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Magnetic domains and waves in the Skyrme - Faddeev model

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Abstract. The Skyrme-Faddeev model admits exact analytical non localized solutions, which describe magnetic domain wall solutions when multivalued singularities appear or, differently, always regular periodic nonlinear waves, which may degenerate into linear spin waves or solitonic structures. Here both classes of solutions are derived and discussed and a general discusssion about the existence of integrable subsectors of the model is addressed.

1. Introduction

The present contribution is mainly based on the developments of the results reported in the recent paper [1], where several reductions of the Skyrme-Faddeev model [2] were solved implicitly or in terms of special functions. In particular we consider the 4-dimensional relativistic ($g_{\mu\nu} = \mathrm{diag}\,(+,-,-,-)$) Skyrme–Faddeev model [6] for the unimodular three vector $\phi \in \mathcal{S}^2$ given by the Lagrangian density

$$\mathcal{L} = \frac{1}{32\pi^2} \left(\partial_\mu \phi \cdot \partial^\mu \phi - \frac{\lambda}{4} \left(\partial_\mu \phi \times \partial_\nu \phi \right) \cdot \left(\partial^\mu \phi \times \partial^\nu \phi \right) \right) - \kappa \left(1 - \phi \cdot \phi \right), \tag{1.1}$$

where $\lambda > 0$ is a scaling parameter, determining the breaking of the conformal symmetry, and κ is a Lagrangian multiplier, implementing the constraint on ϕ. The considered reductions lead to extended infinite energy solutions in the 3-dimensional space, which are rarely considered in the current literature on the subject, that being focused on the hopfion solutions [2] - [13], mainly by numerical integration of the equations of motion. Thus, there are at least three different motivations to the present approach: i) find analytical solutions to the system by exploiting its hidden symmetries, ii) give a suitable mathematical description of the of the extended phases observed in experimentsin ferromagnets and multiferroincs [14, 15, 16, 17], iii) find a method in order to extract as as we can from the general model special subsectors completely integrables. Moreover, numerical investigations concerning the anelastic scattering of hopfions show the excitation of wave-like modes disperding the energy, which are far to be studied. From our view point, a strong connection with the d'Alembert-Eikonal system has been obtained. This observation leads to find large classes of solutions of the Cauchy problem. Here special classes are described and interpreted as magnetic domains for the considered model. In Sec. 3 we will consider certain exact analytic periodic spin waves solutions for the Skyrme-Faddeev model, given in terms of elliptic integrals of third kind, specifying their behavior in some particular limits. In general the obtained solutions are quasi-periodic, but periodic dispersive wave-train solutions, one can show that the Lagrangian can be averaged by the Whitham method. This provides a Lagrangian for a set of parameters, describing the modulation of periodic waves in terms of a quasilinear system of partial derivatives of the first order. Finally, we remark that the imposed constraints is similar to the restrictions imposed by [18] in order to select an integrable sector of the general non integrable system.

2. Domain-Walls

First, let os observe that the geometric constraint $\phi \cdot \phi = 1$ can be realized in several way, for instance by resorting to the stereographic projection of the sphere on the complex plane, but loosing some evident symmetries here it seems useful to introduce the polar representation

$$\phi = (\sin w \cos u, \sin w \sin u, \cos w)\,, \tag{2.1}$$

where w and u are suitable functions on the variables $\left(x^0, \ldots, x^3\right)$ to be determined. The Lagrangian (1.1) becomes

$$\mathcal{L}_p = \frac{1}{32\pi^2}\left\{w_\mu w^\mu + \sin^2 w\left[u_\mu u^\mu - \frac{\lambda}{2}\left(w_\mu w^\mu u_\nu u^\nu - w_\mu w_\nu u^\mu u^\nu\right)\right]\right\}, \tag{2.2}$$

and the Euler–Lagrange equations read

$$\partial_\mu w^\mu = \tfrac{1}{2}\sin(2w) u_\nu u^\nu + \tfrac{\lambda}{2}\sin w\; u_\nu\, \partial_\mu[\sin w(w^\mu u^\nu - w^\nu u^\mu)], \tag{2.3}$$

$$w_\mu u^\mu \sin(2w) + \sin^2 w[\partial_\mu u^\mu + \tfrac{\lambda}{2} w_\nu \partial_\mu (u^\mu w^\nu - u^\nu w^\mu)] = 0. \tag{2.4}$$

The first equation can be interpreted as a quadratic constraint among first and second derivatives of the function u, while the second equation is a linear combination of first and second derivatives of the function u. In both cases the coefficients contain trigonometric functions and derivatives of w.

Such an asymmetry suggest to deal in different way u and w For instance, A first observation comes from the assumption $w = \mathit{const}$, which drastically reduces a d'Alembert equation constrained by the homogeneous Eikonal equations, i.e.

$$\partial_\mu u^\mu = 0, \qquad u_\nu u^\nu = 0. \tag{2.5}$$

This overdetermined system was investigated in many papers [19, 20] and for which the general solution was given in the implicit form

$$G\left(u, A_\mu\left(u\right) x^\mu, B_\mu\left(u\right) x^\mu\right) = 0, \qquad A_\mu A^\mu = B_\mu B^\mu = A_\mu B^\mu = 0, \tag{2.6}$$

with G, A_μ and B_μ arbitrary real regular functions. The process to provide explicit form for u may suggest that multi-valued functions appear. In fact, for sake of simplicity, let us impose $B_\mu \equiv 0$ and restrict to two spacial dimensions coordinated by $\left(x^1, x^2\right)$. Then, under the suitable assumptions of regularity of the function G, one can write the solution in the form $u = F\left(x^0 - n_i\left(u\right)_i x^i\right)$, where F and n_i are differentiable functions, such that $\sum_{i=1}^2 n_i^2 = 1$. Thus, setting $n_1\left(u\right) = \cos\left(f\left(u\right)\right)$, for a suitable function f, one can introduce the auxiliary variable $y = \cos\left(f\left(u\right) - z\right)$, being $z = \arccos\left(\frac{x^1}{r}\right)$ and $r = \sqrt{\left(x^1\right)^2 + \left(x^2\right)^2}$. Thus the equation defining u is reduced to

$$\arccos y = f\left(F\left(x^0 - ry\right)\right) - z. \tag{2.7}$$

For suitable bounded continuous on $\mathbf{R}$ functions f and F, the above equation admits continuous solutions in $\left(x^0, x^1, x^2\right)$ variables, possibly multiple coincident or not. If F goes to finite value at infinity, asymptotic values, may depending on the direction, are taken by u. This value is preserved in time evolution, while other localized structures, caustics or singularities, move at constant velocity along the characteristics. Such a behavior suggests the existence of solutions domain wall in the Skyrme-Faddeev model. However, such a reduction has few of interest, for two reasons: i) asymptotics in time lead to the same boundary value and the solution becomes uniform, ii) the energy density is vanishing everywhere. So one has to look to less drastic reduction, for example imposing that

$$w_\mu u^\mu = 0, \qquad u_\nu u^\nu = \alpha \quad . \tag{2.8}$$

Then, the system (2.4) reduces to the separable set of equations

$$\partial_\mu u^\mu = 0, \qquad u_\nu u^\nu = \alpha, \tag{2.9}$$

$$w_\mu u^\mu = 0, \qquad \partial_\mu w^\mu = \frac{\alpha}{2}\frac{\sin(2w)}{1-\frac{\lambda\alpha}{2}\sin^2 w}(1+\frac{\lambda}{2}w^\mu w_\mu), \tag{2.10}$$

being the last one highly nonlinear for the w field, but depending on the general solution of the d'Alembert-Eikonal system (2.9). For that one the general solution is given in the implicit form by [20]

$$u = A_\mu(\tau)\, x^\mu + R_1(\tau), \tag{2.11}$$

$$B_\mu(\tau)\, x^\mu + R_2(\tau) = 0, \tag{2.12}$$

$$A_\mu A^\mu = \alpha, \quad A_\mu B^\mu = A'_\mu B^\mu = B_\mu B^\mu = 0, \tag{2.13}$$

where the function τ is implicitly defined by the (2.12), the real differentiable functions A_μ, B_μ satisfy the constraints (2.13) and, finally, the function R_i are differentiable up to second order at least.

If in (2.9) it results that $\alpha < 0$, setting $\alpha = -\eta^2$ the solution can be put in the form

$$u = x_k A_k(\tau) + A_0(\tau), \quad t = x_k B_k(\tau) + B_0(\tau), \tag{2.14}$$

$$(A_i) = \eta\hat{\mathbf{A}}\left(f(\tau), g(\tau)\right)\right) \left(\hat{\mathbf{A}}^2 = 1\right), \quad (B_i) = \hat{\mathbf{B}} = \pm\frac{\hat{\mathbf{A}}\times\hat{\mathbf{A}}'}{|\hat{\mathbf{A}}\times\hat{\mathbf{A}}'|}\left[f(\tau), g(\tau)\right)\right] \tag{2.15}$$

where $f(\tau)$, $g(\tau)$ (the $[\ldots]$ in (2.15) indicate the dependency also on their first derivatives), $A_0(\tau)$ and $B_0(\tau)$ are arbitrary differentiable functions. Thus, choosing them large classes of solutions can be found for the d'Alembert-Eikonal system (2.9). To give some insight on it, one may consider the simplest situation in which $\hat{\mathbf{A}}$ rotate uniformly, tracing a maximal circle on the unit sphere embedded into an auxiliary 3-space. The circle axis is given by the constant $\hat{\mathbf{B}}$. Then, accordingly with the properties of invertibility of $B_0(\tau)$, at least locally τ is a function of the linear combination $t - \hat{\mathbf{B}}\cdot\mathbf{x}$. Neglecting for sake of clarity the function A_0 in (2.14), the function u oscillates, increasing its amplitude like r. But the oscillations are strongly dependent on the direction and, in particular, they move in the space according to the time dependency of τ seen above. Moreover, similar static solutions exist. If $B_0(\tau)$ is not an injective continuous function, caustics and singularities of the wave front may appear. We know that in general the singularities of the wave fronts are classified by the Coxeter groups [21].

Similar considerations concern a different reduction, simply obtained from (2.4) by setting to zero the coefficients of all functions of w. After some work one prove that the set of the independent constraints reduces to the overdetermined quasilinear system of the first order in (u^μ, w^μ) fields (we will name it the reduced Skyrme–Faddeev system)

$$\partial_\mu w^\mu = 0, \qquad w_\mu w^\mu = -\epsilon^2, \tag{2.16}$$

$$u_\mu w^\mu = 0, \qquad u_\nu\partial_\mu(w^\mu u^\nu - w^\nu u^\mu) = 0, \tag{2.17}$$

where $\epsilon^2 = \frac{2}{\lambda}$ for notational clarity. Now the d'Alembert-Eikonal system (2.16) has the general solution (2.11)-(2.13) after the change $u \to w$ and $\alpha \to -\epsilon^2$. The first equation in (2.17) is the orthogonality condition among the gradients of the two fields, and the second one is a quadratic differential constraint among the derivatives of the function u. Actually it can be rewritten as two divergence terms (namely $\partial_\mu (u_\nu u^\nu w^\mu)$ and $\partial_\mu (u_\nu w^\nu u^\mu)$) balanced by a contribution of the form $(w^\mu u^\nu - w^\nu u^\mu)\, \partial_\mu u_\nu$. However, the second divergence is vanishing, because of the orthogonality condition, while the balancing contribution is the trace of the product of an antisymmetric tensor with a symmetric one, then it vanishes. Finally, recalling that w satisfies the d'Alembert equation, the equation we considered can be concisely written as

$$a_\mu w^\mu = 0 \quad \text{with} \quad a = u^\nu u_\nu, \tag{2.18}$$

in complete analogy with the first equation in (2.17).

Cross differentiation of those equations and systematic substitution of the x^0-derivatives

$$w_0 = \sqrt{w_m^2 - \epsilon^2},\ u_0 = \frac{u_k w_k}{\sqrt{w_m^2 - \epsilon^2}}, \tag{2.19}$$

lead to a set of compatibility conditions. For the d'Alembert-Eikonal equations it is well known [20] that such a set is finite and any further compatibility condition is satisfied only by the Monge-Ampére equation Det $[w_{ij}] = 0$,. On the other hand, the compatibility for the remaining equations leads to the two quadratic constraints

$$(w_s^2 - \epsilon^2) u_m u_k w_{km} + (u_k w_k)^2 w_{mm} = 2 u_s w_s u_m w_k w_{km}, \tag{2.20}$$

$$4 u_k w_k u_s w_{sp} (w_m w_{pm} - w_p w_{mm}) + 2 (u_s w_m w_{sm})^2 + \tag{2.21}$$
$$(u_s w_s)^2 (w_{mm} w_{pp} - w_{pm}^2) = 2 (w_p^2 - \epsilon^2)(u_s w_{sm})^2. \tag{2.22}$$

Such constraints can be simplified and possibly solved, if one can find a first order linear system of the form

$$u_0 = A u_1, \quad u_2 = B u_1, \quad u_3 = C u_1, \tag{2.23}$$

where the functions $A,\ B,\ C$ depend on first and second order derivative of field variable w only. This formulation is similar to look at solutions of the Skyrme-Faddeev system (2.4) by the method of the hydrodynamic reductions [22, 23], involving a finite number of Riemann invariants. This can be certainty done in two space dimensions, where the equations of the form (2.23)

$$u_0 = \frac{w_1 w_{11} + w_2 w_{12}}{w_0 w_{11}} u_1, \quad u_2 = \frac{w_{12}}{w_{11}} u_1, \tag{2.24}$$

allow to identically satisfy the constraints (2.20) - (2.22). Moreover, it is easy to prove also that a solution of the system (2.24) is given by

$$u = F\,[w_1, w_2] \tag{2.25}$$

where F is an arbitrary real differentiable function of its arguments, depending on the initial data. Finally, by direct computation the equation (2.18) is identically satisfied, because of the vanishing of the quantity a. Thus, we have the general solution for the reduced Skyrme-Faddeev system in 2 space dimensions in terms of the four arbitrary functions involved in the solution of the d'Alembert-Eikonal system.

In three space dimensions, a similar analysis is much more complicated. In fact, substitution of (2.23) into (2.20) and (2.22) one is lead to an algebraic system of 4th degree. Thus, in principle the functions A, B, C in (2.23) can be explicitly determined, but their long expressions are intractable. On the other hand, from the orthogonality condition and (2.19), it is easy to prove that a class of solutions for u is given by $u = F[w_1, w_2, w_3]$, being F an arbitrary differentiable real function, constrained by the second equation in (2.17), which is not longer an identity as in 2-dimensions.

A direct substitution of a general solution for w, in the form (2.14)-(2.15), into the orthogonality condition (2.17-i) leads to the linear PDE

$$[X_m(B'_m(\tau)A_p(\tau) - A'_m(\tau)B_p(\tau)) + B'_0(\tau)A_p(\tau) - A'_0(\tau)B_p(\tau)]u_{X_p} = 0, \tag{2.26}$$

having used the coordinate transformation

$$(x_m B'_m(\tau) + B'_0(\tau))d\tau = dt - B_k(\tau)dx_k,\ X_k = x_k. \tag{2.27}$$

By the methods of characteristics it is equivalent to a 3 ODE system, for the unknown $X_k(s)$, being s an auxiliary variable. Depending the coefficient only on τ, one has a constant coefficient linear system nilpotent of order 2, because of the peculiar symmetry of the constraints (2.14)-(2.15). Thus, the solution is linear in s, from which one can extract two integrals of motion, in terms of which express the solution of (2.26). Then a general solution for the complete system of equations is determined by a sole function of a single variable only and the formula $u = F[w_1, w_2, w_3]$, is certainly redundant.

Finally, Let us notice that the eikonal - like condition was used in [18] to reduce the Skyrme-Faddeev model.

3. Waves

Now, we look at the invariant solutions of any 2-dimensional sub-algebra of the translational symmetries of the Skyrme-Faddeev model. Actually, by using the adjoint action of the space-time rotational subgroup, only 3 parameters will be left arbitrary. The invariant reduction is of the form

$$w = \Theta[\theta],\ u = \Phi[\theta] + \tilde{\theta}, \quad \text{where } \theta = \alpha_\mu x^\mu, \tilde{\theta} = \beta_\mu x^\mu \tag{3.1}$$

in which one distinguishes θ as the *phase* from the *pseudo-phase* $\tilde{\theta}$. Thus the equations of motion reduce to the announced 3-parametric family

$$\left[2B_3 - \frac{\lambda}{4}\mathcal{B}\sin^2\Theta\right]\Theta_{\theta\theta} = \sin 2\Theta\left(\frac{\lambda}{8}\mathcal{B}\,\Theta_\theta^2 + B_3\Phi_\theta^2 + B_2\Phi_\theta + B_1\right) \tag{3.2}$$

$$2B_3\sin^2\Theta\ \Phi_{\theta\theta} + \Theta_\theta \sin 2\Theta\,(2B_3\Phi_\theta + B_2) = 0, \tag{3.3}$$

where $B_1 = -\beta_\mu\beta^\mu$, $B_2 = -2\alpha_\mu\beta^\mu$ $B_3 = -\alpha_\mu\alpha^\mu$ and $\mathcal{B} = B_2^2 - 4B_1B_3$. These equations provide trigonometric plane waves, for instance by setting $B_3 = 0$ and $B_2 \neq 0$.

To obtain the general situation, one uses the expression of the energy-stress tensor $T^{\mu\nu} = (w^\nu\partial_{w_\mu} + u^\nu\partial_{u_\mu})\mathcal{L}_p - g^{\mu\nu}\mathcal{L}_p$ and build the conserved quantities $\mathcal{E}^\mu = T^{\mu\nu}\alpha_\nu$, given by

$$\mathcal{E}^0 = \frac{-1}{32\pi^2}\left\{B_3\alpha_0\Theta_\theta^2 + \sin^2\Theta\left[2\vec{\alpha}\cdot\vec{\beta}\,\beta_0 + \left(B_1 - 2\vec{\beta}^2\right)\alpha_0\right.\right. \tag{3.4}$$

$$\left.\left.+B_3\,(2\beta_0 + \alpha_0\Phi_\theta)\,\Phi_\theta - \frac{\lambda\mathcal{B}}{8}\alpha_0\Theta_\theta^2\right]\right\}, \tag{3.5}$$

$$\mathcal{E}^i = \frac{-1}{32\pi^2}\left\{B_3\alpha_i\Theta_\theta^2 + \sin^2\Theta\left[B_2\beta_i - B_1\alpha_i + B_3\,(2\beta_i + \alpha_i\Phi_\theta)\,\Phi_\theta - \frac{\lambda\mathcal{B}}{8}\alpha_i\Theta_\theta^2\right]\right\}. \tag{3.6}$$

These equations can be used to find an expression of Θ_θ^2 and Φ_θ. Precisely, assuming $B_3 \neq 0$ one finds

$$\Theta_\theta^2 = \frac{8B_3\left(B_1\sin^2\Theta + U_3\right) - 2B_2^2\left(\sin^2\Theta + U_2^2\csc^2\Theta\right)}{B_3\left(8B_3 - \lambda\mathcal{B}\sin^2\Theta\right)} \tag{3.7}$$

$$\Phi_\theta = -\frac{B_2\left(U_2\csc^2(\Theta)+1\right)}{2B_3}, \tag{3.8}$$

where the U_i's are two constants completely defining the quantities in (3.5) by the expressions $\mathcal{E}^\mu = U_3\alpha_\mu + \frac{B_2U_2}{2}\left(\frac{B_2}{B_3}\alpha_\mu - 2\beta_\mu\right)$. The equation (3.7) can be set in algebraic form by the transformation

$$\Theta = \arcsin\sqrt{\psi}, \tag{3.9}$$

yielding the equation

$$\psi_\theta^2 = \frac{64(\psi-1)\left(\psi-A_1\right)\left(\psi-A_2\right)}{\lambda^2\mathcal{B}\psi_1\left(\psi_1-\psi\right)}, \quad (0 \leq \psi \leq 1) \tag{3.10}$$

where one has defined the constants $A_{1,2} = \frac{2B_3U_3 \pm \sqrt{4B_3^2U_3^2 - \mathcal{B}U_2^2}}{\mathcal{B}}$ and $\psi_1 = \frac{8B_3}{\lambda\mathcal{B}}$, related 1-1 to the values of the integrals of motion. Assuming for instance $0 < A_1 < A_2 < 1$, by a continuous variation of ψ_1 one obtains that only one oscillating solution exists for ψ, bounded between two of the three zeros of the numerator in (3.10), even if real unbounded solution may appear or complex ones. Analytically, equation (3.10) can be integrated in terms of incomplete elliptic integrals of the third kind. Precisely, by introducing a parametric variable Z one obtains the parametric form

$$\begin{aligned}
\theta(\psi) &= \theta_0 + \frac{1}{4}\sqrt{\frac{\mathcal{B}\lambda^2\psi_1\left(\psi_1-A_1\right)^2}{\left(A_1-1\right)\left(A_2-\psi_1\right)}}\,\Pi\left[\frac{A_1-A_2}{\psi_1-A_2}; Z | \frac{\left(\psi_1-1\right)\left(A_1-A_2\right)}{\left(A_1-1\right)\left(\psi_1-A_2\right)}\right], \\
\psi &= -\frac{A_2\psi_1\sin^2 Z + A_1\left(\psi_1\cos^2 Z - A_2\right)}{A_1\sin^2 Z + A_2\cos^2 Z + \psi_1}
\end{aligned} \tag{3.11}$$

which can be expressed in terms of Weierstrass $\mathcal{P}$ function. Furthermore, from (3.8) the function Φ can be expressed again in terms of incomplete elliptic integrals, namely

$$\begin{aligned}
\Phi &= -\frac{B_2U_2}{2B_3}\left[\int\frac{d\theta}{\psi(\theta)} + \theta\right] + \Phi_0 = \\
&- \frac{s_1}{2\psi_1}\left[\sqrt{\frac{2\psi_1\left(A_1-\psi_1\right)^2\left(B_1\lambda\psi_1+2\right)}{\left(A_1-1\right)\left(A_2-\psi_1\right)}}\,\Pi\left(\frac{A_2-A_1}{A_2-\psi_1}; Z\left|\frac{\left(A_1-A_2\right)\left(\psi_1-1\right)}{\left(A_1-1\right)\left(\psi_1-A_2\right)}\right.\right)\right. \\
&\left. +2s_2\sqrt{\frac{A_2\psi_1\left(A_1-\psi_1\right)^2}{A_1\left(A_1-1\right)\left(A_2-\psi_1\right)}}\,\Pi\left(\frac{\left(A_1-A_2\right)\psi_1}{A_1\left(\psi_1-A_2\right)}; Z\left|\frac{\left(A_1-A_2\right)\left(\psi_1-1\right)}{\left(A_1-1\right)\left(\psi_1-A_2\right)}\right.\right)\right],
\end{aligned} \tag{3.12}$$

where $s_1 = \text{sign } B_2$, $s_2 = \text{sign } U_2$.

This general solution leads to three different linear harmonic branches in the limit of ψ_1 is approaching A_1, A_2 and 1 respectively. For instance, one has

$$\psi_1 \to 1: \quad \psi \to \frac{1}{2}\left(\left(A_1-A_2\right)\cos\left(\frac{8}{\sqrt{\mathcal{B}}\lambda}\theta\right) + A_1 + A_2\right), \tag{3.13}$$

from which the dispersion relation for such waves can be explicitly derived. The wavelength is affected by the presence of the pseudo-phase by the factor in front of θ. At the opposite, one can notice from the general solution above that the length-wave can be made very large when $A_2 \to 1$ and $\psi_1 \to \infty$. In this limit the solution can be expressed in terms of elementary hyperbolic functions. A similar situation occurs when $A_1 = A_2$, then the the elliptic module is 0 and the solutions are given in terms of trigonometric functions. Thus one can conjecture that varying in a suitable way all the parameters, a set of periodic solutions can be found. This suggests to consider slowly deformations of them. The lowest order of modulation approximation is found by substituting the multi-parametric family of periodic solutions (3.10) (see also (3.7) and (3.8)) into the Skyrme-Faddeev Lagrangian (see (1.1) and (2.2)), introducing an averaged Lagrangian, say $L(\gamma, \omega, \vec{\beta}, \vec{k})$, depending on the slowly changing parameters as new dynamic variables $\gamma, \omega, \vec{\beta}, \vec{k}$, which correspond to the derivatives with respect to space-time variables of phase θ and pseudo-phase $\tilde{\theta}$, now not necessarily linear as in (3.1). It means that $\omega = -\theta_{X^0}, k_i = \theta_{X^i}$ and $\gamma = -\tilde{\theta}_{X^0}, \beta_i = \tilde{\theta}_{X^i}$, where X^0, X^1, X^2, X^3 are the so called "slow" variables in comparison with "fast" variables x^0, x^1, x^2, x^3 (see detail in [24]).

Thus, one immediately derives the four-dimensional quasilinear system (below we use $\partial_k \equiv \partial_{X^k}$)

$$\partial_0 L_\omega = \partial_i L_{k^i}, \quad \partial_0 L_\gamma = \partial_i L_{\beta^i}, \tag{3.14}$$

with the compatibility conditions

$$\begin{array}{rcll} \partial_0 k^1 + \partial_i \omega &=& 0, \quad , & \partial_j k^i = \partial_i k^j \quad i \neq j, \\ \partial_0 \beta^i + \partial_i \gamma &=& 0, \quad , & \partial_j \beta^i = \partial_i \beta^j \quad i \neq j. \end{array} \tag{3.15}$$

After the substitution of the family of periodic solutions (3.7) and (3.8) into the Lagrangian density (2.2), one obtains

$$\hat{\mathcal{L}}_p = \sin^2(\Theta)\left(-\frac{1}{2}\lambda\left(\frac{B_2^2}{4} - B_1 B_3\right)\Theta_\theta^2 + B_3\Phi_\theta^2 + B_2\Phi_\theta + B_1\right) + B_3\Theta_\theta^2, \tag{3.16}$$

which is a function only on θ. Thus, performing an integration over a finite space-like region, contributions from θ-independent coordinates are just time-independent finite multiplicative factors. Then, on a period of the wave, one leads to the Lagrangian

$$L \equiv \frac{1}{2\pi}\oint \hat{\mathcal{L}}_p \, d\theta, \tag{3.17}$$

which, generalizing the standard Whitham approach, is supplemented by the introduction of two natural normalizations (or constraints)

$$\oint d\theta = 2\pi, \qquad < \Phi_\theta >= \oint \Phi d\theta = 2\pi m, \tag{3.18}$$

where the integer "m" is the number of rotations of the vector $\vec{\phi}$ around a value determined by a given pseudo-phase $\tilde{\theta}$. This situation is very similar to to the spin wave configurations called cyclon and extra-cyclon in multiferroic materials ([17]). Then the corresponding averaged Lagrangian is

$$L = \left(B_1 - \frac{B_2^2}{4B_3}\right)\left(A_1 + A_2 + W\sqrt{\frac{\lambda}{2}B_3}\right) + \frac{B_2 + 2mB_3}{2B_3}\sqrt{A_1 A_2 (B_2^2 - 4B_1 B_3)}, \tag{3.19}$$

where we introduced the function

$$W = \frac{1}{2\pi}\oint \sqrt{\frac{(\psi - A_1)(\psi - A_2)(\psi - \psi_1)}{1 - \psi}}\frac{d\psi}{\psi}. \tag{3.20}$$

One can immediately check that two equations $L_{A_1} = 0$ and $L_{A_2} = 0$ (see [24]) coincide with normalizations (3.18), while the Euler–Lagrange equations lead to four dimensional quasilinear system of the first order (3.15).

4. Conclusions

We have found several reductions of the Skyrme-Faddeev model, which can be solved or implicitly or via special functions. We have shown the strong connection with the d'Alembert-Eikonal system, which allows to find large classes of solutions. Even in the more complicated system (2.16)-(2.18), we found large classes of solution in terms of two arbitrary functions. Finally, we have shown that the overdetermined system for the field u can be formulated as a first order system of linear pde's, which can be solved by the "method of hydrodynamic reductions ". However, the analysis of the special constraint 2.18) deserves several technical complexities and for this only particular solutions are given. Only, we remark that the constraint is a further restriction on the subsector of the solution space described in [18]. In Sec. 3 we have found exact analytic periodic spin waves for the Skyrme-Faddeev model, determined in terms of elliptic integrals of third kind. Assuming that there exist such periodic dispersive wave-train solutions, we have shown that the Lagrangian can be averaged by the Whitham method. This provides a Lagrangian for a set of parameters, describing the modulation of periodic waves in terms of a quasilinear system of partial derivatives of the first order

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