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Topological phase states of the $SU(3)$ QCD

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Abstract. We consider the topologically nontrivial phase states and the corresponding topological defects in the $SU(3)$ d-dimensional quantum chromodynamics (QCD). The homotopy groups for topological classes of such defects are calculated explicitly. We have shown that the three nontrivial groups are $\pi_3 SU(3) = \mathbb{Z}$, $\pi_5 SU(3) = \mathbb{Z}$, and $\pi_6 SU(3) = \mathbb{Z}_6$ if $3 \leq d \leq 6$. The latter result means that we are dealing exactly with six topologically different phase states. The topological invariants for d=3,5,6 are described in detail.

Introduction. - Topological invariants of field configurations are the fundamental objects in the quantum field theory and condensed matter physics, which classify topological defects and possible phase states [1, 2]. The well-known examples of topological field distributions are vortices, hedgehogs, and instantons. They are a direct consequence of the nontrivial homotopy groups $\pi_n \mathbb{S}^n = \mathbb{Z}$ for the spheres $\mathbb{S}^n$ with $n = 1, 2$, $\pi_3 SU(2) = \mathbb{Z}$, for the spatial dimensions $d = 1, 2, 3$, respectively. Here, $\mathbb{Z}$ is a group of integers. Recent progress in the theory of topologically ordered phase states [3, 4] is associated with the classification of the systems, in which the D-dimensional surface $\mathbb{S}^D$ surrounds a defect in d-dimensional topological insulators or superconductors [5, 6, 7, 8] and $D \neq d$. In this case, the first nontrivial example $\pi_3 \mathbb{S}^2 = \mathbb{Z}$ is the well-known Hopf mapping of the three-dimensional sphere $\mathbb{S}^3$ into the two-dimensional one $\mathbb{S}^2$. The corresponding topological invariant Q is called the helicity in magnetohydrodynamics or the Abelian Chern-Simons action in the $(2+1)$-dimensional topological field theory. The integer Q means the knotting degree of the field distributions and determines, in particular, the lower bound [9, 10] of the energy of the two-component Ginzburg-Landau model expressed [11] in the form of the Skyrme-Faddeev-Niemi model [12, 13]. In this $O(3)$ σ-model, the $U(1)$ two-form $dA = \mathbf{n}[\mathrm{d}\mathbf{n} \wedge \mathrm{d}\mathbf{n}]$ is parametrized the Hopf invariant $Q = \frac{1}{16\pi^2} \int_{\mathbb{S}^3} A \wedge dA \in \mathbb{Z}$ by the unit 3d vector $\mathbf{n}$ which maps the base space $\mathbb{S}^3$ into $\mathbb{S}^2$. The target sphere $\mathbb{S}^2$ of the map is topologically equivalent to the coset $SU(2)/U(1) \cong \mathbb{S}^2$. The $\mathbf{n}$-field is also a relevant on-shell variable [15, 14, 16, 18] in the infrared limit of the $SU(2)$ QCD.

In this paper, we use the $SU(3)$ group instead of the $SU(2)$ one. The change in the value of the rank is due to several reasons. Primarily, to the three colors of the QCD. From the point of view of the knot theory, this choice is also due to an attempt to extend the low-dimensional topology of the standard knot theory to higher dimensions of the $SU(3)$ QCD target space

d	0	1	2	3	4	5	6	7	8	9	10
$\pi_d F_2$	0	0	$\mathbb{Z}\times\mathbb{Z}$	$\mathbb{Z}$	0	$\mathbb{Z}$	$\mathbb{Z}_6$	0	$\mathbb{Z}_{12}$	$\mathbb{Z}_3$	$\mathbb{Z}_{30}$

Table 1. A list of the homotopy groups $\pi_d F_2$ for dimensions $d \leq 10$.

d	0	1	2	3	4	5	6	7
$\pi_d CP^2$	0	0	$\mathbb{Z}$	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$

Table 2. A list of the homotopy groups $\pi_d CP^2$ for dimensions $d \leq 7$.

[17]. One approach to this problem is to use many-valued functionals [20] in accordance with the conjecture given in Ref. [21]. Another elegant method is based on the results obtained in Ref. [22]. However, it is more expedient to describe the target spaces of the $SU(3)$ QCD as a generalization of the $SU(2)$ target sphere $\mathbb{S}^2$.

Flag space. - To generalize the $SU(2)$ group target space to the $SU(3)$ one, we consider the coset $SU(3)/(U(1) \times U(1)) = F_2$ that is now the flag space F_2 [23, 24]. The remained freedom of the maps is the dimension of the base space having a nontrivial homotopy group. For simplicity, we restrict our consideration to the spheres $\mathbb{S}^n$ as the base spaces. Therefore, we focus on such n of the homotopy groups $\pi_n F_2$, which yields nontrivial results. For comparison, in addition to the maximal torus $U(1)^2$ of $SU(3)$ that results in the general orbit F_2 [25], we calculate the homotopy groups of the degenerate orbit CP^2, which are equivalent to the coset space $SU(3)/U(2) = SU(3)/\left(SU(2) \times U(1)\right) = CP^2$.

It should be noted that we are restricted to the framework of the homotopy group approach. Therefore, we would like to determine the constraints on the type of the possible topological phase states and topological defects only. We will describe the geometry of the flag space F_2 and topological features in the last two sections. The results of our calculations are presented in two tables.

It is seen in Table I that the nontrivial homotopy groups for $d \leq 6$ are $\pi_2 F_2$, $\pi_3 F_2$, $\pi_5 F_2$, and $\pi_6 F_2$.

(i) It is known [24] that nontriviality of $\pi_2 F_2 = \mathbb{Z} \times \mathbb{Z}$ accounts for the presence of two different monopoles in the theory (cf. $\pi_2 CP^2 = \mathbb{Z}$ in Table II which means that we deal with a monopole of one type). The second homotopy group is nontrivial due to the fact that the simply connected flag space F_2 is a compact symplectic manifold.

(ii) The integers in the RHS of $\pi_3 F_2 = \mathbb{Z}$ have the meaning of $SU(3)$ instanton topological charges because $\pi_3 F_2 = \pi_3 SU(3) = \pi_3 SU(2)$.

(iii) The integers in the RHS of $\pi_5 F_2 = \mathbb{Z}$ describe some textures and the corresponding phases. The nature of these textures is difficult to guess now.

(iv) The most interesting answer $\pi_6 F_2 = \mathbb{Z}_6$ means that there are only six phase states with the labels $\{0, 1, \ldots 5\}$. They are usually ordered as three quark doublets. We can topologically distinct the quark states because of the gauge invariant coupling of the fermions to the gauge potential. This takes place on the scales where we can consider the six-dimensional base space as the sphere $\mathbb{S}^6$. Note that some additional parameters of the $(3 + 1)$d gauge theory can add dimensions in order to have finally 6 dimensions of the base space [31]. We encounter these phenomena in some topologically ordered phases of condensed matter [4]. In our case, the best natural choice for the interpretation of the base space $\mathbb{S}^6$ corresponds to the standard six-dimensional space-momentum phase space. We are free also to interpret the six-dimensional compact space $\mathbb{S}^6$ as a complement to the $(3 + 1)$-dimensional space-time, but the previous suggestion is much better.

Gauge fields on the flag space. - Let us describe the flag space F_2 in detail to explain in particular at the end, why we addressed to homotopy theory approach. It is a compact Kähler manifold which is a homogeneous nonsymmetric space of dimension $\dim F_2 = 6$. Since the flag manifold F_2 is the Kähler one, it possesses the complex local coordinates w_α, $\alpha = 1, 2, 3$, the Hermitian Riemanian metric, $ds^2 = g_{\alpha\bar{\beta}} dw^\alpha d\bar{w}^\beta$, and the closed two-form (field strength) $\Omega_K = i g_{\alpha\bar{\beta}} dw^\alpha \wedge d\bar{w}^\beta$, i.e., $d\Omega_K = 0$. Here, $d = \partial + \bar{\partial} = dw_\alpha \frac{\partial}{\partial w_\alpha} + d\bar{w}_\beta \frac{\partial}{\partial \bar{w}_\beta}$ denotes the exterior derivative, while the operators ∂ and $\bar{\partial}$ are called the Dolbeault operators.

According to the Poincaré lemma, any closed form Ω_K is *locally* exact, i.e., $\Omega_K = d\omega$, where ω is the gauge potential. The condition $d\Omega_K = 0$ is equivalent to $g_{\alpha\bar{\beta}} = \frac{\partial}{\partial w^\alpha} \frac{\partial}{\partial \bar{w}^\beta} K$, where $K = K(w, \bar{w})$ is the Kähler potential:

$$K(w, \bar{w}) \quad = \quad \ln[(\Delta_1(w, \bar{w}))^m (\Delta_2(w, \bar{w}))^n] \,, \tag{1}$$

$$\Delta_1(w, \bar{w}) \quad = \quad 1 + |w_1|^2 + |w_2|^2 \,, \tag{2}$$

$$\Delta_2(w, \bar{w}) = 1 + |w_3|^2 + |w_2 - w_1 w_3|^2 \,. \tag{3}$$

By means of three complex variables w_α, the flag space F_2 is realized as a set of triangular matrices of the form

$$\begin{pmatrix} 1 & w_1 & w_2 \\ 0 & 1 & w_3 \\ 0 & 0 & 1 \end{pmatrix}^t \in F_2 = SU(3)/U(1)^2 \,. \tag{4}$$

The Kähler one-form and the two-form are $\omega = \frac{i}{2}(\partial - \bar{\partial})K, \Omega_K = i\partial\bar{\partial}K$. The explicit forms of the gauge potential ω and the field strength Ω_K are given by

$$\begin{aligned} \omega \quad = \quad & im \frac{w_1 d\bar{w}_1 + w_2 d\bar{w}_2}{\Delta_1(w, \bar{w})} + in \frac{w_3 d\bar{w}_3}{\Delta_2(w, \bar{w})} \\ & + in \frac{w_2 - w_1 w_3)(d\bar{w}_2 - \bar{w}_1 d\bar{w}_3 - \bar{w}_3 d\bar{w}_1}{\Delta_2(w, \bar{w})} \,, \end{aligned} \tag{5}$$

$$\begin{aligned} \Omega_K \quad = \quad & d\omega = im(\Delta_1)^{-2}[(1 + |w_1|^2) dw_2 \wedge d\bar{w}_2 \\ & - \bar{w}_2 w_1 dw_2 \wedge d\bar{w}_1 \\ & - w_2 \bar{w}_1 dw_1 \wedge d\bar{w}_2 + (1 + |w_2|^2) dw_1 \wedge d\bar{w}_1] \\ & + in(\Delta_2)^{-2}[\Delta_1 dw_3 \wedge d\bar{w}_3 \\ & - (w_1 + \bar{w}_3 w_2) dw_3 \wedge (d\bar{w}_2 - \bar{w}_3 d\bar{w}_1) \\ & - (\bar{w}_1 + w_3 \bar{w}_2)(dw_2 - w_3 dw_1) \wedge d\bar{w}_3 \\ & + (1 + |w_3|^2)(dw_2 - w_3 dw_1) \\ & (d\bar{w}_2 - \bar{w}_3 d\bar{w}_1)] \,. \end{aligned} \tag{6}$$

Calculation of the Poincaré polynomial $P_{F_2}(t) = \sum_{i=0}^{6} b_i t^i$ of F_2 (see [26, 27]) with the Betti numbers b_i yields $P_{F_2}(t) = 1 + 2t^2 + 2t^4 + t^6$, i.e., $b_0 = b_6 = 1, b_2 = b_4 = 2$. We see that the cohomology class is not zero because all even Betti numbers are nonzero.

The Kähler potential for CP^2 is given by

$$K(w, \bar{w}) = \ln[(\Delta_1)^m], \tag{7}$$

which is obtained as a special case of F_2 by specifying the coordinate $w_3 = 0$ and the parameter $n = 0$ in Eq. (1). Hence, we have

$$\omega = im \frac{w_1 d\bar{w}_1 + w_2 d\bar{w}_2}{\Delta_1(w, \bar{w})} \tag{8}$$

up to the total derivative and

$$\begin{aligned} \Omega_K &= d\omega = im(\Delta_1)^{-2}[(1 + |w_1|^2)dw_2 \wedge d\bar{w}_2 \\ &\quad -\bar{w}_2 w_1 dw_2 \wedge d\bar{w}_1 \\ &\quad -w_2 \bar{w}_1 dw_1 \wedge d\bar{w}_2 \\ &\quad +(1 + |w_2|^2)dw_1 \wedge d\bar{w}_1]. \end{aligned} \tag{9}$$

This should be compared to the case $F_1 = CP^1$, where

$$e^{ws_+} = \begin{pmatrix} 1 & w \\ 0 & 1 \end{pmatrix} \in F_1 = CP^1 = SU(2)/U(1) \cong \mathbb{S}^2 \tag{10}$$

and $s_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. Here, the complex variable w is the CP^1 variable written as $w = e^{i\phi} \cot \frac{\theta}{2}$ in terms of the polar coordinates of the unit vector $\mathbf{n}$ in $\mathbb{S}^2$. The results for $SU(2)$ are well-known: $K(w, \bar{w}) = m \ln[(1 + |w|^2)], \omega = im \frac{wd\bar{w}}{1+|w|^2}$, and $\Omega_K = ig_{w\bar{w}} dw \wedge d\bar{w} = im \frac{dw \wedge d\bar{w}}{(1+|w|^2)^2}$.

It is seen from these equations that the degenerate orbit CP^2 is the four-dimensional feature inside the six-dimensional flag space F_2. Therefore, two-form (6) is closed, but not *globally* exact. One can say that Ω_K is an element of the second cohomology group of F_2. This means that we cannot define the gauge connection ω everywhere in F_2, because the one-form ω is not well defined on the manifold. This is the reason why it is difficult to directly determine the Hopf-like topological invariants in the F_2 case.

Topological invariants . - Let us proceed with the analysis of topological invariants by calculating the homotopy groups of $SU(3)$. The flag space $F_2 = SU(3)/U(1)^2$ is the base space of the $U(1)^2$-fiber bundle $SU(3) \to F_2$. We have the following exact sequences:

$$\begin{aligned} &0 \to \pi_d \left(U(1)^2\right) \to \pi_d SU(3) \overset{\cong}{\to} \pi_d F_2 \\ &\to \pi_d \left(U(1)^2\right) \to 0, \qquad \text{for } d \geq 3\,, \end{aligned} \tag{11}$$

$$\begin{aligned} &0 \to \pi_2 SU(3) \to \pi_2 F_2 \to \pi_1 \left(U(1)^2\right) \\ &\to \pi_1 SU(3) \to \pi_1 F_2 \to 0\,, \end{aligned} \tag{12}$$

where $\pi_1 \left(U(1)^2\right) = \mathbb{Z} \times \mathbb{Z}$ and $\pi_1 SU(3) = \pi_2 SU(3) = 0$. Thus, we have

$$\begin{cases} \pi_0 F_2 = \pi_1 F_2 = 0, \\ \pi_2 F_2 = \mathbb{Z} \times \mathbb{Z}, \\ \pi_d F_2 = \pi_d SU(3), \qquad \text{for } d \geq 3\,. \end{cases} \tag{13}$$

We summarized the results in Table I. It presents the nontrivial homotopy groups of F_2, which are in accord with previous studies [28] (see also Ref. [29]). For completeness and comparison, we also have shown a list of the homotopy groups $\pi_d CP^2$ for $d \leq 7$ in Table II.

Table I is proved by the results of Ref. [30] and references therein. In particular, Ref. [30] presents two theorems that account for the 5^{th} and 6^{th} homotopy groups of SU(3). *Theorem 1*: $\pi_{2n-1}U(N) = \mathbb{Z}$ for $N \geq n$ and *Theorem 2*: $\pi_{2n}U(n) = \mathbb{Z}_{n!}$ for $n \geq 2$. We will study now the 3^{rd}, 5^{th}, and 6^{th} homotopy groups in more detail.

1. *The* 3^{rd} *homotopy group of SU(3).* The exact sequence of the fibration $SU(3) \to SU(3)/SU(2) \cong \mathbb{S}^5$ is

$$\pi_{d+1}\mathbb{S}^5 \to \pi_d SU(2) \xrightarrow{i_*} \pi_d SU(3) \to \pi_d \mathbb{S}^5 \,. \tag{14}$$

Let $d = 3$ and, since $\pi_4\mathbb{S}^5 = \pi_3\mathbb{S}^5 = 0$, let the inclusion i: $SU(2) \hookrightarrow SU(3)$ induce an isomorphism i_*: $\pi_3 SU(2) \xrightarrow{\cong} \pi_3 SU(3)$. A generator for $\pi_3 SU(2)$ is given by

$$g_2(\mathbf{r}) = r^0\mathbf{1} + ir^j\sigma_j, \tag{15}$$

$$\text{for } \mathbf{r} = (r^0, r^1, r^2, r^3) \text{ and } \quad |\mathbf{r}| = 1 \,, \tag{16}$$

where $\mathbf{1}$ is the identity matrix and σ_j are the Pauli matrices. Thus, the generator for $\pi_3 SU(3)$ is

$$g_3(\mathbf{r}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^0 + ir^3 & ir^1 + r^2 \\ 0 & ir^1 - r^2 & r^0 - ir^3 \end{pmatrix} . \tag{17}$$

Given any continuous function g: $\mathbb{S}^3 \to SU(3)$, the topological invariant, i.e., the winding degree $[g] \in \pi_3 SU(3) = \mathbb{Z}$, is determined by the integral formula

$$[g] = \frac{1}{24\pi^2}\int_{\mathbb{S}^3} \mathrm{Tr}\left[(gdg^\dagger)^3\right] =$$

$$\frac{1}{24\pi^2}\int_{\mathbb{S}^3} d^3x\varepsilon_{\mu\nu\lambda}\,\mathrm{Tr}(g^\dagger\partial_\nu g g^\dagger\partial_\mu g g^\dagger\partial_\lambda g)\,. \tag{18}$$

2. *The* 5^{th} *homotopy group of SU(3).* Using exact sequence (14), we have

$$\pi_5 SU(2) \to \pi_5 SU(3) \to \pi_5\mathbb{S}^5$$

$$\to \pi_4 SU(2) \to \pi_4 SU(3)\,. \tag{19}$$

It is known that $\pi_5 SU(2) = \pi_5\mathbb{S}^3 = \mathbb{Z}_2$ and $\pi_5 SU(3) = \mathbb{Z}$ (from theorem 1), and thus the first arrow must be the zero map ($\mathbb{Z}_2 \xrightarrow{0} \mathbb{Z}$). We also know that $\pi_4 SU(3) = 0$, as the homotopy group $\pi_4 SU(N) = 0$ stabilizes after $N \geq 3$. Finally, we need $\pi_4 SU(2) = \pi_4\mathbb{S}^3 = \mathbb{Z}_2$. We now have

$$0 \to \pi_5 SU(3) \xrightarrow{\times 2} \pi_5\mathbb{S}^5 \to \mathbb{Z}_2 \to 0\,. \tag{20}$$

This means that given g: $\pi_5\mathbb{S}^5 \to SU(3)$,

$$g(\mathbf{r}) = \begin{pmatrix} | & | & | \\ \mathbf{u}_1(\mathbf{r}) & \mathbf{u}_2(\mathbf{r}) & \mathbf{u}_3(\mathbf{r}) \\ | & | & | \end{pmatrix}, \quad \text{for } \mathbf{r} \in \mathbb{S}^5, \tag{21}$$

the vector $\mathbf{u}_1$: $\mathbb{S}^5 \to \mathbb{S}^5$ has an even winding degree, namely, the winding degree $[\mathbf{u}_1] = 2 \times$ winding degree$[g]$.

The exact sequence for the fibration $SU(N+1) \to SU(N+1)/SU(N) = \mathbb{S}^{2N+1}$ shows that $\pi_5 SU(N) = \mathbb{Z}$ stabilizes after $N \geq 3$. Thus, the winding degree of g can be also deduced by the usual formula

$$[g] = \frac{1}{480\pi^3 i} \int\limits_{\mathbb{S}^5} \mathrm{Tr}\left[(gdg^\dagger)^5\right] . \tag{22}$$

A particular generator of $\pi_5 SU(3)$ can be found in Ref. [30].

3. *The* 6^{th} *homotopy group of SU(3).* Exact sequence (14) yields

$$\pi_7\mathbb{S}^5 \to \pi_6 SU(2) \to \pi_6 SU(3)$$

$$\xrightarrow{0} \pi_6\mathbb{S}^5 \xrightarrow{\cong} \pi_5 SU(2) , \tag{23}$$

where $\pi_7\mathbb{S}^5 = \mathbb{Z}_2$ and $\pi_6 SU(2) = \mathbb{Z}_{12}$. It turns out that $\pi_6 SU(3) = \mathbb{Z}_{3!} = \mathbb{Z}_6$. A generator for $\pi_6 SU(3)$ can be found in [30] in page 6.

Conclusion. - In conclusion, we focus on the nontrivial homotopy groups for $d \leq 6$ $\pi_2 F_2$, $\pi_3 F_2$, $\pi_5 F_2$, and $\pi_6 F_2$ considered so far for the spheres $\mathbb{S}^2$ as the base space. The generalization $\mathbb{S}^2 \to T^n$, where T^n is the n-dimensional torus, is an interesting and more complicated extension even in the $SU(2)$ case. The result of calculations [32] of the mapping class groups in the last case with $T^3 = S^1 \times S^1 \times S^1$ leads to the linear superposition of the topological invariants beginning from the first Chern class to the Hopf invariant (see also [12]). Similar behavior also takes place in the case of $T^3 = \mathbb{S}^2 \times \mathbb{S}^1$. The classification problems of the mappings $T^n \to F_2$ are still totally open.

Up to now, we did not pay any attention to the relation between the existence of strong interaction in the system and homotopy group results. It is well known in the condensed matter community that nontrivial answers $\mathbb{Z}_2$ or $\mathbb{Z}$ for the topological invariant of non-interacting systems change drastically in particular to $\mathbb{Z}_8$ in the case of the interacting system [33]. Considering from this point of view the result $\pi_6(F_2) = \mathbb{Z}_6$, one can say that we deal here with the significant interaction as it takes place in our QCD system.

Thorough understanding of the role of the flag space F_2 in the $SU(3)$ gauge theory is related to the search for an analog of the Hopf number, i.e., the linking number of pullbacks on a space $\mathbb{M}$ of two arbitrary "points" on a target space $\mathbb{N}$ of the map $\mathbb{M} \to \mathbb{N}$. Such an analog can have the form of pre-images of the target points in the codimension two [22]. This could take place if $\mathbb{M} = \mathbb{S}^3$ and $\mathbb{N}$ is the $2d$ complement of CP^2 with respect to the whole space F_2. This is an open question, which is difficult to answer without knowing the details of the map. We leave the problem of describing the details of this map for future work.

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