

# Building Generalized Lax Integrable KdV and MKdV Equations with Spatiotemporally Varying Coefficients

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**Abstract.** We present a technique based on extended Lax Pairs to derive variable-coefficient generalizations of various Lax-integrable NLPDE hierarchies. As illustrative examples, we consider generalized KdV equations, and three variants of generalized MKdV equations. It is demonstrated that the technique yields Lax- or S-integrable NLPDEs with both time- AND space-dependent coefficients which are thus more general than almost all cases considered earlier via other methods such as the Painlevé Test, Bell Polynomials, and various similarity methods. Some solutions are also presented for the generalized KdV equation derived here by the use of the Painlevé singular manifold method. Current and future work is centered on generalizing other integrable hierarchies of NLPDEs similarly, and deriving various integrability properties such as solutions, Backlund Transformations, and hierarchies of conservation laws for these new integrable systems with variable coefficients.

**Key Words:** Generalizing Lax or S-integrable equations, spatially and temporally-dependent coefficients, solutions.

## 1. Introduction

Variable Coefficient Korteweg de Vries (vcKdV) and Modified Korteweg de Vries (MKdV) equations have a long history dating from their derivation in various applications[1]-[10].

Almost all studies, including those which derived exact solutions by a variety of techniques, as well as those which considered integrable sub-cases and various integrability properties by methods such as Painlevé analysis, Hirota's method, and Bell Polynomials treat vcKdV equations with coefficients which are functions of the time only. For instance, for generalized variable coefficient NLS (vcNLS) equations, a particular coefficient is usually taken to be a function of  $x$ [11], as has also been sometimes done for vcMKdV equations[12]. The papers [13]-[14] are somewhat of an exception in that they treat vcNLS equations having coefficients with general  $x$  and  $t$  dependences. Variational principles, solutions, and other integrability properties have also been considered for some of the above variable coefficient NLPDEs in cases with time-dependent coefficients.

In applications, the coefficients of vcKdV equations may include spatial dependence, in addition to the temporal variations that have been extensively considered using a variety of techniques. Both for this reason, as well as for their mathematical interest, extending integrable hierarchies of nonlinear PDEs (NLPDEs) to include *both* spatial and temporal dependence of the coefficients is worthwhile.



Given the above, we consider a direct method for deriving the integrability conditions of both a general form of variable-coefficient MKdV (vcMKdV) equation, as well as a general, variable-coefficient KdV (vcKdV) equation. In both cases, the coefficients are allowed to vary in space AND time.

The method used is based on directly establishing Lax integrability (or S-integrability to use the technical term) as detailed in the following section. As such, it is rather general, although subject to the ensuing equations being solvable. We should stress that the computer algebra involved is quite challenging, and an order of magnitude beyond that encountered for integrable, constant coefficient NLPDEs.

However, there is an additional important proviso or qualifier that needs to be stressed. In order that Lax integrability ensues from the compatibility of the Lax Pair, i.e., the compatibility condition(s) for the Lax Pair contain the NLPDE under question, a form for each entry in the Lax Pair must be guessed *a priori*. Since there is no strict algorithm for doing this, one must rely on something like inspired guesswork. The inspiration comes from basing the Lax Pair on the known one for the existing constant coefficient version of the NLPDE under investigation. However, non-trivial guesswork must now go into adding additional terms and coefficients in each entry of this Lax Pair, and also generalizing some of the constants (related to the spectral parameter of the constant coefficient NLPDE) to functions.

We shall explain the above paragraph, which may seem rather dense at the first reading, in more detail in the context of the actual examples in the following two sections.

The outline of this paper is as follows. In Section 2, we briefly review the Lax Pair method and its modifications for variable-coefficient NLPDEs, and then apply it to three classes of generalized vcMKdV equation. Section 3 considers an analogous treatment of a generalized vcKdV equation. Some solutions of the generalized vcKdV equations of Section 3 are then derived in Section 4 via the use of truncated Painlevé expansions. Section 5 briefly considers the results and conclusions of the paper, and discusses the natural next steps or directions for further work.

## 2. Extended Lax Pair method and application to three generalized vcMKdV equations

In the Lax pair method [15] - [16] for solving and determining the integrability conditions for nonlinear partial differential equations (NLPDEs) a pair of  $n \times n$  matrices,  $\mathbf{U}$  and  $\mathbf{V}$  needs to be derived or constructed. The key component of this construction is that the integrable nonlinear PDE under consideration must be contained in, or result from, the compatibility of the following two linear Lax equations (the Lax Pair)

$$\Phi_x = U\Phi \tag{1}$$

$$\Phi_t = V\Phi \tag{2}$$

where  $\Phi$  is an eigenfunction, and  $\mathbf{U}$  and  $\mathbf{V}$  are the time-evolution and spatial-evolution (eigenvalue problem) matrices. From the cross-derivative condition (i.e.  $\Phi_{xt} = \Phi_{tx}$ ) we get

$$U_t - V_x + [U, V] = \dot{0} \tag{3}$$

known as the zero-curvature condition where  $\dot{0}$  is contingent on  $v(x, t)$  being a solution to the nonlinear PDE. A Darboux transformation can then be applied to the linear system to obtain solutions from known solutions and other integrability properties of the integrable NLPDE.

We first consider the following three variants of generalized variable-coefficient MKdV (vcMKdV) equations:

$$v_t + a_1 v_{xxx} + a_2 v^2 v_x = 0 \tag{4}$$

$$v_t + b_1 v_{xxxxx} + b_2 v^2 v_{xxx} + b_3 v v_x v_{xx} + b_4 v_x^3 + b_5 v^4 v_x = 0 \tag{5}$$

$$v_t + c_1 v_{xxxxxxx} + c_2 v^2 v_{xxxxx} + c_4 v v_{xx} v_{xxx} + c_5 v_x^2 v_{xxx} + c_6 v_x v_{xx}^2 + c_7 v^4 v_{xxx} + c_8 v^3 v_x v_{xx} + c_9 v^2 v_x^3 + c_{10} v^6 v_x = 0. \tag{6}$$

These equations, which we shall always call the physical (or field) NLPDEs to distinguish them from the many other NLPDEs we encounter, will be Lax-integrable or S-integrable if we can find a Lax pair whose compatibility condition (3) contains them.

One expands the Lax pair  $\mathbf{U}$  and  $\mathbf{V}$  in powers of  $v$  and its derivatives with unknown functions as coefficients. This results in a VERY LARGE system of coupled NLPDEs for the variable coefficient functions. Upon solving these (and a solution is not guaranteed, and may prove to be impossible to obtain in general for some physical NLPDEs), we simultaneously obtain the Lax pair and integrability conditions on the  $a_i, b_i,$  and  $c_i$  for which (4) - (6) are Lax-integrable.

The results are given in the following three subsections. The details of derivations are omitted for the sake of readability.

### 2.1. Conditions on the $a_i$

$$6a_1 a_{2x}^3 - 6a_1 a_2 a_{2x} a_{2xx} + a_1 a_2^2 a_{2xxx} - \frac{K_t}{K} a_2^3 + a_2^2 a_{2t} - a_2^3 a_{1xxx} + v + 3a_{1xx} a_2^2 a_{2x} - 6a_{1x} a_2 a_{2x}^2 + 3a_{1x} a_2^2 a_{2xx} = 0 \tag{7}$$

where  $K(t)$  is an arbitrary function of  $t$ .

### 2.2. Conditions on the $b_i$

$$b_3 = 2(b_2 + b_4), \quad b_5 = H_1(t)(2b_2 - b_4). \tag{8}$$

where  $H_1(t)$  is an arbitrary function of  $t$  and  $b_2$  or  $b_4$  are considered psuedo-arbitrary. We also do not display one more condition which we omit because it is very long. Now, once  $b_2$  or  $b_4$  is given all other coefficients can be found. The latter equation may look a bit daunting but is quite easily managed with the aid of a CAS (in this case MAPLE) once  $b_2$  and  $b_4$  are given. One example is presented here:

### 2.3. Conditions on the $c_i$

$$c_4 = -10c_2 + 5c_3 - 4c_5 + 2c_6 \tag{9}$$

$$c_8 = \frac{2}{3}c_9 + 4c_7 \tag{10}$$

$$c_{10x}c_6 - c_6c_{10x} - 2c_5c_{10x} + 2c_{5x}c_{10} + c_{10x}c_3 - c_{3x}c_{10} = 0 \tag{11}$$

$$-12c_{10x}c_7 + 12c_{7x}c_{10} + c_{10x}c_9 - c_{9x}c_{10} = 0 \tag{12}$$

$$5Gc_{10}^3 - 2Hc_{10x}c_{7x}c_{10} + Hc_{10}^2c_{7xx} + 2Hc_{10x}^2c_7 - Hc_7c_{10}c_{10xx} = 0 \tag{13}$$

$$14c_3c_{10x}^2 - 7c_3c_{10}c_{10xx} - 60c_2c_{10x}^2 + 30c_2c_{10}c_{10xx} + 60c_{2x}c_{10}c_{10x} - 30c_{2xx}c_{10}^2 + 4c_{5x}c_{10}c_{10x} - 14c_{3x}c_{10}c_{10x} + 7c_{3xx}c_{10}^2 - 4c_5c_{10x}^2 + 2c_5c_{10}c_{10xx} - 2c_{5xx}c_{10}^2 = 0 \tag{14}$$

$$18c_3c_{10x}^2 - 18c_3c_{10}c_{10x}c_{10xx} + 3a_3c_{10}^3c_{10xxx} - 18c_{3x}c_{10}c_{10x}^2 + 9c_{3x}c_{10}^2c_{10xx} - 60c_2c_{10x}^3 + 60c_2c_{10}c_{10x}c_{10xx} - 10c_2c_{10}^2c_{10xxx} + 60c_{2x}c_{10}c_{10x}^2 - 30c_{2x}c_{10}^2c_{10xx} - 30c_{2xx}c_{10}^2c_{10x} + 10c_{2xxx}c_{10}^3 + 6c_{5x}c_{10}c_{10x}^2 - 3c_{5x}c_{10}^2c_{10xx} - 3c_{5xx}c_{10}^2c_{10x} + 9c_{3xx}c_{10}^2c_{10x} - 3c_{3xxx}c_{10}^3 - 6c_5c_{10x}^3 + c_5c_{10}c_{10x}c_{10xx} - c_5c_{10}^2c_{10xxx} + c_{5xx}c_{10}^3 = 0 \tag{15}$$

$$\begin{aligned}
 & c_{1xxxxxx}c_{10}^7 + 5040c_{1x}c_{10}c_{10x}^6 - 7c_{1xxxxx}c_{10}^6c_{10x} - c_{10}^6c_{10xxxxxx} - 630c_{1x}c_{10}^4c_{10xx}^3 \\
 & + 140c_{1x}c_{10}^5c_{10xxx}^2 - 7c_{1x}c_{10}^6c_{10xxxxx} - 2520c_{1xx}c_{10}^2c_{10x}^5 - 21c_{1xx}c_{10}^6c_{10xxxx} \\
 & + 840c_{1xxx}c_{10}^3c_{10x}^4 + 210c_{1xxx}c_{10}^5c_{10xx}^2 - 35c_{1xxx}c_{10}^6c_{10xxxx} - 210c_{1xxxx}c_{10}^4c_{10x}^3 \\
 & - 35c_{1xxxx}c_{10}^6c_{10xxx} + 42c_{1xxxx}c_{10}^5c_{10x}^2 - 21c_{1xxxx}c_{10}^6c_{10xx} + 5040c_{10}^3c_{10x}^2c_{10xx}c_{10xxx} \\
 & - 630c_{10}^4c_{10x}c_{10xx}c_{10xxx} - 2520c_{1x}c_{10}^4c_{10x}c_{10xx}c_{10xxx} + \frac{H_t}{H}c_{10}^7 - c_{10}^6c_{10t} - 5040c_{10}c_{10x}^7 \\
 & + 15210c_{10}c_{10x}^5c_{10xx} - 4200c_{10}^2c_{10x}^4c_{10xxx} + 840c_{10}^3c_{10x}^3c_{10xxxx} - 630c_{10}^4c_{10x}^2c_{10xxx} \\
 & - 126c_{10}^4c_{10x}^2c_{10xxxx} + 70c_{10}^5c_{10xxx}c_{10xxxx} + 42c_{10}^5c_{10xx}c_{10xxxx} \\
 & + 14c_{10}^5c_{10x}c_{10xxxx} - 12600c_{10}^2c_{10x}^3c_{10xx}^2 + 2520c_{10}^3c_{10x}c_{10xx}^3 - 420c_{10}^4c_{10x}c_{10xx}^2 \\
 & - 12600c_{1x}c_{10}^2c_{10x}^4c_{10xx} + 7560c_{1x}c_{10}^3c_{10x}^2c_{10xx}^2 + 3360c_{1x}c_{10}^3c_{10x}^3c_{10xxx} \\
 & - 630c_{1x}c_{10}^4c_{10x}^2c_{10xxx} + 210c_{1x}c_{10}^5c_{10xx}c_{10xxx} + 84c_{1x}c_{10}^5c_{10x}c_{10xxxx} \\
 & + 5040c_{1xx}c_{10}^3c_{10x}^3c_{10xx} - 1890c_{1xx}c_{10}^4c_{10x}c_{10xx}^2 - 1260c_{1xx}c_{10}^4c_{10x}^2c_{10xxx} \\
 & + 420c_{1xx}c_{10}^5c_{10xx}c_{10xxx} + 210c_{1xx}c_{10}^5c_{10x}c_{10xxxx} - 1260c_{1xxx}c_{10}^4c_{10x}^2c_{10xx} \\
 & + 280c_{1xxx}c_{10}^5c_{10x}c_{10xxx} + 210c_{1xxx}c_{10}^5c_{10x}c_{10xx} = 0
 \end{aligned} \tag{16}$$

$$\begin{aligned}
 & -240c_3c_{10x}^5 + 720c_2c_{10x}^5 + 120c_5c_{10x}^5 + 40c_{3xxx}c_{10}^3c_{10x}^2 - 20c_{3xxx}c_{10}^4c_{10xx} + 6c_2c_{10}^4c_{10xxxx} \\
 & - 180c_{2x}c_{10}^3c_{10xx}^2 + 30c_{2x}c_{10}^4c_{10xxx} + 360c_{2xx}c_{10}^2c_{10x}^3 + 60c_{2xx}c_{10}^4c_{10xxx} - 120c_{2xxx}c_{10}^3c_{10x}^2 \\
 & + 60c_{2xxx}c_{10}^4c_{10xx} - 30c_{5x}c_{10}^3c_{10xx}^2 + 5c_{5x}c_{10}^4c_{10xxxx} + 60c_{5xx}c_{10}^2c_{10x}^3 + 10c_{5xx}c_{10}^4c_{10xxx} \\
 & - 20c_{5xxx}c_{10}^3c_{10x}^2 + 10c_{5xxx}c_{10}^4c_{10xx} + c_5c_{10}^4c_{10xxxx} - 120c_{3xx}c_{10}^2c_{10x}^3 - 20c_{3xx}c_{10}^4c_{10xxx} \\
 & - 2c_3c_{10}^4c_{10xxxx} + 60c_{3x}c_{10}^3c_{10xx}^2 - 10c_{3x}c_{10}^4c_{10xxxx} + 30c_{2xxx}c_{10}^4c_{10x} + 5c_{5xxx}c_{10}^4c_{10x} \\
 & - 10c_{3xxx}c_{10}^4c_{10x} - 6c_{2xxxx}c_{10}^5 + 2c_{3xxxx}c_{10}^5 - c_{5xxxx}c_{10}^5 - 720c_{2x}c_{10}^4c_{10x} - 120c_{5x}c_{10}^4c_{10x}^4 \\
 & + 240c_{3x}c_{10}^4c_{10x} + 480c_3c_{10}c_{10x}^3c_{10xx} - 180c_3c_{10}^2c_{10x}c_{10xx}^2 - 120c_3c_{10}^2c_{10x}^2c_{10xxx} \\
 & + 40c_3c_{10}^3c_{10xx}c_{10xxx} + 20c_3c_{10}^3c_{10x}c_{10xxxx} - 360c_{3x}c_{10}^2c_{10x}^2c_{10xx} + 80c_{3x}c_{10}^3c_{10x}c_{10xxx} \\
 & + 120c_{3xx}c_{10}^3c_{10x}c_{10xx} - 1440c_2c_{10}c_{10x}^3c_{10xx} + 540c_2c_{10}^2c_{10x}c_{10xx}^2 + 360c_2c_{10}^2c_{10x}^2c_{10xxx} \\
 & - 120c_2c_{10}^3c_{10xx}c_{10xxx} - 60c_2c_{10}^3c_{10x}c_{10xxxx} + 1080c_{2x}c_{10}^2c_{10x}^2c_{10xx} - 240c_{2x}c_{10}^3c_{10x}c_{10xxx} \\
 & - 360c_{2xx}c_{10}^3c_{10x}c_{10xx} + 180c_{5x}c_{10}^2c_{10x}^2c_{10xx} - 40c_{5x}c_{10}^3c_{10x}c_{10xxx} - 60c_{5xx}c_{10}^3c_{10x}c_{10xx} \\
 & - 240c_5c_{10}c_{10x}^3c_{10xx} + 90c_5c_{10}^2c_{10x}c_{10xx}^2 + 60c_5c_{10}^2c_{10x}^2c_{10xxx} \\
 & - 20c_5c_{10}^3c_{10xx}c_{10xxx} - 10c_5c_{10}^3c_{10x}c_{10xxxx} = 0
 \end{aligned} \tag{17}$$

where  $G(t)$  and  $H(t)$  are arbitrary functions of  $t$ .

### 3. Generalized vcKdV Equations

Here, we will applying the technique of the last section in exactly the same fashion to generalized vcKdV equations, but will omit the details for the sake of brevity. Please note that *the coefficients  $a_i$ , as well as the quantities labeled  $c_i$  in this section are totally distinct or different from those given the same symbols in the previous section. All equations in this section are thus to be read independently of those in the previous one.*

Consider the generalized vcKdV equation in the form

$$u_t + a_1uu_{xxx} + a_2u_xu_{xx} + a_3u^2u_x + a_4uu_x + a_5u_{xxx} + a_6u_{xxxx} + a_7u + a_8u_x = 0 \tag{18}$$

As before, we consider the generalized variable-coefficient KDV equation to be integrable if we can find a Lax pair which satisfies (3). In the method given in [14] one expands the Lax pair  $\mathbf{U}$  and  $\mathbf{V}$  in powers of  $u$  its derivatives with unknown function coefficients and require (3) to be equivalent to the nonlinear system. This results in a system of coupled PDEs for

the unknown coefficients for which upon solving we simultaneously obtain the Lax pair and integrability conditions on the  $a_i$ . The results, for which the details are similar to the previous section and so are omitted here, are as follows

$$a_{1-4} = \frac{H_{1-4}}{p}, \quad a_7 = \frac{pt + (pa_5)_{xxx} + (pa_6)_{xxxx} + (pa_8)_x}{p} \tag{19}$$

where  $H(t)$  is an arbitrary function of  $t$ ,  $c_1, c_2$  are arbitrary constants and  $a_1, a_5, a_6, a_8$  and  $p(\neq 0)$  are taken to be arbitrary functions of  $x$  and  $t$ . This form helps to give integrability conditions in specific cases. For example an integrable variable-coefficient KDV equation would require that  $H_{1-3}(t) = a_{6-8}(x, t) = 0$ . For  $a_7(x, t) = 0$  we would need to further require that (through a little algebraic manipulation) the following be satisfied

$$\left(\frac{H(t)}{a_1}\right)_t + H(t)\left(\frac{a_5}{a_1}\right)_{xxx} = 0 \tag{20}$$

Note that with the choices  $p(x, t) = e^{\int m(t)dt}$ ,  $H_1(t) = H_2(t) = \frac{1}{2\mu_2}H_4(t) = a(t)e^{\int m(t)dt}$ ,  $H_3(t) = \mu_1 e^{2\int m(t)dt}$ ,  $a_5(x, t) = 5\mu_2 a_6(x, t) = \frac{\mu_2 a(t)}{\mu_1 e^{\int m(t)dt}}$  and  $a_8(x, t) = n(t)$ , we see that  $a_7(x, t) = m(t)$  and thus we get back exactly the integrability conditions found in earlier papers on generalized vKdV systems.

#### 4. Painleve Analysis Method

Given a nonlinear partial differential equation in  $(n + 1)$ -dimensions, without specifying initial or boundary conditions, we may find a solution about a movable singular manifold  $\phi - \phi_0 = 0$  as an infinite Laurent series given by

$$u(x_1, \dots, x_n, t) = \phi^{-\alpha} \sum_{m=0}^{\infty} u_m \phi^m. \tag{21}$$

Note that when  $m \in (\mathbb{Q} - \mathbb{Z})$  (21) is more commonly known as a Puiseux series. One can avoid dealing with Puiseux series if proper substitutions are made, as we will see a little later on. Plugging this infinite series into the NLPDE yields a recurrence relation for the  $u_m$ 's. As with most series-type solution methods for NLPDEs we will seek a solution to our NLPDE as (21) truncated at the constant term. Plugging this truncated series into our original NLPDE and collecting terms in decreasing order of  $\phi$  will give us a set of determining equations for our unknown coefficients  $u_0, \dots, u_\alpha$  known as the Painleve-Backlund equations. We now define new functions

$$C_0(x_0, \dots, x_n, t) = \frac{\phi_t}{\phi_{x_0}} \tag{22}$$

$$C_1(x_0, \dots, x_n, t) = \frac{\phi_{x_1}}{\phi_{x_0}} \tag{23}$$

$$\vdots \tag{24}$$

$$C_n(x_0, \dots, x_n, t) = \frac{\phi_{x_n}}{\phi_{x_0}} \tag{25}$$

$$V(x_0, \dots, x_n, t) = \frac{\phi_{x_0 x_0}}{\phi_{x_0}} \tag{26}$$

which will allow us to eliminate all derivatives of  $\phi$  other than  $\phi_{x_0}$ . For simplicity it is common to allow  $C_i(x_0, \dots, x_n, t)$  and  $V(x_0, \dots, x_n, t)$  to be constants, thereby reducing a system of PDEs (more than likely nonlinear) in  $\{C_i(x, t), V(x, t)\}$  to an algebraic system in  $\{C_i, V\}$  for  $(i = 0, \dots, n)$ .

4.1. Exactly Solvable Examples

Consider the following example

$$\begin{aligned}
 u_t + \frac{H_1}{F(\frac{x}{C} + t)} uu_{xxx} + \left( \frac{c_1 H_3 - 2H_1 + 1}{F(\frac{x}{C} + t)} \right) u_x u_{xx} + \frac{H_3}{F(\frac{x}{C} + t)} u^2 u_x + \frac{H_4(t)}{F(\frac{x}{C} + t)} uu_x \\
 + \left( \frac{c_1(c_1 H_3(t) + 2H_4 - 2H_1)}{8F(\frac{x}{C} + t)} \right) u_{xxx} + \left( \frac{c_1(c_1 H_3 - 2)}{40F(\frac{x}{C} + t)} \right) u_{xxxx} \\
 + \left( \frac{(5c_1(H_1 - H_4) - c_1 - 3c_1^2 H_3 - 20F(\frac{x}{C} + t)C)}{20F(\frac{x}{C} + t)} \right) u_x = 0
 \end{aligned} \tag{27}$$

Note that in this example we have  $a_7 = 0$ . The leading order analysis yields  $\alpha = 2$ . Therefore we seek a solution of the form

$$u(x, t) = \frac{u_0}{\phi(x, t)^2} + \frac{u_1}{\phi(x, t)} + u_2(x, t) \tag{28}$$

This forms an auto-Bäcklund transformation. For simplicity we will allow our initial solution  $u_2(x, t)$  to be 0. Plugging this into our pde yields the following determining equations for  $\phi(x, t), u_0(x, t), u_1(x, t), V$  and  $C_1$  :

$$O(\phi^{-7}) : 2u_0 \phi_x [(9c_1^2 H_3 + 18c_1) \phi_x^4 + (6H_3 c_1 + 6) u_0 \phi_x^2 + H_3 u_0^2] = 0 \tag{29}$$

$$\begin{aligned}
 O(\phi^{-6}) : & -10H_1 u_0 u_1 \phi_x^3 - 5H_3 u_0^2 u_1 \phi_x + 60c_1 u_0 \phi_x^3 \phi_{xx} + 15c_1^2 H_3 u_0 \phi_x^4 - 3c_1^2 H_3 u_1 \phi_x^5 \\
 & + 10H_1 u_0^2 \phi_x \phi_{xx} - 10H_1 u_0 u_0 x \phi_x^2 + 30c_1^2 H_3 u_0 \phi_x^3 \phi_{xx} + 14c_1 H_3 u_0 u_0 x \phi_x^2 \\
 & + 4c_1 H_3 u_0^2 \phi_x \phi_{xx} - 10c_1 H_3 u_0 u_1 \phi_x^3 + 30c_1 u_0 \phi_x^4 - 6c_1 u_1 \phi_x^4 + 14u_0 u_0 x \phi_x^2 \\
 & + 4u_0^2 \phi_x \phi_{xx} - 10u_0 u_1 \phi_x^3 + H_3 u_0^2 u_0 x = 0
 \end{aligned} \tag{30}$$

$$\begin{aligned}
 O(\phi^{-5}) : & H_3^2 u_0^2 u_{1x} - 6H_4 u_0 \phi_x^3 - 6u_0 \phi_x^3 (c_1 H_3 - 2H_1 + 1) + 3u_0 \phi_x^3 \\
 & + 6H_1 u_0 \phi_x^3 (c_1 H_3 - 2H_1 + 1) - 4c_1 H_3^2 u_0 u_1 \phi_x \phi_{xx} - 16H_1 H_3 u_0 u_1 \phi_x \phi_{xx} \\
 & + 9c_1^2 H_3^2 u_0 \phi_x \phi_{xx}^2 + 4H_3 u_0^2 \phi_x + 2H_3 u_1^2 \phi_x^3 - 6H_1 H_3 u_0 u_1 \phi_x^2 \\
 & + 14H_1 H_3 u_0 u_{1x} \phi_x^2 - 10c_1 H_3^2 u_0 u_{1x} \phi_x^2 - 6c_1 H_3^2 u_0 u_1 \phi_x^2 + 36c_1 H_3 u_0 \phi_x^2 \phi_{xx} \\
 & + 12c_1 H_3 u_0 \phi_x^2 \phi_{xxx} + 2H_1 H_3 u_0 u_0 x \phi_{xx} - 4H_3 u_0 u_1 \phi_x \phi_{xx} - 6c_1^2 H_3^2 u_1 \phi_x^3 \phi_{xx} \\
 & + 6c_1^2 H_3^2 u_0 \phi_x^2 \phi_{xxx} + 18c_1^2 H_3^2 u_0 \phi_x^2 \phi_{xx} + 2c_1 H_3^2 u_0 u_0 x \phi_{xx} + 2H_1 H_3 u_0 u_0 x \phi_x \\
 & - 12c_1 H_3 u_1 \phi_x^3 \phi_{xx} + 2c_1 H_3^2 u_0 u_0 x \phi_x + 18c_1 H_3 u_0 \phi_x \phi_{xx}^2 + 2H_1 H_3 u_1^2 \phi_x^3 \\
 & + 4H_3^2 u_0 u_1^2 \phi_x + 2H_3 H_4 u_0^2 \phi_x - 6c_1 H_3 u_1 \phi_x^4 - 3c_1^2 H_3^2 u_{1x} \phi_x^4 - 8H_1 H_3 u_0^2 \phi_x \\
 & - 10H_3 u_0 u_{1x} \phi_x^2 + 2c_1 H_3^2 u_1^2 \phi_x^3 - 6H_3 u_0 u_1 \phi_x^2 + 2H_3 u_0 u_0 x \phi_x + 4c_1 H_3^2 u_0^2 \phi_x \\
 & + 6H_4 u_0 \phi_x^3 (c_1 H_3 - 2H_1 + 1) + 2H_1 H_3 u_0^2 \phi_{xxx} + 12c_1 H_3 u_0 u_0 x \phi_x^3 \\
 & + 6c_1^2 H_3^2 u_0 u_0 x \phi_x^3 - 2H_3^2 u_0 u_0 x u_1 + 12H_1 H_4 u_0 \phi_x^3 + 2H_3 u_0 u_0 x \phi_{xx} \\
 & + 3u_0 \phi_x^3 (c_1 H_3 - 2H_1 + 1)^2 - 6H_1 u_0 \phi_x^3 = 0
 \end{aligned} \tag{31}$$

Upon solving the  $O(\phi^{-7})$  and  $O(\phi^{-6})$  equations for  $u_0$  and  $u_1$  respectively we find that

$$u_0(x, t) = -3c_1 \phi_x^2, \quad u_1(x, t) = 3c_1 \phi_{xx} \tag{32}$$

which lends itself nicely to a representation of the solution as  $u(x, t) = 3c_1 \log[\phi(x, t)]_{xx}$ . Further with the choice  $V = 1$  the choices for coefficients the remaining orders of  $\phi$  are identically satisfied. Now solving the system for  $\phi$  given in the previous section we find that  $\phi(x, t) = c_2 + c_3 e^{x+Ct}$ . Therefore we have the solution

$$u(x, t) = \frac{3c_1 c_2 c_3 e^{x+Ct}}{(c_2 + c_3 e^{x+Ct})^2} \tag{33}$$

which for the selection  $c_2 = c_3$  reduces to the solution  $u(x, t) = \frac{3c_1}{4} \operatorname{sech}^2\left(\frac{x}{2} + \frac{Ct}{2}\right)$

The next example is similar to the first however in this case we don't have  $a_7 = 0$  and we will not force the  $u_2$  term to be the trivial solution. We thus consider the following example

$$\begin{aligned}
 u_t + \frac{10H_1\xi(t)}{F(\int \eta(t)dt + x)}uu_{xxx} + \frac{2(3 + 2H_1)\xi(t)}{F(\int \eta(t)dt + x)}u_xu_{xx} + \\
 + \frac{6H_1 - 1}{F(\int \eta(t)dt + x)}u^2u_x + \frac{10H_4\xi(t)}{F(\int \eta(t)dt + x)}uu_x + \frac{10H_5\xi(t)}{F(\int \eta(t)dt + x)}u_{xxx} + \\
 + \frac{4(3H_1 + 2)\xi(t)^2}{5F(\int \eta(t)dt + x)}u_{xxxxx} - \left(\frac{1}{\xi(t)}\right)' F\left(\int \eta(t)dt + x\right)u + \\
 + \left(H_6(t) + \frac{\xi(t)^2(c_2^4(8H_1 - 3) - 2500c_1(H_4 + 30c_1H_1 - 5c_1))}{5F(\int \eta(t)dt + x)}\right)u_x = 0 \quad (34)
 \end{aligned}$$

where  $\xi(t) = H_5/(10c_1H_1 - 10c_1 + H_4)$  and  $H_1(t), H_4(t), H_5(t)$  and  $\eta(t)$  are arbitrary functions of  $t$  and  $c_1, c_2$  are arbitrary constants. As with our last example the leading order analysis yields  $\alpha = 2$ . Unlike our last example we will not force the  $u_2$  term to be 0 initially. The first orders of  $\phi$  which determine the  $u_i$  are as follows :

$$O(\phi^{-7}) : -576(3H_1 + 2)\xi(t)^2u_0\phi_x^5 - 2(6H_1 - 1)u_0^3\phi_x - 24H_1u_0^2\phi_x^3 - 24(3 + 2H_1)\xi(t)u_0^2\phi_x^3 = 0 \quad (35)$$

$$\begin{aligned}
 O(\phi^{-6}) : \quad & 1440H_1H_5^2\phi_x^4u_{0x} - 288H_1H_5^2\phi_x^5u_1 + 600c_1^2H_1^3u_0^2u_{0x} - 1300c_1^2H_1^2u_0^2u_{0x} \\
 & + 800c_1^2H_1u_0^2u_{0x} + 6H_1H_4^2u_0^2u_{0x} + 500c_1^2\phi_xu_0^2u_1 + 20c_1H_4u_0^2u_{0x} + 5H_4^2\phi_xu_0^2u_1 \\
 & - 840c_1H_5\phi_x^2u_0u_{0x} + 84H_4H_5\phi_x^2u_0u_{0x} + 600c_1H_5\phi_x^3u_0u_1 - 60H_4H_5\phi_x^3u_0u_1 \\
 & - 3000c_1^2H_1^3\phi_xu_0^2u_1 + 6500c_1^2H_1^2\phi_xu_0^2u_1 + 120c_1H_1^2H_4u_0^2u_{0x} - 4000c_1^2H_1\phi_xu_0^2u_1 \\
 & - 140c_1H_1H_4u_0^2u_{0x} - 30H_1H_4^2\phi_xu_0^2u_1 - 100c_1H_4\phi_xu_0^2u_1 - 192H_5^2\phi_x^5u_1 \\
 & - 100c_1^2u_0^2u_{0x} - H_4^2u_0^2u_{0x} + 960H_5^2\phi_x^4u_{0x} + 1920H_5^2\phi_x^3\phi_{xx}u_0 \\
 & + 2880H_1H_5^2\phi_x^3\phi_{xx}u_0 - 240c_1H_5\phi_x\phi_{xx}u_0^2 + 24H_4H_5\phi_x\phi_{xx}u_0^2 \\
 & + 1960c_1H_1^2H_5\phi_x\phi_{xx}u_0^2 - 1720c_1H_1H_5\phi_x\phi_{xx}u_0^2 + 196H_1H_4H_5\phi_x\phi_{xx}u_0^2 \\
 & + 2360c_1H_1^2H_5\phi_x^2u_0u_{0x} - 1520c_1H_1H_5\phi_x^2u_0u_{0x} + 236H_1H_4H_5\phi_x^2u_0u_{0x} \\
 & - 3400c_1H_1^2H_5\phi_x^3u_0u_1 + 2800c_1H_1H_5\phi_x^3u_0u_1 - 340H_1H_4H_5\phi_x^3u_0u_1 \\
 & - 600c_1H_1^2H_4\phi_xu_0^2u_1 + 700c_1H_1H_4\phi_xu_0^2u_1 = 0 \quad (36)
 \end{aligned}$$

and another equation at  $O(\phi^{-5})$  which we omit for reasons of length.

Upon solving the  $O(\phi^{-7}), O(\phi^{-6})$  and  $O(\phi^{-5})$  equations for  $u_0, u_1$  and  $u_2$  respectively we find that

$$u_0(x, t) = -\frac{12H_5\phi_x^2}{10c_1H_1 - 10c_1 + H_4} = -12\xi(t)\phi_x^2 \quad (37)$$

$$u_1(x, t) = \frac{12H_5\phi_{xx}}{10c_1H_1 - 10c_1 + H_4} = 12\xi(t)\phi_{xx} \quad (38)$$

$$u_2(x, t) = -\frac{(4\phi_x\phi_{xxx} - 50c_1\phi_x^2 - 3\phi_{xx}^2)H_5}{(10c_1H_1 - 10c_1 + H_4)\phi_x^2} = \frac{(4\phi_x\phi_{xxx} - 50c_1\phi_x^2 - 3\phi_{xx}^2)\xi(t)}{\phi_x^2} \quad (39)$$

which similarly lends itself nicely to a representation of the solution as

$$u(x, t) = 12\xi(t)[\log(\phi(x, t))]_{xx} + u_2(x, t)$$

Further, if we let  $C(x, t) = B(t)$  and once again  $V(x, t) = 1$  the choices for coefficients reduce the remaining orders of  $\phi$  to an identically satisfied system. Solving the determining equations for  $\phi(x, t)$  we have that  $\phi(x, t) = c_2 + c_3 e^{\int V B(t) dt + V x}$ . Therefore we have the solution

$$u(x, t) = - \frac{\xi(t) \left( c_2^2 (V^2 - 50c_1) - 10c_2 c_3 (V^2 + 10c_1) e^{\int V B(t) dt + V x} + c_3^2 (V^2 - 50c_1) e^{2 \int V B(t) dt + 2V x} \right)}{\left( c_2 + c_3 e^{\int V B(t) dt + V x} \right)^2} \quad (40)$$

## 5. Conclusions and Future Work

We have used a direct method to obtain very significantly extended Lax- or S-integrable families of generalized KdV and MKdV equations with coefficients which may in general vary in both space and time. Some solutions for the generalized inhomogeneous KdV equations have also been presented here.

Future work will address the derivation of additional solutions by various methods, as well as detailed investigations of other integrability properties of these novel integrable inhomogeneous NLPDEs such as Backlund Transformations and conservation laws.

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