*, J. Math. Phys. **12** (1971) 419-436;
J. Moser, *Three integrable hamiltonian systems connected with isospectrum deformations*, Adv. Math. **16** (1976) 354-370.
[10] E. Mukhin, V. Tarasov and A. Varchenko, *Gaudin Hamiltonians generate the Bethe algebra of a tensor power of vector representation of gl_N*, St. Petersburg Math. J. **22** (2011) 463-472 [arXiv:0904.2131];
E. Mukhin, V. Tarasov and A. Varchenko, *KZ characteristic variety as the zero set of classical Calogero-Moser Hamiltonians*, SIGMA **8** (2012) 072 (11 pages) [arXiv:1201.3990].
[11] N. Nekrasov, A. Rosly and S. Shatashvili, "Darboux coordinates, Yang-Yang functional, and gauge theory", Nucl. Phys. Proc. Suppl. **216** (2011) 69-93 [arXiv:1103.3919].
[12] D. Gaiotto and P. Koroteev, *On three dimensional quiver gauge theories and integrability*, JHEP **05** (2013) 126 [arXiv:1304.0779].
[13] M. Sato and Y. Sato, *Soliton equations as dynamical systems on infinite dimensional Grassmann manifold*, Lect. Notes in Num. Appl. Anal. **5** (1982) 259-271.
[14] E. Date, M. Jimbo, M. Kashiwara and T. Miwa, *Transformation groups for soliton equations*, in "Nonlinear integrable systems – classical and quantum", eds. M. Jimbo and T. Miwa, World Scientific, pp. 39-120 (1983);
M. Jimbo and T. Miwa, *Solitons and infinite dimensional Lie algebras*, Publ. RIMS, Kyoto Univ. **19** (1983) 943-1001.
[15] I. Krichever, O. Lipan, P. Wiegmann and A. Zabrodin, *Quantum Integrable Models and Discrete Classical Hirota Equations*, Commun. Math. Phys. **188** (1997) 267-304 [arXiv:hep-th/9604080].
[16] A. Zabrodin, *Discrete Hirota's equation in quantum integrable models*, Int. J. Mod. Phys. **B11** (1997) 3125-3158;
A. Zabrodin, *Hirota equation and Bethe ansatz*, Teor. Mat. Fyz., **116** (1998) 54-100 (English translation: Theor. Math. Phys. **116** (1998) 782-819).
[17] V. Kazakov, A. S. Sorin and A. Zabrodin, *Supersymmetric Bethe ansatz and Baxter equations from discrete Hirota dynamics*, Nucl. Phys. B **790** (2008) 345-413 [arXiv:hep-th/0703147];
A. Zabrodin, *Bäcklund transformations for difference Hirota equation and supersymmetric Bethe ansatz*, Teor. Mat. Fyz. **155** (2008) 74-93 (English translation: Theor. Math. Phys. **155** (2008) 567-584) [arXiv:0705.4006].
[18] R. Hirota, *Discrete analogue of a generalized Toda equation*, J. Phys. Soc. Japan **50** (1981) 3785-3791.
[19] K. Takasaki and T. Takebe, *Integrable hierarchies and dispersionless limit*, Rev. Math. Phys. **7** (1995) 743-808;
K. Takasaki and T. Takebe, *$\hbar$-Dependent KP hierarchy*, Theor. Math. Phys. **171** (2012) 683-690 [arXiv:1105.0794].
[20] B. Feigin, E. Frenkel and N. Reshetikhin, *Gaudin model, Bethe Ansatz and critical level*, Commun. Math. Phys. **166** (1994) 27-62.
[21] B. Enriques and V. Rubtsov, *Hitchin systems, higher Gaudin operators and R-matrices*, Math. Res. Lett. **3** (1996) 343-357.
[22] D. Talalaev, *Quantization of the Gaudin system*, Funct. Anal. Appl. **40**(1) (2006) 86-91.
[23] V. Kazakov and P. Vieira, *From characters to quantum (super)spin chains via fusion*, JHEP **0810** (2008) 050 [arXiv:0711.2470 [hep-th]].
[24] I. Macdonald, *Symmetric functions and Hall polynomials*, 2nd ed., Oxford University Press, 1995.
[25] A. Orlov and T. Shiota, *Schur function expansion for normal matrix model and associated discrete matrix models*, Phys. Lett. **A343** (2005) 384-396.
[26] V. Enolski and J. Harnad, *Schur function expansions of KP tau functions associated to algebraic curves*, Uspekhi Mat. Nauk **66:4** (2011) 137-178 (Russian Math. Surveys **66:4** (2011) 767-807) [arXiv:1012.3152].
[27] K. Sawada and T. Kotera, *Integrability and a solution for the one-dimensional N-particle system with inversely quadratic pair potential*, J. Phys. Soc. Japan, **39** (1975) 1614-1618.
[28] S. Wojciechowski, *New completely integrable Hamiltonian systems of N particles on the real line*, Phys. Lett. **59A** (1976) 84-86.
[29] M. Olshanetsky and A. Perelomov, *Classical integrable finite dimensional systems related to Lie algebras*, Phys. Reps. **71** (1981) 313-400.