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On singular elliptic equations involving critical Sobolev exponent

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Abstract. Given a $n-$dimensional compact Riemannian manifold (M,g) with $n \geq 5$, we consider the following semi-linear elliptic equation :

$$P_g(u) := \Delta_g^2 u + div_g\left(a(x)\nabla_g u\right) + b(x)u = f(x)\left|u\right|^{N-2}u + \lambda h(x)\left|u\right|^{q-2}u$$

where the functions a, b and h are in suitable Lebesgue spaces, $2 < q < N$ and $\lambda > 0$ a real parameter, f is a smooth positive function and the operator P_g is coercive. Under some additional conditions, we obtain results concering the existence of strong solutions of the above equation in $H_2^2(M)$.

1. Introduction

In 1983 Paneitz discovered a particular conformally fourth-order operator defined on $4-$dimensional smooth Riemannian manifolds [1]. In 1987, Branson generalized the definition to higher dimensions in [2] as follows. Let (M,g) be smooth, compact $n-$dimensional Riemannian manifold with $n \geq 5$, and $u \in C^4(M)$. The Paneitz-Branson operator P_g^n is then defined via [2]:

$$P_g^n(u) = \Delta_g^2 u - div_g\left(a_n(x)du\right) + Q_g^n u$$

where

$$a_n(x) = \frac{(n-2)^2+4}{2(n-2)(n-1)}S_g.g - \frac{4}{n-2}Ric_g,$$
$$Q_g^n = \frac{1}{(n-1)(n-4)}\Delta_g S_g + \frac{n^3-4n^2+16n-16}{4(n-1)^2(n-2)^2(n-4)}S_g^2 - \frac{4}{(n-4)(n-2)^2}\left|Ric_g\right|^2,$$

being Δ_g, S_g and Ric_g the Laplace-Beltrami operator, the scalar and the Ricci curvatures of g, respectively. The Paneitz-Branson operator enjoys interesting conformal properties that are very similar to those of the conformal Laplacian operator. Remark that if $\tilde{g} = \varphi^{\frac{4}{n-4}}g$, with φ a positive function of class $C^4(M)$, is a conformal metric to g, then for all $u \in C^4(M)$, $P_g^n(u\varphi) = \varphi^{\frac{n+4}{n-4}}P_{\tilde{g}}^n(u)$. In particular, if $u \equiv 1$ then $P_g^n(\varphi) = Q_{\tilde{g}}^n\varphi^{\frac{n+4}{n-4}}$.

Many interesting results on Paneitz-Branson operator and related topics have been recently obtained by several authors, we refer the reader to Refs. [3]- [10]. Here we recall a few of these results that are pertinent to our investigation, see the list P1)-P3) below.

Let (M, g) be an $n-$dimensional compact, smooth and oriented Riemannian manifold with $n \geq 5$, $H_2^2(M)$ be the standard Sobolev space consisting of function in $L^2(M)$ whose derivatives up to the second order are in $L^2(M)$, and let $N = \frac{2n}{n-4}$ be the associated Sobolev critical exponent. Now, we define the best constant $K_\circ$ of the embedding $H_2^2(\mathbb{R}^n) \subset L^N(\mathbb{R}^n)$ given by

$$\frac{1}{K_o} = \frac{n(n^2-4)(n-4)\omega_n^{\frac{4}{n}}}{16}$$

where ω_n is the volume of the unit Euclidean $n-$sphere (S^n, h).

P1) In 2002, F. Robert and P. Esposito in [10] considered the following equation

$$\Delta_g^2 u + div_g\left(a(x)\nabla_g u\right) + b(x)u = f(x)\,|u|^{N-2}\,u + h(x)\,|u|^{q-2}\,u$$

where: i) $a \in \Lambda_{(2,0)}^{+\infty}(M)$ is a smooth symmetric $(2,0)$-tensor field, ii) b, h, f are smooth functions in M, with f positive, and iii) $2 < q < N$. They established the following remarkable result:

Theorem 1 *Let (M, g) be an $n-$dimensional compact Einsteinian manifold with $n \geq 8$. Assume that P_g^n is coercive and let $f \in C^\infty(M)$, $f > 0$ such that there exists $x_\circ \in M$ with $f(x_\circ) = \max_{x\in M} f(x)$, $\Delta f(x_\circ) = 0$ and*

$$\frac{4(n^2-4n-4)}{3(n+2)}\left|Weyl_g(x_\circ)\right|^2 + (n-6)(n-8)\frac{\Delta_g^2 f(x_\circ)}{f(x_\circ)} + 2(n-6)(n-8)\frac{\langle\nabla_g f(x_\circ), Ric_g(x_\circ)\rangle}{f(x_\circ)} > 0.$$

Then, there exists $\tilde{g}$ conformal to g such that $Q_{\tilde{g}}^n(x) = f(x)$.

P2) In 2010, M. Benalili in [5] considered the equation:

$$\Delta_g^2 u + div_g\left(a(x)\nabla_g u\right) + b(x)u = f(x)\,|u|^{N-2}\,u \qquad (1)$$

where f is a positive $C^\infty-$function on M, $a \in L^r(M)$ and $b \in L^s(M)$, with $r > \frac{n}{2}$, $s > \frac{n}{4}$. He established the following result:

Theorem 2 *Let (M, g) is an $n-$dimensional compact manifold with $n \geq 8$ and for $2 < p < 5$, $\frac{9}{4} < s < 11$ or $n = 7$, $\frac{7}{2} < p < 9$, $\frac{7}{4} < s < 9$ assume that there exists $x_\circ \in M$ such that $f(x_\circ) = \max_{x\in M} f(x)$ and*

$$\frac{n^2+4n-20}{6(n-6)(n^2-4)}S_g(x_\circ) + \frac{(n-4)}{2n(n-2)}\frac{\Delta_g f(x_\circ)}{f(x_\circ)} > 0.$$

For $n = 6$, $\frac{3}{2} < p < 2$, $3 < s < 4$, assume that $S_g(x_\circ) > 0$. Then, the equation (1) has a weak solution in $H_2^2(M)$.

P3) Recently, M. Benalili and the author proved in [7], the following result:

Theorem 3 *Let (M, g) be a compact manifold of dimension $n \geq 6$, $a \in L^r(M)$, $b \in L^s(M)$, with $r > \frac{n}{2}$, $s > \frac{n}{4}$, $0 < q < 2$ and f a positive $C^\infty-$function on M. We suppose that P_g is coercive and the existence of a point $x_\circ \in M$ such $f(x_\circ) = \max_{x\in M} f(x)$ and*

$$\begin{cases} \frac{\Delta_g f(x_\circ)}{f(x_\circ)} < \frac{1}{3}\left(\frac{(n-1)n\left(n^2+4n-20\right)}{(n-6)(n-4)(n^2-4)}\left(1 + \|a\|_r + \|b\|_s\right)^{-\frac{4}{n}} - \right) S_g(x_\circ) & \text{if } n > 6 \\ S_g(x_\circ) > 0 & \text{if } n = 6 \end{cases}$$

Then, there exists $\lambda^* > 0$ such that for any $\lambda \in (0, \lambda^*)$, the equation

$$\Delta_g^2 u + div_g \left(a(x)\nabla_g u\right) + b(x)u = f(x)\left|u\right|^{N-2} u + \lambda \left|u\right|^{q-2} u$$

has a weak solution in $H_2^2(M)$.

In this paper, we look for solutions to the following semi-linear elliptic equation:

$$P_g(u) := \Delta_g^2 u + div_g \left(a(x)\nabla_g u\right) + b(x)u = f(x)\left|u\right|^{N-2} u + \lambda h(x)\left|u\right|^{q-2} u \tag{2}$$

where $a \in L^r(M)$, $b \in L^s(M)$ and $h \in L^d(M)$, with $r > \frac{n}{2}$, $s > \frac{n}{4}$ and $d > \frac{N}{N-q} := \alpha$, $2 < q < N$ and $\lambda > 0$ a real parameter. In doing so, we assume the following conditions:

(h^1) The operator P_g is coercive, that is: $\exists \Lambda > 0 : \langle P_g(u); u\rangle \geq \Lambda \left\|u\right\|^2_{H_2^2(M)}, \forall u \in H_2^2(M)$;
(h^2) The function h doesn't vanish almost everywhere on M.

(h^3) The parameter λ fulfills $0 < \lambda < \lambda_1$ with

$$\lambda_1 := \frac{q\,(N-2)}{2\,(N-q)}\Lambda^{\frac{q}{2}}(\max((1+\varepsilon)K_\circ, A_\varepsilon))^{-\frac{q}{2}}\left\|h\right\|_\alpha^{-1}.$$

Our main results state as follows:

Theorem 4 *Let (M, g) be an $n-$dimensional compact, smooth and oriented Riemannian manifold with $n > 6$ and f a smooth positive function on M. Let $a \in L^r(M)$, $b \in L^s(M)$ and $h \in L^d(M)$, with $r > \frac{n}{2}$, $s > \frac{n}{4}$, $d > \frac{N}{N-q}$ and $2 < q < N$. We assume that the conditions (h^1), (h^2) and (h^3) are satisfied and that there exists $x_\circ \in M$ such that $f(x_o) = \max_{x\in M} f(x)$ and*

$$\left(\frac{n(n-2\sqrt{6}+2)(n+2\sqrt{6}+2)-(n-6)(n-4)^3(n+2)}{3(n+2)(n-4)^2(n-6)\left(1+\left\|a\right\|_r+\left\|b\right\|_s\right)^{\frac{4}{n}}}S_g(x_\circ) - \frac{(n-4)\Delta f(x_\circ)}{2f(x_\circ)}\right) > 0.$$

Then, the equation (2) possesses a nontrivial solution in $H_2^2(M)$.

Theorem 5 *Let (M, g) be a compact, smooth and oriented Riemannian manifold of dimension $n = 6$ under the same conditions of theorem 4 with*

$$S_g(x_\circ) > 0$$

Then, the equation (2) possesses a nontrivial solution in $H_2^2(M)$.

2. Generic existence result

Throughout this section, we consider the energy functional J_λ, for each $u \in H_2^2(M)$,

$$J_\lambda(u) = \frac{1}{2}\int_M \left((\Delta_g u)^2 - a(x)\left|\nabla_g u\right|^2 + b(x)u^2\right)dv(g) - \frac{\lambda}{q}\int_M h(x)\left|u\right|^q dv(g) - \frac{1}{N}\int_M f(x)\left|u\right|^N dv(g)$$

First, we have the following lemma, whose proof is easy and can be found in [7].

Lemma 6 $\|u\| = \left(\int_M \left((\Delta_g u)^2 - a(x)\left|\nabla_g u\right|^2 + b(x)u^2\right)dv(g)\right)^{\frac{1}{2}}$ *is an equivalent norm of the usual one of $H_2^2(M)$ if only if the operator P_g is coercive.*

The main tool to prove our result is the Mountain-Pass lemma of Ambrossetti-Rabinowitz given by the following lemma:

Lemma 7 *Let $J \in C^1(E, \mathbb{R})$ where $(E, \|.\|)$ is a Banach space. We assume that:*
(i) $J(0) = 0$.
(ii) $\exists r, R > 0$ such that $J(u) \geq R > 0$ for all $u \in E$ such that $\|u\| = r$.
(iii) $\exists v \in E$ such that $\limsup_{t \longrightarrow +\infty} J(tv) < 0$.

If

$$c = \min_{\eta \in \Gamma} \max_{t \in [0,1]} \left(J\left(\eta(t)\right) \right) \qquad \text{where} \quad \Gamma = \left\{ \eta \in C^1\left([0;1]; E\right) : \eta\left(0\right) = 0, \eta\left(1\right) = v \right\}$$

then there exists a sequence $(u_n)_n$ in E such that:

$$J\left(u_n\right) \longrightarrow c \qquad \text{and} \qquad \nabla J\left(u_n\right) \longrightarrow 0 \qquad \text{in } E^*$$

where E^* is the dual space of E. Moreover, we have that: $c \leq \sup_{t \geq 0} J(tv)$.

It is easily seen that J_λ is a C^1 functional and its Fréchet derivative is given by:

$$\begin{aligned} \left\langle \nabla J_\lambda\left(u\right), v \right\rangle = \int_M \left(\Delta_g u . \Delta_g v - a(x) g(\nabla_g u, \nabla_g v) + b(x) uv \right) dv(g) \quad + \\ -\lambda \int_M h(x) \left|u\right|^{q-2} uv dv(g) - \int_M f(x) \left|u\right|^{N-2} uv dv(g). \end{aligned}$$

Moreover, the functional J_λ verifies the Mountain-Pass conditions, namely:

Lemma 8 *Suppose that the conditions of (h^1), (h^2) and (h^3) of section 1 are satisfied. Then J_λ fulfills the following properties*
1-There exist constants $r, R > 0$ such that $J_\lambda(u) \geq R > 0$, $\|u\| = r$.
2-There exists $v \in H_2^2(M)$, with $\|v\| > r$, such that $J_\lambda(v) < 0$.

Lemma 9 *Let (M, g) be a $n-$dimensional compact, smooth and oriented Riemannian manifold with $n \geq 5$ and suppose that conditions (h^1)- (h^2) are satisfied. Then each Palais-Smale sequence at level c_λ is bounded in $H_2^2(M)$.*

Proof. The proof follows from the coerciveness of the operator P_g, the Sobolev's inequality and the condition (h^2). ■

Theorem 10 *Let (M, g) is an $n-$dimensional compact, smooth and oriented Riemannian manifold with $n \geq 5$. Let $(u_m)_m$ be a Palais-Smale sequence at level c_λ. Assume that conditions (h^1)-(h^2) and (h^3) are satisfied and that*

$$c_\lambda < \frac{1}{(1+\varepsilon)^{\frac{n}{n-4}} K_\circ^{\frac{n}{n-4}} \max_{x \in M} f(x)}.$$

Then, there is a subsequence of $(u_m)_m$ converging strongly in $H_2^2(M)$.

Proof. We follows closely the method used in [7]. ■

3. The sharp case

Let $P \in M$, we define the distance function ρ on M by

$$\rho_P(Q) = \begin{cases} d(P,Q) & \text{if} \quad d(P,Q) < i_g(M) \\ \delta(M) & \text{if} \quad d(P,Q) \geq i_g(M) \end{cases}$$

and $i_g(M)$ is the injectivity radius of M. Furthermore, we define the space $L^p(M, \rho^\gamma)$ as follows.

Definition 11 *Let (M, g) be a compact $5 \leq n$-dimensional Riemannian manifold. We consider the space $L^p(M, \rho^\gamma)$ where $1 \leq p \leq +\infty$ of measurable functions u on M such that $\rho^\gamma |u|^p$ is integrable, i.e.*

$$\|u\|_{p,\rho^\gamma}^p := \int_M \rho^\gamma |u|^p \, dv(g) < +\infty$$

Now, we use the following Hardy-Sobolev inequalities proven in [5] (the Hardy-Sobolev inequalities for the singular Yamabe equation was proven in [9]).

Theorem 12 *[5] Let (M, g) be a compact $5 \leq n$-dimensional Riemannian manifold and p, q and γ three real numbers satisfying $\frac{\gamma}{p} = \frac{n}{q} - \frac{n}{p} - 2$ and $2 \leq p \leq \frac{2n}{n-4}$.*
For any $\epsilon > 0$, there is a constant $A(\epsilon, q, \gamma)$ such that

$$\forall u \in H_2^2(M) : \|u\|_{p,\rho^\gamma}^2 \leq (1+\epsilon) K(n, 2, \gamma)^2 \|\Delta_g u\|_2^2 + A(\epsilon, q, \gamma) \|u\|_2^2$$

In particular: $K(n, 2, 0)^2 = K_\circ$ is the optimal constant of Sobolev inequality.

Theorem 13 *[5] Let (M, g) be a compact $5 \leq n$dimensional Riemannian manifold and p, q and γ three real numbers satisfying: $1 \leq q \leq p \leq \frac{nq}{n-2q}$ and $\gamma < 0$.*
•.If $\frac{\gamma}{p} = n(\frac{1}{q} - \frac{1}{p}) - 2$, then the imbedding $H_2^q(M) \subset L^p(M, \rho^\gamma)$ is continuous.
•.If $\frac{\gamma}{p} > n(\frac{1}{q} - \frac{1}{p}) - 2$, then the imbedding $H_2^q(M) \subset L^p(M, \rho^\gamma)$ is compact.

We consider the following equation:

$$\Delta_g^2 u + div_g \left(\frac{a(x)}{\rho^\sigma} \nabla_g u \right) + \frac{b(x)}{\rho^\mu} u = f(x) |u|^{N-2} u + \lambda \frac{h(x)}{\rho^\beta} |u|^{q-2} u \tag{3}$$

where a, b and h are three smooth functions and the distance function defined before in section 1, $2 < q < N$ and $\lambda > 0$ a real parameter. The energy functional J_λ: $H_2^2(M) \longrightarrow \mathbb{R}$ associated to equation (3) is defined as:

$$\begin{aligned} J_\lambda(u) = \; & \frac{1}{2} \int_M \left((\Delta_g u)^2 - \frac{a(x)}{\rho^\sigma} |\nabla_g u|^2 + \frac{b(x)}{\rho^\mu} u^2 \right) dv(g) + \\ & - \frac{\lambda}{q} \int_M \frac{h(x)}{\rho^\beta} |u|^q \, dv(g) - \frac{1}{N} \int_M f(x) |u|^N \, dv(g), \end{aligned}$$

where $u \in H_2^2(M)$ and it is well-known that the critical points of J_λ are the weak solutions of (3).

Theorem 14 *Let $0 < \sigma < \frac{n}{r} < 2$, $0 < \mu < \frac{n}{s} < 4$ and $0 < \beta < \frac{N}{d} < N - q$. We suppose that the conditions (h^1), (h^2) and (h^3) are satisfied and*

$$\sup_{u \in H_2^2(M)} J_\lambda^{\sigma,\mu,\beta}(u) < \frac{2}{n K_\circ^{\frac{n}{4}} (f(x_\circ))^{\frac{n-4}{4}}}$$

Then, the equation (3) has a non trivial solution $u_{\sigma,\mu,\beta} \in H_2^2(M)$.

Proof. The result follows in that if we put $\tilde{a} = \frac{a(x)}{\rho^\sigma}$, $\tilde{b}(x) = \frac{b(x)}{\rho^\mu}$ and $\tilde{h}(x) = \frac{h(x)}{\rho^\beta}$, then $\tilde{a} \in L^r(M)$, $\tilde{b} \in L^s(M)$ and $\tilde{h} \in L^d(M)$, with $r > \frac{n}{2}$, $s > \frac{n}{4}$ and $d > \frac{N}{N-q}$. ■

4. Critical cases

Strategies developed in [7] and [8] enable us to derive another result, that refers to the critical cases when $\sigma = 2$, $\mu = 4$, and $\beta = \frac{n(q-2)}{2} - 2q$.

Theorem 15 *Let (M, g) be an $n-$dimensional compact, smooth and oriented Riemannian manifold with $n \geq 5$ and suppose that the conditions (h^1), (h^2) and (h^3) are satisfied. In addition, let $(u_m)_m := (u_{\sigma_m,\mu_m,\beta_m})_m$ be a sequence in $H_2^2(M)$ such that:*

$$\begin{cases} J_\lambda^{\sigma,\mu,\beta}(u_m) \to c_\lambda^{\sigma,\mu,\beta} & \text{for all} \quad n \in \mathbb{N} \\ \nabla J_\lambda^{\sigma,\mu,\beta}(u_m) \to 0 & \text{weakly in} \quad H_2^2(M) \end{cases} \quad \text{with } c_\lambda^{\sigma,\mu,\beta} < \frac{2}{nK_\circ^{\frac{n}{4}} (f(x_\circ))^{\frac{n-4}{4}}} \tag{4}$$

and

$$1 + a^- \max(K(n, 2, \sigma); \quad A(\epsilon, \sigma)) + b^- \max(K(n, 2, \mu); \quad A(\epsilon, \mu)) > 0. \tag{5}$$

Then, the equation

$$\Delta_g^2 u + div_g \left(\frac{a(x)}{\rho^\sigma} \nabla_g u \right) + \frac{b(x)}{\rho^\mu} u = f(x) \left| u \right|^{N-2} u + \lambda \frac{h(x)}{\rho^\beta} \left| u \right|^{q-2} u$$

has a nontrivial solution $u_{\sigma,\mu,\beta} \in H_2^2(M)$.

Proof. We follow closely the method used in [7] and [8]. First by using the condition (5) we obtain, as in [7], that the sequence $(\Lambda_{\alpha,\mu})_{\alpha,\mu}$ of constants of coerciveness of the operator $u \to \Delta_g^2 u + div_g \left(\frac{a(x)}{\rho^\sigma} \nabla_g u \right) + \frac{b(x)}{\rho^\mu} u$ is bounded below by a constant $\Lambda > 0$ as $(\alpha, \mu) \to (2^-, 4^-)$. Let $(u_m)_m \subset H_2^2(M)$, such that :

$$J_\lambda^{\sigma,\mu,\beta}(u_m) = c_\lambda^{\sigma,\mu,\beta} + o(1) \qquad \text{and} \qquad \nabla J_\lambda^{\sigma,\mu,\beta}(u_m) = o(1) \qquad \text{in } \left(H_2^2(M) \right)^*$$

Then we have:

$$J_\lambda^{\sigma,\mu,\beta}(u_m) - \frac{1}{N} \left\langle J_\lambda^{\sigma,\mu,\beta}(u_m), u_m \right\rangle = \left(\frac{1}{2} - \frac{1}{N} \right) \|u_m\|^2 - \lambda \left(\frac{1}{q} - \frac{1}{N} \right) \int_M h(x) \left| u_m \right|^q dv(g)$$

By Hölder and Sobolev inequalities, we get that

$$J_\lambda^{\sigma,\mu,\beta}(u_m) - \frac{1}{N} \left\langle \nabla J_\lambda^{\sigma,\mu,\beta}(u_m), u_m \right\rangle = c_\lambda^{\sigma,\mu,\beta} + o(1)$$

and

$$c_\lambda^{\sigma,\mu,\beta} + o(1) \geq \left(\frac{1}{2} - \frac{1}{N} \right) \|u_m\|^2 - \left(\frac{1}{q} - \frac{1}{N} \right) (\max((1+\varepsilon)K_\circ, A_\varepsilon))^{\frac{q}{2}} \|h\|_\alpha \|u_m\|_{H_2^2(M)}^q$$

In addition the hypothesis (h^1) and (h^2) are satisfied and if we have $\|u_n\| \geq 1$, then we obtain

$$\|u_m\| \leq \left[\left(\frac{N-2}{2} - \lambda \frac{N-q}{q} \Lambda^{-\frac{q}{2}} (\max((1+\varepsilon)K_\circ, A_\varepsilon))^{\frac{q}{2}} \|h\|_\alpha \right)^{-1} N c_\lambda^{\sigma,\mu,\beta} \right]^{\frac{1}{q}} + o(1)$$

Then $(u_m)_m$ is bounded in $H_2^2(M)$. The rest of the proof is the same as in Theorem 10. ∎

Concluding remark. To prove main Theorems given in the Introduction, let $\delta \in \left(0, \frac{i_g(M)}{2}\right)$ and $\eta \in C^\infty(M)$ such that:

$$\eta(x) = \begin{cases} 1 & \text{if} \quad x \in B(x_\circ, \delta) \\ 0 & \text{if} \quad x \in M - B(x_\circ, 2\delta) \end{cases}$$

For $\epsilon > 0$, we define the radial function u_ϵ by:

$$u_\epsilon(x) := \frac{\eta(x)}{\left(\epsilon^2 + (\xi\rho)^2\right)^{\frac{n-4}{2}}} \qquad \text{with} \quad \xi = (1 + \|a\|_r + \|b\|_s)^{\frac{1}{n}}\,. \tag{6}$$

We next point out that, by resorting to the strategy outlined in [7, 8], the function given by (6) can be proved to verify condition (4) of the generic theorem. This step completes our discussion on the solutions of Equation (2).

References

[1] Paneitz S 2008 A quartic conformally covariant differential operator for arbitrary peudo Riemannian manifolds *SIGMA* **4**
[2] Branson T P 1987 Group representation arising from Lorentz conformal geometry *J. Funct. Anal.* **74** 199
[3] Benalili M 2009 Existence and multiplicity of solutions to elliptic equations of fourth order on compact manifolds *Dynamics of PDE*, **3** 203
[4] Benalili M 2010 Existence and multiplicity of solutions to fourth order elliptic equations with critical exponent on compact manifolds *Bull. Belg. Math. Soc. Simon Stevin* **17**
[5] Benalili M 2013 On singular Q-curvature type equations *J. Diff. Eq.* **254** 547
[6] Benalili M and Tahri K 2011 Nonlinear elliptic fourth order equations existence and multiplicity results *Nonlin. Differ. Equ. Appl.* **18** 539
[7] Benalili M and Tahri K 2013 Existence of solutions to singular fourth-order elliptic equations *Electron. J. Diff. Equ.* **2013** No. 63, pp. 1
[8] Benalili M and Tahri K 2012 Multiple solutions to singular fourth order elliptic equations, (preprint: arXiv:1209.3764v1 [math.DG] 16 Sep 2012)
[9] Madani F 2008 Le problème de Yamabé avec singularités, (preprint: ArXix: 1717v1 [mathAP]).
[10] Robert F and Esposito P 2002 Mountain-Pass critical points for Paneitz-Branson operators *Calc. of Variations and Partial Diff. Eq.* **15** 493