

# A macroscopic system with undamped periodic compressional oscillations

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**Abstract.** A system consisting of an arbitrary number of particles of equal masses interacting via an arbitrary potential of homogeneity degree  $-2$  and confined by an isotropic harmonic potential has the property of sustaining undamped isochronous compressional oscillations, as has been shown earlier. In this paper, we review and generalize this finding, and also the concept of thermodynamic equilibrium for such systems. It turns out that these compressional oscillations are *adiabatic*, and that correspondingly, the temperature varies when the size of the system does (in the specific case stated above, this dependence is one of inverse proportionality). It is also shown that some of these results extend to the quantal case.

## 1. Introduction

Oscillations arise in a large variety of physical systems. When the systems contain few particles or are in some other way highly idealized, it may happen that the oscillation does not decay. However, whenever the system contains a macroscopic number of particles, and is capable of displaying irreversible behaviour, one expects that any macroscopic oscillation arising in such a system will eventually be damped. In the following, we show analytically how systems of an arbitrary (finite) number of particles obeying classical Hamiltonian dynamics may display an *undamped* finite amplitude compressional oscillation. By this we mean that the oscillation amplitude never decays, even though its amplitude is arbitrary. Thus the size of such a system varies with time. As we shall show in greater detail, so do also various thermodynamic variables characterizing it, such as temperature and pressure.

This leads us to the most peculiar feature of these systems: they never reach thermal equilibrium. They are thus a counterexample to the generally accepted principle that all (sufficiently chaotic) systems must somehow relax to equilibrium, see for example [1]. Note that these systems are *not* integrable. Rather, their dynamics separate in a chaotic part, describing the motion of all degrees of freedom except two. These two are collective in nature and satisfy an equation of motion which does not couple to the other degrees of freedom. Approach to equilibrium has been extensively studied in recent years, see for example [2, 3]. Presumably, our systems satisfy the “relaxation theorem” proved there, if one previously factors out the oscillatory motion, as discussed in Section 3.



Let us tersely describe in informal terms the class of systems we consider. A quite particular case is described as follows: an arbitrary number of equal-mass particles moving according to classical (i.e., nonrelativistic) dynamics, are confined by a common, external isotropic harmonic potential centered at the origin (representing a container with “soft” walls). These particles further interact among themselves via conservative repulsive forces scaling as the inverse cube of distance (but which are otherwise *arbitrary*: such as, for instance, pairwise forces with *arbitrary* coupling constants). The main peculiarity of this system, first pointed out in [4] (for similar results obtained by a related approach, see [5]) is that the dynamical and thermodynamical behaviors of this system display the peculiar features mentioned above: in particular, for generic initial data the radius of gyration of the system oscillates isochronously, that is, with a fixed frequency independent of the initial data. As we shall see, this oscillation is *adiabatic*, and implies a corresponding variation of all other thermodynamic variables. Perhaps the most remarkable aspect of this finding is its validity for systems living in spaces of *arbitrary* dimension  $d$  (including, of course,  $d = 3$ ): indeed, for  $d \geq 2$  the many-body system under consideration is *not* integrable, as is readily seen by numerical simulations, which show clearly a non-vanishing maximal Lyapunov exponent. This system is also generally *not* integrable in the  $d = 1$  case, also in the case with two-body forces unless these are restricted to feature the same coupling constant for every particle pair [6, 7]. One therefore expects that such general systems should evolve towards thermodynamic equilibrium. We shall see that there is, in fact, a sense in which this presumably happens, and numerics presented in [8, 9] certainly seems to confirm this.

It should be pointed out that the results stated above were published by us in [10] without realizing that these same results had been shown a decade earlier in [4, 8] by D. Lynden-Bell and R. M. Lynden-Bell, as was kindly pointed out to us by M. Kiessling. We have since submitted another paper recognizing this fact [11]. Indeed, the underlying property of systems having homogeneous interactions of degree  $-2$  relies on a certain kind of dilation symmetry presumably first pointed out in [5].

In the following Section 2 the many-body problem treated in this paper is described in the context of classical (Hamiltonian and Newtonian) dynamics and the notation used throughout is introduced. In Section 2.2 we describe the main results of this paper: we obtain a new derivation of the results obtained in [4, 8] which lead to a rather extensive generalization of their claims to a broader class of Hamiltonians than those described in the preceding paragraph. In particular, an extension to time-dependent systems is discussed, which permits to analyse methods for acting externally on such systems. In Section 3 the thermodynamics of these systems is described and several of its peculiar features are discussed. In Section 4 we discuss the systems’ behaviour with respect to the second law of thermodynamics, making use of the time-dependent Hamiltonians discussed earlier. In Section 5 we shortly discuss the extension of our results to quantum mechanics.

## 2. The many-body problem: notation and main results

### 2.1. Preliminaries and notation

The fundamental building blocks of the Hamiltonians we shall be considering are the following

$$Q \equiv \sum_{n=1}^N r_n^2 = \sum_{n=1}^N \sum_{j=1}^d x_{nj}^2, \quad (1a)$$

$$D \equiv D(\vec{r}, \vec{p}) = \sum_{n=1}^N \vec{r}_n \cdot \vec{p}_n, \quad (1b)$$

$$H_0(\vec{r}; \vec{p}) = \frac{1}{2} \sum_{n=1}^N p_n^2 + V^{(-2)}(\vec{r}). \quad (1c)$$

*Notation 2.1.* Here and hereafter the (arbitrary) positive integer  $N$  denotes the number of (point) particles; the index  $n$  (and likewise  $m$ , see below) ranges generally from 1 to  $N$ ; the “coordinates”  $d$ -vector  $\vec{r}_n \equiv (x_{n1}, \dots, x_{nd})$  identifies the position in  $d$ -dimensional space (with  $d$  an arbitrary positive integer) of the  $n$ -th particle and depends of course on the “time” variable  $t$ ,  $\vec{r}_n \equiv \vec{r}_n(t)$  (but often the time dependence of the various quantities is not explicitly displayed); the notation  $\underline{\vec{r}}$  denotes the  $N \times d$ -matrix with elements  $x_{nj}$  where  $n = 1, \dots, N$  and  $j = 1, \dots, d$ ; likewise for the “momentum” (or “velocity”)  $N$ -vector  $\vec{p}_n \equiv (p_{n1}, \dots, p_{nd}) \equiv \vec{p}_n(t)$  and the  $N \times d$ -matrix  $\underline{\vec{p}} \equiv \underline{\vec{p}}(t)$ ; of course  $p_n^2 = \vec{p}_n \cdot \vec{p}_n \equiv \sum_{j=1}^d p_{nj}^2$  and likewise  $r_n^2 = \vec{r}_n \cdot \vec{r}_n \equiv \sum_{j=1}^d x_{nj}^2$ ; superimposed dots indicate time-differentiations; and we trust the rest of the notation to be self-evident (note that for simplicity we set all particle masses to *unity*). Throughout this paper we work of course with real functions of real variables.

We always assume the potential  $V^{(-2)}(\underline{\vec{r}})$  to be *nonnegative* for all values of its  $Nd$  arguments,

$$V^{(-2)}(\underline{\vec{r}}) \geq 0, \quad (2)$$

and to scale as the *inverse-square* of its arguments,

$$V^{(-2)}(\alpha \underline{\vec{r}}) = \alpha^{-2} V^{(-2)}(\underline{\vec{r}}). \quad (3)$$

This scaling entails that the forces and the potential energy diverge when the particle positions all tend to zero. Moreover, whenever the potential energy has a singularity, it diverges to *positive* infinity; hence, for any initial conditions, the moving particles are kept away by a finite amount from any singularity. Hence, as long as attention is restricted to the real domain (corresponding to the physical situation), *all* solutions  $\underline{\vec{r}}(t)$  and  $\underline{\vec{p}}(t)$  of the Hamiltonian and Newtonian equations written above are uniquely defined and finite for *all* time.

A subcase of the class of potentials we just described is given as follows

$$V^{(-2)}(\underline{\vec{r}}) = \frac{1}{4} \sum_{n,m=1; m \neq n}^N \frac{g_{nm}^2}{r_{nm}^2}, \quad (4a)$$

entailing

$$\vec{F}_n^{(-3)}(\underline{\vec{r}}) = \sum_{m=1; m \neq n}^N \frac{g_{nm}^2 \vec{r}_{nm}}{r_{nm}^4}. \quad (4b)$$

Of course in these formulas

$$\vec{r}_{nm} \equiv \vec{r}_n - \vec{r}_m, \quad r_{nm}^2 \equiv \vec{r}_{nm} \cdot \vec{r}_{nm} = \sum_{j=1}^d (x_{nj} - x_{mj})^2. \quad (4c)$$

This case is mentioned here because of its physically more interesting character. Note that the “coupling constants”  $g_{nm}^2$  may be different (for different pairs of interacting particles; of course  $g_{nm}^2 \equiv g_{mn}^2$ , see (4a)), but they should all be nonnegative,  $g_{nm}^2 \geq 0$  (as suggested by their notation), to avoid that the corresponding (singular) interaction be attractive. But the restriction (4) to pair interactions is not required for the validity of our main results.

## 2.2. Derivation of the main results

The crucial remark is now the following: the various quantities  $H_0$ ,  $Q$  and  $D$  close as follows under Poisson brackets:

$$\{D, H_0\} = -2H_0, \quad (5a)$$

$$\{D, Q\} = 2Q, \quad (5b)$$

$$\{H_0, Q\} = 2D. \quad (5c)$$

The Casimir operator  $\Xi$  corresponding to the Lie algebra defined by (5), is given by

$$\Xi = QH_0 - \frac{D^2}{2} , \quad (6)$$

which clearly commutes with  $Q$ ,  $D$  and  $H_0$ .

From this follows that any Hamiltonian of the form

$$H(\vec{r}; \vec{p}; t) = \Phi(H_0, Q, D; t) \quad (7)$$

has a dynamics for the collective variables  $H_0$ ,  $Q$  and  $D$  given by

$$\dot{Q} = \{H, Q\} = 2Q \frac{\partial \Phi}{\partial D} + 2D \frac{\partial \Phi}{\partial H_0} , \quad (8a)$$

$$\dot{D} = \{H, D\} = -2Q \frac{\partial \Phi}{\partial Q} + 2H_0 \frac{\partial \Phi}{\partial H_0} , \quad (8b)$$

$$\dot{H}_0 = \{H, H_0\} = -2D \frac{\partial \Phi}{\partial Q} - 2H_0 \frac{\partial \Phi}{\partial D} . \quad (8c)$$

These equations always have  $\Xi$  as an integral of motion, whether the system be time-dependent or not. In the time-independent case, these equations are additionally an *integrable* Hamiltonian system, since they possess the energy as an additional integral of motion apart from  $\Xi$ . In the following, we shall limit ourselves to the time-independent case, returning to the general case in Section 4, where we shall discuss the possibility of using such time-dependent variations of the Hamiltonians to describe compression and expansion of the system, whether adiabatic or sudden.

Whenever the orbit is bound, we therefore have systems for which the variable  $Q$ , which describes the size of the system, varies periodically. The simplest case is the following:

$$H(\vec{r}; \vec{p}) = H_0 + \frac{\Omega^2 Q}{2} = \frac{1}{2} \sum_{n=1}^N (p_n^2 + \Omega^2 r_n^2) + V^{(-2)}(\vec{r}) , \quad (9a)$$

$$\Omega \equiv \Omega(N) = \omega N^{-1/d} , \quad (9b)$$

which was discovered in a slightly different manner in [4] and further discussed in [8, 9]. A somewhat more general case but still yielding “reasonable” equations of motion is

$$H(\vec{r}; \vec{p}) = H_0 + \Psi(Q, D) , \quad (10)$$

for which the equations of motion can be interpreted as the ones corresponding to  $H_0$  modified by a harmonic potential and a dilational dynamics (determined by  $D$ ) the parameters of which are determined by the collective variables  $Q$  and  $D$ . The equations of motion for the collective variables are

$$\dot{Q} = 2D + 2Q \frac{\partial \Psi}{\partial D} , \quad (11a)$$

$$\dot{D} = 2H_0 + 2Q \frac{\partial \Psi}{\partial Q} , \quad (11b)$$

$$\dot{H}_0 = -2D \frac{\partial \Psi}{\partial Q} - 2H_0 \frac{\partial \Psi}{\partial D} \quad (11c)$$

and Newton’s equations read

$$\ddot{\vec{r}}_i = -\frac{\partial V^{(-2)}(\vec{r})}{\partial \vec{r}_i} + 2\vec{r}_i \left[ 2 \left( \frac{\partial \Psi}{\partial D} \right)^2 - \frac{\partial \Psi}{\partial Q} + \frac{\partial^2 \Psi}{\partial D^2} \dot{D} + \frac{\partial^2 \Psi}{\partial D \partial Q} \dot{Q} \right] . \quad (12)$$

These equations of motion are readily interpreted as describing particles interacting both via a general interaction potential of homogeneity degree  $-2$  and oscillating in a self-consistent harmonic potential, the strength of which is determined by the values of the collective coordinates  $H_0$ ,  $D$  and  $Q$ , which themselves evolve according to (11).

If the interaction potential  $V^{(-2)}(\vec{r})$  is additionally assumed to be both rotationally and translationally invariant, as most physically relevant interactions should be, then the above argument can be considerably extended. Indeed, then the components of  $d$ -dimensional angular momentum, the position and momentum of the center of mass,  $D$ ,  $Q$  and  $H_0$  all yield a closed set of commutation relations among themselves. It can then easily be checked that the most general Hamiltonian in this enlarged set of variables needs no more be integrable even in the time-independent case, but bounded motions of the various quantities can still be generated. We may thus have a non-integrable dynamics for the collective variables  $D$  and  $Q$ , these being then coupled to angular motion as well as center-of-mass motion. While these possibilities are no doubt of interest, we leave their exploration to future work and limit ourselves in the following to the case in which the position and momentum of the center of mass as well as the total angular momentum are all zero.

### 3. Thermodynamics

In the following, we aim at making thermodynamic statements concerning systems of the type we have just been describing. Normally, thermodynamics only describes states of equilibrium, that is, states such that the values of macroscopic observables are constant up to fluctuations of relative order  $1/\sqrt{N}$ .

For a system which is strictly speaking out of equilibrium, devising an equivalent of entropy is quite a challenging problem. However, we shall not need to discuss such issues or to develop techniques for systems out of equilibrium. What happens in the systems described above is that there is an additional conserved quantity  $\Xi$  in addition to the total energy of the system given by the Hamiltonian (7), or more specifically (10), as was shown in section 2, see (6). To take into account the existence of  $\Xi$ , it is standard to introduce a *restricted ensemble*, that is, to fix the value of  $\Xi$ . There are, of course, different, but essentially equivalent, possibilities, such as using a generalized canonical ensemble. In the following we shall only consider an ensemble of the microcanonical type, both with respect to the energy and to  $\Xi$ . The reason for this is that, for the systems we consider, the standard theorems concerning equivalence of ensembles derived, say, in [12] do not apply in dimensions above two, as interaction potentials satisfying (3) are then long-range. The microcanonical ensemble, however, is the only one appropriate to describe a perfectly isolated system such as the one we have been describing.

But if we consider such a restricted ensemble, two related problems arise: since the region of phase space in which  $E$  and  $\Xi$  take prescribed values is the part of phase space corresponding to an oscillation of a given amplitude, the system does not explore this region uniformly. Rather, it runs through various parts of this region, corresponding to different values of  $Q$  and  $D$  as the oscillation proceeds. This has the additional consequence that macroscopic variables such as  $D$  and  $Q$  generally vary over a macroscopic range of values in an ensemble characterized by fixing only the value of  $E$  and  $\Xi$ , contrary to the basic property of thermodynamic equilibrium.

Since, however, the oscillation is always *slow* in the thermodynamic limit, that is, the period of the oscillation is always much larger than the collision time, we may ask whether it is not possible to describe what happens in terms of ensembles where the values of  $Q$  and  $D$  are fixed separately. We shall see that this is indeed possible.

Let us therefore consider such an ensemble, restricted in the values of both  $Q$  and  $D$ . The

partition function is then given by

$$W_r(E, Q, D; N) = \int d\vec{r} d\vec{p} \delta(E - \Phi[H_0(\vec{p}_n, \vec{r}_n), Q, D]) \times \delta\left(Q - \sum_{n=1}^N r_n^2\right) \delta\left(D - \sum_{n=1}^N r_n p_n\right). \quad (13)$$

Note that the parameter  $\omega$ , see (9b), is not mentioned explicitly: this corresponds to the fact that it corresponds physically to a pressure, and we are fixing the volume via the value of  $Q$ .

The following remark is elementary: out of the two equations

$$E = \Phi(H_0, D, Q), \quad (14a)$$

$$\Xi = QH_0 - \frac{D^2}{2}, \quad (14b)$$

we may in general eliminate  $H_0$  to obtain a relationship between  $E$  and  $\Xi$  at given values of  $Q$  and  $D$ . Out of this relationship, we may finally express  $D$  as a function of  $\Xi$ ,  $E$  and  $Q$ , say as

$$D = \mathcal{D}(E, \Xi, Q). \quad (15)$$

We can thus express the restricted partition function in terms of  $E$ ,  $\Xi$  and  $Q$ : we define

$$\overline{W}_r(E, \Xi, Q; N) = W[E, Q, \mathcal{D}(E, \Xi, Q); N]. \quad (16)$$

Let us now assume that we have started an ensemble of restricted systems all at the same values  $Q(0)$  and  $D(0)$  of  $Q$  and  $D$  and distributed according to the corresponding generalized microcanonical ensemble having the partition function  $\overline{W}_r(E, \Xi(0), Q(0), N)$ . We now ask what happens when the restriction is simultaneously lifted for all systems of the ensemble. They then start to oscillate and all the systems belonging to the ensemble have at time  $t$  the same values of  $Q(t)$  and  $\Xi(t) = \Xi(0)$  defined by (8). Since the systems belonging to the ensemble all obey a dynamics defined by a Hamiltonian, Liouville's theorem holds. From this follows that, since the systems were originally uniformly distributed with respect to the Liouville measure restricted to the values  $Q(0)$  and  $D(0)$  of  $Q$  and  $D$  respectively, they will remain thus uniformly distributed for all times with respect to the Liouville measure similarly restricted to the values  $Q(t)$  and  $D(t)$ .

Hence, defining restricted thermodynamic equilibrium as the one determined by the restricted generalized microcanonical ensemble for fixed values of  $Q$  and  $D$  as described above, it follows from the considerations just made that, if the system starts out in a restricted equilibrium, it stays in such an equilibrium forever.

Since it follows from (14) that the restricted partition function  $\overline{W}_r$  depends on  $E$ ,  $\Xi$ ,  $Q$  and  $N$ , and since, of those four variables, only  $Q$  varies in the course of an oscillation, it follows that  $\overline{W}_r$  can in fact only depend on  $\Xi$  and  $N$ :

$$W_r(E, Q, D; N) = \overline{W}_r(E, \Xi, Q; N) = \Psi_r(E, \Xi, N), \quad (17)$$

where  $\Psi_r(K, N)$  is defined by the last equality.

It thus follows that the oscillation can be described as an *adiabatic* oscillation, that is, it has constant entropy. Here we only need a quite mild generalization of the concept of entropy, and by no means require to consider true non-equilibrium properties. We are dealing, in fact, not with a true non-equilibrium system, but with a system in ordinary thermodynamic equilibrium, coupled with a hydrodynamic oscillation. The nature of the coupling is, however, peculiar: no

energy exchange takes place between the oscillation and the microscopic motion. Indeed, no irreversible processes that arise in the microscopic motion can damp the collective oscillation. That such processes may well occur is clear enough: if initially, for example, a temperature or pressure gradient were present, the fact that the restricted system tends to equilibrium tells us that these gradients will dissipate. Such relaxation processes will presumably satisfy the relaxation theorem described in [2, 3] and have in fact been observed in numerical work, see [8]. However, as we have seen, the dynamics of the macroscopic oscillation is unaffected by anything that happens at the microscopic level, so the dissipation of initial temperature gradients will take place without causing any damping of the macroscopic oscillation.

It thus follows that the entropy of the system only depends on  $\Xi$ :

$$s(E, Q, D) = \frac{1}{N} \ln \bar{W}_r(E, \Xi, Q; N) = \frac{1}{N} \ln \Psi_r(E, \Xi; N) = \sigma(E, \Xi) \quad (18)$$

where the final equality *defines* the function  $\sigma(E, \Xi)$ . The fact that the  $N$  dependence vanishes in the thermodynamic limit is standard, see for example [12]. The temperature is thus given by

$$\frac{1}{T} = N \left( \frac{\partial s(E, Q, D)}{\partial E} \right)_{Q, D} = N \left[ \left( \frac{\partial \sigma(E, \Xi)}{\partial E} \right)_{\Xi} + \left( \frac{\partial \sigma(E, \Xi)}{\partial \Xi} \right)_E \left( \frac{\partial \Xi}{\partial E} \right)_{Q, D} \right] \quad (19)$$

which means that the temperature will generally depend on  $Q$  and  $D$ , that is, it will vary during the oscillation, leading to the main conclusion of this paper, that such systems do not approach a physical state of equilibrium.

This leads to the remarkable fact that the systems described here do not tend to thermodynamic equilibrium, since the temperature oscillates indefinitely. Ordinarily, relaxation to equilibrium is shown to follow from appropriate ergodicity and mixing properties [1], which can then presumably be assumed in the case of sufficiently complex systems. What we see here is an instance of a system that behaves in a somewhat peculiar and unexpected manner. It has indeed a conservation law distinct from energy, which contradicts the traditional statement of ergodicity. However, in such cases, it generally suffices to limit consideration to a surface in which the additional conservation law has a fixed value to recover the usual ergodic results and relaxation to equilibrium. This is, for example, what happens if angular momentum is conserved. It is also a useful approach, say, when adiabatically conserved quantities exist [13]. Here, however, the situation is different, because two macroscopic variables, namely  $Q$  and  $D$ , follow a separate, non-trivial, conservative dynamics uninfluenced by the other degrees of freedom; on the other hand, these two variables do affect the microscopic motion, at least to the extent of affecting the system temperature.

Finally, let us note that, since no damping of the radial oscillation described above takes place, it follows that compression and dilation of the system cannot be accompanied by any dissipative processes. Specifically, this implies that the bulk viscosity of the class of systems we consider must vanish. While this is an immediate consequence of our work, this does not seem to have been noticed before. It is, of course, a straightforward consequence of the Green-Kubo formula for the bulk viscosity [14] in the case of systems such as the ones described by (9).

#### 4. Generalization to a time-dependent Hamiltonian

In this section we investigate whether a system which fails to approach thermodynamic equilibrium in the manner described above might conceivably provide a counterexample to the second law of thermodynamics. In the sense that this law states the impossibility of perpetual motion machines of the second kind, we found no way to violate the second law using the above class of systems; nor do we believe this to be possible. What we show below is that the simplest attempts to obtain energy out of this class of systems remain consistent with the second law.

The class of systems described here presumably provides an interesting test of any proofs of the second law, such as the one provided in [15].

The Hamiltonian (7) in its most general form provides various possibilities of acting on the system in a mechanical manner by an external agency. Indeed, allowing the external harmonic potential in (9) to depend on time, one can compress or expand the system, either suddenly, adiabatically or following any other protocol. Similarly, one can use such a potential to extract work out of an existing oscillation, while decreasing its amplitude. The Hamiltonian (7) in its most general form presumably allows many possibilities of acting on the system in a mechanical manner. Of course, one cannot describe in this way processes involving heat flow, such as isothermal expansion or compression, or putting the system in contact with a heat bath at a given temperature.

We now describe a mechanical cyclic process as follows: we start from a system without an oscillation and at thermodynamic equilibrium in the usual sense. The system is subjected to a time evolution  $H(t)$ , where the Hamiltonian is only time-dependent for  $0 \leq t \leq T$ , such that  $H(0) = H(T)$  and  $H(t)$  is at all times of the form (7). Such a cyclic process can, as is readily seen, describe fairly arbitrary mechanical operations on the system, but no operations involving heat transfer. Let us further assume that the system returns to its initial state, that is, the initial values of  $Q$  and  $D$  are recovered at the end of the cycle. Since  $\Xi$  is conserved also for the case of a time-dependent Hamiltonian, the fact that  $Q$  and  $D$  return to their original values means that  $H_0$  does so as well. Since the final Hamiltonian is the same as the initial one, the total energy has remained constant. Since by definition no transfer of heat took place, the total work done on the system in such a cyclic evolution vanishes. This means, of course, that no violations of the second law can be obtained in this manner, consistently with the result proved in [15].

Thus, for example, if starting from an initial condition in which the system is in equilibrium with a frequency  $\omega_1$  one suddenly decreases the frequency to  $\omega_2 < \omega_1$ , an oscillation will be set in motion. It is then possible to get work by damping the oscillation, and finally one may get back to the initial configuration by an adiabatic compression. From the above result we see that the total work performed on the system is zero. This means that, in such a free expansion, contrary to what is well-known in the case of the ideal gas, the entropy does not grow irreversibly.

## 5. The quantum case

In this section we show that our main finding remains true in the *quantal* context; i. e., also in that context the time evolution of the effective volume occupied by the  $N$ -body system whose dynamics is determined by the Hamiltonian (9) oscillates isochronously without any damping.

A straightforward way to show this is by working in the Heisenberg representation. The Heisenberg equations of motion for the observables corresponding to the classical quantities  $Q$  and  $D$  then reduce to (8), which must, however, be interpreted as operator equations. In the specific case where the Hamiltonian is of the form (9), these equations reduce to

$$\dot{Q} = 2D, \quad (20a)$$

$$\dot{D} = -2\Omega^2 Q + 2H. \quad (20b)$$

Since  $H$  is a constant, the equations for  $D$  and  $Q - H/\Omega^2$  are *linear*, so that their classical solution can be immediately reinterpreted as an operator solution in quantum mechanics. A more detailed proof in the Schrödinger representation can be found in [10].

The generalization to Hamiltonians of more general form, such as, for example, (7) or (10) is not so straightforward, since the Heisenberg equations of motion are not linear any more, and therefore the classical solutions for the system of ODE's have no relation to the corresponding solution of the operator equations.



## 6. Conclusions

To summarize, we have studied a rather general class of systems including those in which particles of equal masses interact with each other according to a potential of degree of homogeneity  $-2$  and these particles are additionally confined by a harmonic external potential: these have the property of showing *undamped* compressional oscillations whenever the initial conditions deviate from thermodynamic equilibrium. These oscillations are adiabatic, and lead therefore to variations in temperature and pressure. If the system starts out in (restricted) thermodynamic equilibrium at a given stage of the oscillation, it remains in equilibrium, meaning that no pressure or temperature gradients can be generated. Further, the oscillation remains uncoupled from the microscopic motion, in the sense that dissipative processes arising in the system cannot affect the amplitude of the oscillation. In other words, energy cannot pass from the microscopic degrees of freedom to the collective variables characterizing the oscillation or viceversa. On the other hand, as we have seen, the oscillation modifies the temperature of the system, so that the system really does not approach thermal equilibrium.

The reason for the phenomenon of undamped oscillations is that a set of three collective coordinates, of which the radius of gyration of the system is one, form a closed algebra under Poisson brackets, and that the full Hamiltonian is a function of these.

Finally, it was shown that the main results remain valid in the quantal context. The extension of the thermodynamic considerations to the quantum case has not been done. We do not expect that it should present any significant difficulties.

We also generalized the approach to include a fairly general time-dependent Hamiltonian, so that we could model mechanical manipulations on the system. This allowed to show, for example, that it is possible to reverse a free expansion made by suddenly decreasing the strength of the confining harmonic potential. As expected, it was not possible to display any counterexamples to the second law of thermodynamics.

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