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Generalized Lenard chains and multi-separability of the Smorodinsky–Winternitz system

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Abstract. We show that the notion of generalized Lenard chains allows to formulate in a natural way the theory of multi–separable systems in the context of bi–Hamiltonian geometry. We prove that the existence of generalized Lenard chains generated by a Hamiltonian function and by a Nijenhuis tensor defined on a symplectic manifold guarantees the separation of variables. As an application, we construct such a chain for the case I of the classical Smorodinsky–Winternitz model.

1. Introduction: superintegrability and separation of variables

This paper is part of a joint research program with P. Tempesta started in [1], where we have established a new connection between the theory of integrable and superintegrable systems on one side, and that of bi–Hamiltonian separation of variables on the other side. In fact, we have provided a theoretical framework for studying separation of variables for classical systems, by means of the notion of generalized Lenard (GL) chains. These chains, jointly with a couple of compatible Poisson tensors, are the main geometrical objects for our bi–Hamiltonian description of classical mechanics. These structures guarantee the separation of variables in a suitable bi–structured manifold.

In classical mechanics, superintegrable systems are Hamiltonian systems that possess more than N integrals of motion functionally independent, globally defined in a $2N$–dimensional phase space (see e.g. [2] for a monograph on the topic). These systems are also called noncommutatively integrable [3], [4]. Especially important are the maximally superintegrable ones, i.e. those having $2N - 1$ integrals. It turns out that for these systems all bounded orbits are closed and the motion is periodic [5]. Among the physically most relevant superintegrable potentials are the harmonic oscillator and the Kepler potential, the Calogero–Moser potential, the Smorodinsky–Winternitz systems, the Euler top, etc. [2]–[7].

In quantum mechanics, superintegrable systems are also particularly interesting: they possess accidental degeneracy of the energy levels. This degeneracy can be removed by considering the quantum numbers associated to the additional integrals of motion. A paradigmatic example is offered by the Coulomb atom [8]. Recently, new examples of superintegrable systems have been discovered [9]–[12], both classical and quantum.

One of the most effective methods to solve Hamiltonian systems is to find a complete integral of the corresponding Hamilton–Jacobi equation through the technique of separation of variables. For the sake of clarity, we will recall briefly the geometric setting of Hamiltonian dynamics [13].

Let (M, ω) be a symplectic manifold, i.e. a $2n$–dimensional manifold endowed with a non degenerate closed 2–form ω, said to be a symplectic form. Such a geometrical structure selects a privileged dynamics on M, the one given by Hamiltonian vector fields defined by

$$i_{X_H}\omega = -\mathrm{d}H$$

(i_{X_H} denotes the contraction operator w.r.t. the vector field X_H and d denotes the exterior derivative operator) or, equivalently

$$X_H = (\omega^\flat)^{-1}\mathrm{d}H \ ,$$

where $\omega^\flat : TM \to T^*M$ denotes the fiber bundles isomorphism induced by ω. The function H is said to be the Hamiltonian function of the vector field X_H. A symplectic form acting on vector fields is equivalent to a non degenerate Poisson bracket defined as

$$\{F, G\} := \omega(X_F, X_G) =< \mathrm{d}F, X_G > \ , \tag{1}$$

($<,>$ denotes the natural pairing between 1–forms and vector fields), i.e. as a skew–symmetric derivation on the ring $C^\infty(M)$, fulfilling the Jacobi identity. As we wish to study separable Hamiltonian systems that are Liouville–integrable, in principle we can start with a set of n independent Hamiltonian functions in involution w.r.t. the Poisson brackets (1). In this framework, in the tradition of the *Italian school* [14], an important result has been obtained by Benenti in [15]. It gives a characterization of separated coordinates in terms of Poisson bracket.

The Hamiltonian functions $\{H_i\}_{1\le i\le n}$ are separable in a set of canonical coordinates $(\boldsymbol{q}, \boldsymbol{p})$ if and only if they are in separable involution, i.e. if and only if they satisfy

$$\{H_i, H_j\}_{|k} = \frac{\partial H_i}{\partial q_k}\frac{\partial H_j}{\partial p_k} - \frac{\partial H_i}{\partial p_k}\frac{\partial H_j}{\partial q_k} = 0 \ , \quad 1 \le k \le n, \tag{2}$$

where no summation over k is understood. However, such a condition is not intrinsic as it requires to know *a priori* the coordinates $(\boldsymbol{q}, \boldsymbol{p})$ in order to be applied.

Recently, a new geometric approach to SoV has been developed, based on the bi–Hamiltonian theory ([16], [17]) and on GL chains ([18]–[21]). It has succeeded in giving intrinsic and constructive criteria of separability and has connected the classical theory of SoV with the modern theory by Sklyanin [22]. The bi–Hamiltonian theory of SoV is formulated in phase spaces represented by manifolds endowed with two geometric structures satisfying two suitable compatibility conditions. Such structures are a symplectic form ω and a Nijenhuis (or hereditary) operator N acting on the tangent bundle of M. For this reason such manifolds have been called ωN manifolds. Whilst the symplectic form defines the algebra of Hamiltonian vector fields, the Nijenhuis operator defines sets of distinguished coordinates that are separated coordinates for a special class of Hamiltonian vector fields, those belonging to GL chains, so called as they are extensions of classical Lenard chains, widely known in soliton literature [23], [24].

In this paper we extend a theorem given in [1] proving that, given a generic integrable system on the cotangent bundle of the Euclidean plane, the existence of a GL chain ensures separation of variables on a ωN manifold, in a set of coordinates more general than those of [1]. Moreover, we will complete the study of an important physical model, namely the case I of the Smorodinsky–Winternitz (SWI) system, constructing explicitly a new bi–Hamiltonian structure related to elliptic separated coordinates in the plane.

The paper is organized as follows. In Section 2, the theory of bi–Hamiltonian manifolds is briefly reviewed. In Section 3, the main geometrical object of our theory, i.e. the GL chains, are introduced. The above mentioned general theorem on separation of variables for systems in T^*E^2 is proposed. In Section 4, the previous theory is applied to the study of the general class of integrable mechanical systems in the Euclidean plane, separable in elliptic coordinates. In Section 5, a GL chain is constructed for the classical SWI system. Some discussions are drawn in the final Section 6.

2. Bi–Hamiltonian manifolds and ωN manifolds

Generally, a Poisson bracket (see, e.g., [25]) can be defined by a Poisson bi–vector field, i.e. a skew–symmetric linear map $P : T^*M \mapsto TM$, eventually not invertible, such that

$$\{F, G\}_P :=< \mathrm{d}F, P\mathrm{d}G >$$

with vanishing Schouten bracket

$$0 = [P, P]\,(\alpha, \beta) := \mathcal{L}_{P\beta}(P)\alpha + P(i_{P\alpha}\mathrm{d}\beta) \qquad \forall\, \alpha, \beta \in T^*M, \tag{3}$$

($\mathcal{L}$ denotes the Lie derivative). In the special case of symplectic manifolds, $P := (\omega^\flat)^{-1}$ is a Poisson bi-vector. Generalizing (1) the vector field $X_G := P\,\mathrm{d}G$ is said to be the Hamiltonian vector field with Hamiltonian function G.

Bi–Hamiltonian manifolds were introduced by Magri [26] as models of phase space for soliton equations.

Definition 1 *A bi–Hamiltonian manifold (M, P_0, P_1) is a manifold M endowed with* two *Poisson bi-vectors fields such that*

$$0 = 2\,[P_0, P_1]\,(\alpha, \beta) := \mathcal{L}_{P_0\beta}(P_1)\alpha + P_1(i_{P_0\alpha}\mathrm{d}\beta) + \mathcal{L}_{P_1\beta}(P_0)\alpha + P_0(i_{P_1\alpha}\mathrm{d}\beta) \quad \forall\, \alpha, \beta \in T^*M\ . \tag{4}$$

Such a condition assures that the linear combination $P_1 - \lambda P_0$ is a Poisson *pencil*, i.e. it is a Poisson bi-vector for each $\lambda \in \mathbb{C}$, and therefore the corresponding bracket $\{,\}_{P_1} - \lambda\{,\}_{P_0}$ is a pencil of Poisson bracket. Condition (4) is known as the compatibility condition between P_0 and P_1.

If one of the Poisson tensor, say P_0, is invertible, and therefore its inverse is a symplectic operator $\omega^\flat := P_0^{-1}$, the bi–Hamiltonian manifold M turns out to be an ωN manifold (see [27]). Indeed, the composed operator $N := P_1 P_0^{-1}$, thanks to the compatibility condition between P_0 and P_1, is a Nijenhuis (or hereditary) operator compatible with the symplectic form ω, induced by $\omega^\flat = P_0^{-1}$.

Definition 2 *A ωN manifold (M, ω, N) is a symplectic manifold endowed with an endomorphism of the tangent bundle of M, $N : TM \mapsto TM$ which satisfies the following conditions:*

- *its Nijenhuis torsion vanishes identically, i.e. $\forall X, Y \in TM$*

$$[NX, NY] - N([X, NY] + [NX, Y]) + N^2[X, Y]) = 0; \tag{5}$$

- *it is compatible with ω, i.e. the tensor $P_1 = N(\omega^\flat)^{-1}$ is again a Poisson tensor and is compatible with $P_0 := (\omega^\flat)^{-1}$, according to Definition 1.*

In short, the condition (5) can be rephrased by saying that the endomorphism N is a Nijenhuis (or hereditary) operator. The adjoint linear map w.r.t. the natural pairing will be denoted by $N^T : T^*M \mapsto T^*M$ and will be defined by

$$< N^T\alpha, X >=< \alpha, NX > .$$

The condition (5) on the operator N, introduced by Nijenhuis [28], has a relevant geometrical meaning: it implies that the distributions of its eigenvectors are integrable according to the Frobenius theorem. Consequently, under suitable completeness assumption to be introduced below, one can select local coordinate charts, in which N takes a diagonal form. We suppose that at each point x (or in a dense open subset) of M, the Nijenhuis tensor field N admits n distinct eigenvalues $\lambda_i(x)$ $(i = 1\ldots, n)$ (maximally distinct). Since in a generic ωN manifold the eigenspaces of N are even–dimensional, belonging to the kernel of the skewsimmetric tensor field $P_1 - \lambda P_0$, from the above assumption it follows that N (and the adjoint tensor N^T) can be put in diagonal form. Then we have the following result, proved in [29]. Let (M, ω, N) be a maximally distinct ωN manifold. In a suitable open neighborhood of each point, there exist a Darboux chart $(\boldsymbol{q}, \boldsymbol{p})$ for ω that, simultaneously, diagonalize N^T

$$N^T \mathrm{d}q_i = \lambda_i \mathrm{d}q_i \qquad\qquad N^T \mathrm{d}p_i = \lambda_i \mathrm{d}p_i \ . \tag{6}$$

As, in such a Darboux chart the Nijenhuis tensor N takes a diagonal form, the coordinates $(\boldsymbol{q}, \boldsymbol{p})$ are said to be Darboux–Nijenhuis (DN) coordinates, and are just separation coordinates in the bi–Hamiltonian theory of SoV. Hereafter, with an abuse of notation, we will identify an operator P with its matrix in a suitable basis.

Remark 1 *If the eigenvalues $\lambda_i(x)(i = 1, \dots, n)$ of N are functionally independent, they can be chosen as n coordinates of a DN chart. In [30], has been proved that the sets of conjugate momenta, denoted as $\mu_i(x)(i = 1, \dots, n)$, can be computed by quadratures. Such special DN coordinates $(\boldsymbol{\lambda}, \boldsymbol{\mu})$ are referred to as sDN coordinates [17].*

Remark 2 *The condition* (5) *is sufficient but not necessary for the integrability of the distribution of the eigenvalues of a generic operator N. In fact, Haantjes proved that the necessary and sufficient condition for such an integrability is the vanishing of the Haantjes tensor of N [31].*

Remark 3 *It can be easily verified that, given a DN chart $(\boldsymbol{q}, \boldsymbol{p})$, the* separated *canonical transformations*

$$\tilde{q}_i = f_i(q_i) \ , \quad \tilde{p}_i = \frac{p_i}{f_i'} \ , \tag{7}$$

with f_i a generic invertible smooth function of a single coordinate q_i, preserve the property (6)*, i.e.,*

$$N^T \mathrm{d}\tilde{q}_i = \lambda_i \mathrm{d}\tilde{q}_i \qquad\qquad N^T \mathrm{d}\tilde{p}_i = \lambda_i \mathrm{d}\tilde{p}_i \ . \tag{8}$$

From a geometrical point of view, we can say that coordinates $(\tilde{\boldsymbol{q}}, \tilde{\boldsymbol{p}})$ and $(\boldsymbol{q}, \boldsymbol{p})$, related by transformations (7) *are DN coordinates adapted to the same coordinate web [32].*

3. Generalized Lenard chains

After having introduced the geometrical structures which define separation coordinates in the bi–Hamiltonian theory of SoV, let us characterize the class of Hamiltonian functions which are separable in DN coordinates. For the sake of concreteness, we will do this in the case of a 4–dimensional manifold, extending the theorem proved in [1] in the special case of sDN coordinates to generic DN coordinates.

Theorem 3 *Let (M, ω, N) be a 4–dimensional maximally distinct ωN manifold and (q_1, q_2, p_1, p_2) a DN local chart. Let H be a smooth function in M. The DN coordinates (q_1, q_2, p_1, p_2) are separated variables for H if and only if there exist two smooth functions f and g such that the one form*

$$\alpha = f\, \mathrm{d}H + g\, N^T \mathrm{d}H \tag{9}$$

is an exact one form, i.e., α is the differential of a function, say H_2

$$\alpha = \mathrm{d}H_2. \tag{10}$$

In this case, the function H_2 is an integral of motion in involution with $H_1 := H$ and the same DN coordinates are separated variables for H_2 as well.

Proof. In the above–mentioned chart, N takes the diagonal form

$$N = \lambda_1(\frac{\partial}{\partial q_1} \otimes \mathrm{d}q_1 + \frac{\partial}{\partial p_1} \otimes \mathrm{d}p_1) + \lambda_2(\frac{\partial}{\partial q_2} \otimes \mathrm{d}q_2 + \frac{\partial}{\partial p_2} \otimes \mathrm{d}p_2) \ , \tag{11}$$

being λ_1 and λ_2 the two double eigenvalues. Let us suppose that eqs. (9) and (10) are fulfilled. Then, it follows that

$$\begin{array}{rcl} \frac{\partial H_2}{\partial q_k} & = & f\frac{\partial H_1}{\partial q_k} + g\,\lambda_k \frac{\partial H_1}{\partial q_k} \ , \\ \frac{\partial H_2}{\partial p_k} & = & f\frac{\partial H_1}{\partial p_k} + g\,\lambda_k \frac{\partial H_1}{\partial q_k} \ . \end{array} \qquad k = 1, 2 \tag{12}$$

Therefore

$$\{H_1, H_2\}_{|k} = \frac{\partial H_1}{\partial q_k}\frac{\partial H_2}{\partial p_k} - \frac{\partial H_1}{\partial p_k}\frac{\partial H_2}{\partial q_k} \overset{(12)}{=} 0 \ , \tag{13}$$

i.e. H_1 and H_2 are in separable involution according to Benenti's theorem (see formulas (2)), w.r.t. a DN chart. Consequently, H_2 is an integral of motion for X_H.

Viceversa, let us suppose that (q_1, q_2, p_1, p_2) are separated variables for H_1 and H_2, i.e. conditions (13) hold, and let us consider the equation

$$f\mathrm{d}H_1 + gN^T\mathrm{d}H_1 = \mathrm{d}H_2 \tag{14}$$

in the unknown functions f and g. In the local chart (q_1, q_2, p_1, p_2), eq. (14) takes the form

$$f + g\lambda_1 = \frac{\frac{\partial H_2}{\partial q_1}}{\frac{\partial H_1}{\partial q_1}}, \qquad f + g\lambda_2 = \frac{\frac{\partial H_2}{\partial q_2}}{\frac{\partial H_1}{\partial q_2}}, \tag{15}$$

$$f + g\lambda_1 = \frac{\frac{\partial H_2}{\partial p_1}}{\frac{\partial H_1}{\partial p_1}}, \qquad f + g\lambda_2 = \frac{\frac{\partial H_2}{\partial p_2}}{\frac{\partial H_1}{\partial p_2}} \ . \tag{16}$$

We observe that the first equations (15) and (16) coincide in virtue of the conditions (13), so do the second equations (15) and (16). Thus the above system of four equations reduces to two equations that admit the unique solution

$$f = \frac{1}{\lambda_2 - \lambda_1}(\lambda_2 \frac{\frac{\partial H_2}{\partial q_1}}{\frac{\partial H_1}{\partial q_1}} - \lambda_1 \frac{\frac{\partial H_2}{\partial q_2}}{\frac{\partial H_1}{\partial q_2}}), \tag{17}$$

$$g = \frac{1}{\lambda_2 - \lambda_1}(-\frac{\frac{\partial H_2}{\partial q_1}}{\frac{\partial H_1}{\partial q_1}} + \frac{\frac{\partial H_2}{\partial q_2}}{\frac{\partial H_1}{\partial q_2}}). \tag{18}$$

Then we will say that Hamiltonian functions related by eqs. (9) and (10) belong to a GL chain generated by (ω, N, H) since, for $(f = 0, g = 1)$, a GL chain reduces to a classical Lenard chain.

Remark 4 *We note that, if $f = -(\lambda_1 + \lambda_2), g = 1$, we get a Quasi–Bi–Hamiltonian (QBH) chain of Pfaffian type generated by the function H [16, 33, 34, 35].*

Remark 5 *Let us observe that, in the proof of theorem 3, only the diagonal form of N has been exploited, without any reference to its null torsion. This implies that theorem 3 holds under the weaker assumption on N that its Haantjes tensor be null (see Rem. 2). Some results along this novel line of research are in preparation.*

Theorem 3 suggests the following procedure in order to classify Hamiltonian systems separable in DN coordinates:

(i) choose a Darboux chart (q_1, q_2, p_1, p_2) in a 4–dimensional symplectic manifold M;
(ii) construct a ωN structure which has (q_1, q_2, p_1, p_2) as DN coordinates;
(iii) search for Hamiltonian function H and for functions f and g such that make the one form (9) locally exact.

This procedure can be considered as an *inverse problem*, with respect to the *direct approach* that starts from a given Hamiltonian and aims to find separation coordinates.

Let us observe that the above method provides the integral of motion H_2 together with a set of separated variables both for H and H_2.

4. Bi–Hamiltonian geometry in T^*E^2: construction of GL chains

In [1], we have applied the procedure previously discussed, to the study of the bi–Hamiltonian properties of systems defined in the cotangent bundle of the Euclidean plane, $M = T^*E^2$ Precisely, we have studied mechanical Hamiltonian functions

$$H = \text{kinetic energy } + \text{potential energy}$$

and we have recovered the most general form of the potential in the Euclidean plane, which makes H separable in cartesian, polar and parabolic coordinates, considered as sDN coordinates of the (ω, N) structure given by the standard symplectic form and by the linear Nijenhuis tensor (11). Here, we recall the cartesian case for the sake of clarity, and complete the analysis of [1] adding the case of elliptic coordinates.

Classical separation of variables in cartesian and elliptic coordinates
Let us consider the natural Hamiltonian function

$$H = \frac{1}{2}\left(p_x^2 + p_y^2\right) + V(x, y), \tag{19}$$

(x, y, p_x, p_y) being cartesian coordinates and conjugate momenta. According to the requirement (11), we choose the linear Nijenhuis tensor $N_{car} : T(T^*E^2) \to T(T^*E^2)$

$$N_{car} = \text{diag}(x, y, x, y).$$

The one–form (9) reads

$$\alpha = f\,(x, y, p_x, p_y)\,dH + g\,(x, y, p_x, p_y)\,N^T dH, \tag{20}$$

and the closure condition $d\alpha = 0$ provides a system of nonlinear PDEs for f, g and V reported in formula (A1) of the Appendix of [1]. From such a system, we have deduced the differential consequence:

$$V_{xy} = 0\ , \tag{21}$$

therefore recovering the known result that

$$V(x, y) = V_1(x) + V_2(y) \tag{22}$$

is the most general potential that makes the Hamiltonian function (19) separable in cartesian coordinates. Furthermore, observing that the functions

$$f = \frac{x}{x-y}\ , \qquad g = -\frac{1}{x-y} \qquad x \neq y\ ,$$

fulfills the above mentioned system (A1), we get a GL chain generated by (ω, N, H). The second integral of motion is given by a primitive function of the 1–form (20) which is just the energy associated with the coordinate y

$$H_2 = \frac{p_y^2}{2} + V_2(y). \tag{23}$$

Let us introduce elliptic coordinates (ξ, η) in the plane

$$x = \frac{\xi\eta}{4c}, \qquad y = \pm\frac{1}{4c}\sqrt{(\xi^2 - 4c^2)(4c^2 - \eta^2)}, \quad \xi \geq 0 \quad \eta \in \mathbb{R} \ ,$$

where $(\pm c, 0)$ are the foci of the confocal ellipses and hyperbolas given by $\xi = const$, $\eta = const$, respectively. In such coordinates, the generic Hamiltonian function (19) reads

$$H = 2\frac{(\xi^2 - 4c^2)p_\xi^2 + (4c^2 - \eta^2)p_\eta^2}{\xi^2 - \eta^2} + V(\xi, \eta)\,, \tag{24}$$

being (p_ξ, p_η) conjugate momenta to (ξ, η). In order to get a QBH chain, thanks to the Rem. 3, we can choose the quadratic Nijenhuis tensor

$$N_{ell} = \mathrm{diag}(\xi^2, \eta^2, \xi^2, \eta^2), \tag{25}$$

with

$$\alpha = f(\xi, \eta, p_\xi, p_\eta)\, dH + g(\xi, \eta, p_\xi, p_\eta)\, N_{ell}^T\, dH. \tag{26}$$

The closure condition for α provides the system (40) of the Appendix. By combining such Eqs. we get the consequence

$$V_{\xi\eta} + 2\frac{\xi V_\eta - \eta V_\xi}{\xi^2 - \eta^2} = 0. \tag{27}$$

The general solution of (27) is

$$V(\xi, \eta) = \frac{V_1(\xi) + V_2(\eta)}{\xi^2 - \eta^2} \qquad \xi \neq \eta \tag{28}$$

which is the most general potential on E^2 that makes (24) separable in elliptic coordinates. Moreover, we deduce a particular solution of the system (40)

$$f = -(\xi^2 + \eta^2)\ , \qquad g = 1\ , \tag{29}$$

which enables us to construct, by means of the same procedure as in the cartesian case, the following GL chain related to the elliptic coordinates

$$H_1 := \frac{2}{\xi^2 - \eta^2}\left((\xi^2 - 4c^2)p_\xi^2 + (4c^2 - \eta^2)p_\eta^2 + V_1(\xi) + V_2(\eta)\right) \tag{30}$$

$$H_2 := \frac{2}{\xi^2 - \eta^2}\left(\eta^2(4c^2 - \xi^2)p_\xi^2 + \xi^2(\eta^2 - 4c^2)p_\eta^2 - \eta^2 V_1(\xi) + \xi^2 V_2(\eta)\right) \tag{31}$$

Remark 6 *Let us note that, thanks to the choice (25), the above GL chain for elliptic coordinates is indeed an example of the QBH chain defined in Rem. 4. In contrast, if one choose a Nijenhuis tensor linear in the elliptic coordinates, one can verify that a genuine GL chain generated by such a Nijenhuis tensor and H exists, but a QBH chain does not exist.*

Remark 7 *Choosing $V_1(\xi) = (b_1 + b_2)\xi$ and $V_2(\eta) = (b_1 - b_2)\eta$ in (28), one recovers the classical Euler problem for the potential of two Coulomb centers, fixed at the foci. Thus, also such a problem admits a QBH formulation in a (ω, N) manifold [36].*

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5. Multi–separation of variables and superintegrable systems

In this Section, we use the bi–Hamiltonian structures constructed in the previous discussion to construct potentials admitting more than a system of separation coordinates. In fact, it can achieve that a Hamiltonian function belongs to GL chains generated by different and incompatible bi–Hamiltonian structures. In this case, we get a Hamiltonian system separable in different coordinates system or a *multi–separable* system together with additional integrals of motion that, if they are independent, assures superintegrability of the model. Thus, we recover in a natural way one of the Smorodinsky–Winternitz potentials in the plane, first discovered in a quantum–mechanical context in [37], [38] and studied again in [39] and [40] from a group theoretical point of view. These are the only potentials that are multi–separable in terms of orthogonal coordinates in E^2.

5.1. Cartesian and Elliptic coordinates

Let us search for the most general potential $V(x, y)$ that admits SoV both in cartesian and in elliptic coordinates. To this end, let us write down every equation in cartesian coordinates. Eq. (27) is equivalent to

$$xy(V_{xx} - V_{yy}) - (x^2 - y^2 - c^2)V_{xy} + 3yV_x - 3xV_y = 0. \tag{32}$$

Thus, the potential has to satisfy the system of the two PDEs (21) and (32). By substituting the solution (19) of the eq. (21) into eq. (32) we get the separated equations

$$V_1'' + \frac{3}{x}V_1' = V_2'' + \frac{3}{y}V_2' = 4a, \tag{33}$$

where a is an arbitrary constant. Their general solution is

$$V_1(x) = \frac{1}{2}ax^2 + \frac{c_1}{x^2}, \tag{34}$$

$$V_2(y) = \frac{1}{2}ay^2 + \frac{c_2}{y^2}. \tag{35}$$

Thus, the general solution of the system (21) and (32) is

$$V(x, y) = \frac{1}{2}a(x^2 + y^2) + \frac{c_1}{x^2} + \frac{c_2}{y^2}, \tag{36}$$

which is nothing but the SWI potential [38], sum of an isotropic elastic potential and an anisotropic Rosochatius potential. The Hamiltonian system with the SWI potential inherits the integral of motion (23) from SoV in cartesian coordinates

$$H_2^{(car)} = \frac{p_y^2}{2} + \frac{a}{2}y^2 + \frac{c_2}{y^2}, \tag{37}$$

together with the integral (31) from SoV in elliptic coordinates. The latter integral written down in cartesian coordinates, reads

$$H_2^{(ell)} = -2\left(c^2p_x^2 + (xp_y - yp_x)^2 + ac^2x^2 + 2c_1\frac{y^2 + c^2}{x^2} + 2c_2\left(\frac{x}{y}\right)^2\right). \tag{38}$$

A simple check shows that the Hamiltonian SWI, $H_2^{(car)}$ and $H_2^{(ell)}$ are independent. Consequently, the potential (36) is maximally superintegrable. Finally, we can state that the SWI Hamiltonian function generates two GL chains, starting from the two incompatible structures (ω, N_{car}) and (ω, N_{ell}), with the operator N_{ell} (25) represented in cartesian coordinates by the simple quadratic matrix

$$N_{ell} \equiv 4 \begin{bmatrix} x^2 & xy & 0 & 0 \\ xy & y^2+c^2 & 0 & 0 \\ 0 & xp_y - yp_x & x^2 & xy \\ -(xp_y - yp_x) & 0 & xy & y^2+c^2 \end{bmatrix} . \tag{39}$$

Indeed, it can be checked that the two Poisson tensors fields

$$P_{car} := N_{car}(\omega^\flat)^{-1} \ , \quad P_{ell} := N_{ell}(\omega^\flat)^{-1}$$

have non vanishing Schouten brackets, i.e.,

$$[P_{car}, P_{ell}] \neq 0 \ .$$

6. Future perspectives

In this work, we have illustrated the general bi–Hamiltonian setting for treating the geometry of both integrable and superintegrable systems on the relevant physical example of the SWI system. The present approach for the sake of concreteness has been formulated in the Euclidean plane. However, there is no theoretical restriction in extending it to higher–dimensional cases. Also, it seems interesting to include in the present analysis integrable and superintegrable systems defined in curved spaces. It would be very interesting to derive a quantum formulation of the present theory. Furthermore, its generalization to recursion operators with vanishing Haantjes tensor is under investigation.

Appendix

We report the explicit expressions of the systems of differential equations quoted in Section 4.

$$\begin{aligned}
&4\,\left(\eta^2-4\,c^2\right)\left(f_{p_\xi}+g_{p_\xi}\eta^2\right)p_\eta+4\,\left(\xi^2-4\,c^2\right)\left(f_{p_\eta}+g_{p_\eta}\xi^2\right)p_\xi=0, \\
&4\,\left(\eta^2-4\,c^2\right)\left(\xi\,\left(f_{p_\eta}+g_{p_\eta}\xi^2\right)\left(p_\xi^2-{p_\eta}^2\right)-\left(\xi^2-\eta^2\right)\left(g_\xi\eta^2+2\,g\xi+f_\xi\right)p_\eta\right)+ \\
&-\left(\xi^2-\eta^2\right)^2\left(f_{p_\eta}+g_{p_\eta}\xi^2\right)V_\xi=0\ , \\
&4\left(\xi^2-4\,c^2\right)\,\left(\eta\,\left(f_{p_\xi}+g_{p_\xi}\eta^2\right)\left({p_\eta}^2-{p_\xi}^2\right)+\left(\xi^2-\eta^2\right)\left(g_\eta\xi^2+2\,g\,\eta+f_\eta\right)p_\xi\right)+ \\
&-\left(\xi^2-\eta^2\right)^2\left(f_{p_\xi}+g_{p_\xi}\eta^2\right)V_\eta=0\ , \\
&4\,\xi\,\left(\eta^2-4\,c^2\right)\left(f_{p_\xi}+g_{p_\xi}\xi^2\right)\left({p_\xi}^2-{p_\eta}^2\right)+4\,\left(\xi^2-4\,c^2\right)\left(\xi^2-\eta^2\right)\left(g_\xi\xi^2+2\,g\xi+f_\xi\right)p_\xi+ \\
&-\left(\xi^2-\eta^2\right)^2\left(f_{p_\xi}+g_{p_\xi}\xi^2\right)V_\xi=0\ , \\
&4\,\eta\,\left(\xi^2-4\,c^2\right)\left(f_{p_\eta}+g_{p_\eta}\eta^2\right)\left({p_\eta}^2-p_\xi^2\right)-4\,\left(\eta^2-4\,c^2\right)\left(\xi-\eta^2\right)\left(g_\eta\eta^2+2\,g\eta+f_\eta\right)p_\eta+ \\
&-\left(\xi^2-\eta^2\right)^2\left(f_{p_\eta}+g_{p_\eta}\eta^2\right)V_\eta=0\ , \\
&4\Big(2\,\xi\,\eta\,\left(8\,c^2-\eta^2-\xi^2\right)g+\,\xi\,\left(4\,c^2-\eta^2\right)\left(f_\eta+\xi^2g_\eta\right)+\eta\,\left(4\,c^2-\xi^2\right)\left(f_\xi+\eta^2g_\xi\right)\Big)\left({p_\eta}^2-{p_\xi}^2\right)+ \\
&-\left(\xi^2-\eta^2\right)^2\left(f_\eta+\xi^2g_\eta\right)\left(V_\xi-V_\eta\right)+g\left(\xi^2-\eta^2\right)^3V_{\eta,\xi}=0\ .
\end{aligned} \tag{40}$$

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