Stability under persistent perturbation by white noise

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Abstract. Deterministic dynamical system which has an asymptotical stable equilibrium is considered under persistent perturbation by white noise. It is well known that if the perturbation does not vanish in the equilibrium position then there is not Lyapunov's stability. The trajectories of the perturbed system diverge from the equilibrium to arbitrarily large distances with probability 1 in finite time. New concept of stability on a large time interval is discussed. The length of interval agrees the reciprocal quantity of the perturbation parameter. The measure of stability is the expectation of the square distance from the trajectory till the equilibrium position. The method of parabolic equation is applied to both estimate the expectation and prove such stability. The main breakthrough is the barrier function derived for the parabolic equation. The barrier is constructed by using the Lyapunov function of the unperturbed system.

1. Statement of the problem

Let the deterministic system be

$$\frac{d\mathbf{y}}{dT} = \mathbf{a}(\mathbf{y}, T), \qquad \mathbf{y} \in \mathbb{R}^n,\ T > 0. \tag{1}$$

The point $\mathbf{y} = 0$ is equilibrium, which means $\mathbf{a}(0, T) \equiv 0$. Let the perturbed system be the stochastic Ito equation [1, 2, 3]

$$d\mathbf{y} = \mathbf{a}(\mathbf{y}, T)dT + \mu\, B(\mathbf{y}, T)\, d\mathbf{w}(T), \qquad T > 0; \quad \mathbf{y}|_{T=0} = \mathbf{x}; \quad 0 < \mu^2 \ll 1. \tag{2}$$

Here $\mathbf{w}(T)$ accounts for the standard Brownian motion in $\mathbb{R}^n$ and $B(\mathbf{y}, T)$ is a matrix $n \times n$ with the property $B(0, T) \not\equiv 0$. The solution $\mathbf{y} = \mathbf{y}_\mu(T; \mathbf{x})$ is a stochastic process in $\mathbb{R}^n$ depending on both the initial point $\mathbf{x}$ and a small parameter μ. The solution depends on the matrix B as well.

Let the equilibrium position $\mathbf{y} = 0$ of the system (1) be asymptotically stable in Lyapunov sense. Consider the problem of stability under white noise as follows: *does the trajectory* $\mathbf{y} = \mathbf{y}_\mu(T; \mathbf{x})$ *remain near equilibrium, if the perturbations* $\mu, |\mathbf{x}|$ *are small and the matrix belongs a fixed ball* $||B|| < M$*?*

It is well known, if $B(0, T) \not\equiv 0$, that there is not any Lyapunov type stability with respect to white noise. Almost all trajectories of the perturbed system diverge from the equilibrium to arbitrarily large distances in finite time. The concept of stability must be modified in appropriate way. The similar problem for autonomous systems was considered in [1], see also [4, 5, 6, 7]. We solve the problem for nonautonomous systems by method of parabolic equation. This approach and the obtained results are different from [1].

2. Definition of stability

The measure of stability is the expectation of the square perturbed solution, i.e.

$$\mathbb{E}[\big(\mathbf{y}_\mu(T;\mathbf{x})\big)^2] = u_\mu(\mathbf{x},T).$$

It is a function on $\mathbf{x}, T$. In the definition we follow Khasminskii [8], Katc and Krasovskii [9, 10].

Definition 1. *Equilibrium $\mathbf{y} = 0$ of the deterministic system (1) is weak stable with respect to white noise under the given estimates $\delta = \delta(\varepsilon)$, $\Delta = \Delta(\varepsilon)$ on an asymptotically large interval uniformly for B from the set $\mathcal{B}$, if there exists $T_0 > 0$, so that for any $\varepsilon > 0$ the expectation along perturbed trajectory is small: $\mathbb{E}[\big(\mathbf{y}_\mu(T;\mathbf{x})\big)^2] < \varepsilon$ uniformly for all: $|\mathbf{x}| < \delta(\varepsilon),\ |\mu| < \Delta(\varepsilon),\ 0 < T < T_0\mu^{-2},\ B \in \mathcal{B}$.*

This definition does not follow the usual mathematical approach [8] by reason of the finite time interval. But it is very useful for physics due to the given estimates and the large time interval under small μ, [9], p.3. We use the expectation instead of probability in the definition. It is not an essential difference due to Chebyshev's inequality [8], p.14, [2], p.65.

3. Reduction to the parabolic equation

Let $f(\mathbf{x})$ be a random function in $\mathbb{R}^n$. The expectation $\mathbb{E}[f(\mathbf{x})]$ is thus a function on $\mathbf{x}$. Consider the parabolic (Kolmogorov) equation

$$\partial_s u + \mathbf{a}(\mathbf{x},s)\partial_{\mathbf{x}} u + \mu^2 \sum_{i,j=1}^{n} \alpha_{i,j}(\mathbf{x},s)\partial_{x_i}\partial_{x_j} u = 0,\ \mathbf{x} \in \mathbb{R}^n,\ 0 \le s < T$$

with the specific final condition

$$u|_{s=T} = \mathbb{E}[f(\mathbf{x})].$$

Here the matrix $A = \{\alpha_{i,j}\}$ is determined by $2A = BB^*$. The equations may be reduced to the initial problem by change of time: $s = T - t$. After that the problem takes the form with specific dependence on T, t:

$$L_{T,\mu} u \equiv \partial_t u - \mathbf{a}(\mathbf{x}, T-t)\partial_{\mathbf{x}} u - \mu^2 \sum_{i,j=1}^{n} \alpha_{i,j}(\mathbf{x}, T-t)\partial_{x_i}\partial_{x_j} u = 0, \quad \mathbf{x} \in \mathbb{R}^n,\ 0 < t \le T;$$
$$u(\mathbf{x}, t; T, \mu)|_{t=0} = \mathbb{E}[f(\mathbf{x})], \quad \mathbf{x} \in \mathbb{R}^n. \tag{3}$$

Connection with the solution $\mathbf{y}_\mu(T;\mathbf{x})$ of the stochastic equation is given by the formula [2, 3]

$$u(\mathbf{x}, t; T, \mu)|_{t=T} = \mathbb{E}[f(\mathbf{y}_\mu(T;\mathbf{x})].$$

In the special case, when $f(\mathbf{y}) = |\mathbf{y}|^2$, the solution of the parabolic problem is taken on the upper boundary: $u(\mathbf{x}, T; T, \mu) = \mathbb{E}[\big(\mathbf{y}_\mu(T;\mathbf{x})\big)^2]$ gives the expectation of the quadrate distance from the random trajectory $\mathbf{y} = \mathbf{y}_\mu(T;\mathbf{x})$ till the point $\mathbf{y} = 0$. In such interpretation the function $u(\mathbf{x}, T; T, \mu)$ may be used as a measure of drift of the perturbed trajectory $\mathbf{y}_\mu(T;\mathbf{x})$ from zero. In the case, when the point $\mathbf{y} = 0$ is the equilibrium position of the unperturbed system (1), the magnitude of the function $u(\mathbf{x}, T; T, \mu)$ as $\mathbf{x}, \mu \to 0$ features stability of equilibrium under white noise perturbation.

4. Barrier function

The prove-out of stability is so reduced to estimate the solution $u(\mathbf{x}, t; T, \mu)$ of the parabolic equation near the point $\mathbf{x} = 0$. *Desirable estimate must be uniform with respect to the parameter T on a long interval:* $T \in (0, \mathcal{O}(\mu^{-2}))$, $\mu \to 0$. We construct a barrier function in order to obtain an appropriate estimate.

Definition 2. *Barrier for the equation* (3) *is the function* $W(\mathbf{x}, t; T, \mu)$*, which possesses continues derivatives involved in differential operator* $L_{T,\mu}$ *from* (3)*, and the value of the operator is nonnegative* $L_{T,\mu} W(\mathbf{x}, t; T, \mu) \geq 0$ *in the layer* $D^T = \{\mathbf{x} \in \mathbb{R}^n,\ 0 < t \leq T\}$*.*

Application of barrier function is based on the maximum principle which says: any solution of the equation (3), which is majorized by barrier at the initial moment $u(\mathbf{x}, t)|_{t=0} \leq W(\mathbf{x}, 0; T, \mu))$, will be majorized at subsequent moments $u(\mathbf{x}, t) \leq W(\mathbf{x}, t; T, \mu))$, $0 < t \leq T$ (see [11], p.14). To construct an appropriate barrier we use a Lyapunov function of the unperturbed system (1).

5. Lyapunov function

Consider the unperturbed system (1). Remind the concept of Lyapunov function [9].

Definition 3. *A function* $U(\mathbf{x}, T)$ *is called a Lyapunov function for* (1) *at* $\mathbf{x} = 0$ *if exists a cylinder* $D_r = \{(\mathbf{x}, T) :\ |\mathbf{x}| < r,\ T \geq 0\}$ *such that 1)* $U(\mathbf{x}, T)$ *is defined, continuous, and differentiable in* D_r*; 2)* $U(\mathbf{x}) > 0$ *if* $\mathbf{x} \neq 0$*,* $U(0) = 0$*, and 3)* $\big[\partial_T + \mathbf{a}(\mathbf{x}, T)\partial_{\mathbf{x}}\big] U(\mathbf{x}, T) < 0$ *in* D_r*.*

If the system (1) possesses a Lyapunov function of such type, then x = 0 is asymptotically stable. Moreover the equilibrium is stable with respect to a persistent smooth deterministic perturbation [9]. A specific of the problem under consideration is that the white noise is not deterministic and the perturbation is not smooth. The perturbed system (2) is not a differential equation in ordinary sense and there is not Lyapunov's stability if $B(0, T) \not\equiv 0$. The trajectories of the stochastic system (2) diverge from the origin to arbitrarily large distances with probability 1 in finite time [1, 2]. But if the diffusion coefficient μ is small then many of trajectories stay near equilibrium for long times. We refer to this property as stability on large interval. The proof of such stability is derived from estimate of the solution of the parabolic equation by means of an appropriate barrier.

Lyapunov function is applied to construct such barrier. The reason is as follows. Let $U(\mathbf{x}, T)$ be a Lyapunov function for (1). Consider the function $V(\mathbf{x}, t; T) = U(\mathbf{x}, T - t)$, $0 \leq t \leq T$. Then for two terms of the parabolic operator the inequality

$$\big[\partial_t - \mathbf{a}(\mathbf{x}, T - t)\partial_{\mathbf{x}}\big] V(\mathbf{x}, t, T) > 0$$

takes place and this suggests to use $V(\mathbf{x}, t; T)$ in the barrier of the equation (3).

There are two obstacle in this way. The first is the diffusion term with a small factor μ^2 in the parabolic operator which gives a negative addend in $L_{T,\mu} V$. The second is the local property of the Lyapunov function while the barrier function must be global. In order to overcome these difficulties we specify the system under consideration.

6. Conditions

Principal restrictions, which are imposed on the differential equations (1), are described by means of conditions on the Lyapunov function $U(\mathbf{x}, T)$. The first part of the conditions has a local nature (in the cylinder D_r) and they are conventional in case of asymptotical stable equilibrium:

$$\partial_t U - \mathbf{a}(\mathbf{x}, T)\partial_{\mathbf{x}} U \geq \gamma U, \tag{4}$$

$$|\mathbf{x}|^2 \leq U(\mathbf{x}, T) \leq M_0 |\mathbf{x}|^2, \quad |\partial_{\mathbf{x}} U|^2 \leq M_1 U, \tag{5}$$

$$\sum_{i,j=1}^{n} a_{i,j}(\mathbf{x},T)\partial_{x_i}\partial_{x_j}U \leq M_2; \quad (\mathbf{x},T)\in D_r;\ \gamma, M_k = \text{const} > 0. \tag{6}$$

The local inequality (4) is a fundamental property. In general case there is not any Lyapunov function with global property (4). In what follows we use a function which is given in the whole half-space $D = \{\mathbf{x} \in \mathbb{R}^n,\ T \geq 0\}$ while the condition (4) holds in the subdomain D_r. One can guess that the local Lyapunov function with properties (4)-(6) may be continued in the wide domain D without (4). The second part of the conditions describes the properties of such continuation. They are easily verified if the Lyapunov function is a quadratic form:

$$|\partial_{\mathbf{x}}U|^2 \leq M_1 U, \quad L_{T,\mu}U \geq -M_3 U, \quad \mathbf{x}, t \in D \setminus D_r,\ M_3 = \text{const} > 0. \tag{7}$$

7. Construction of the barrier function

The barrier function is described by different formulas in different subregions of the layer $D^T = \{\mathbf{x} \in \mathbb{R}^n,\ 0 < t \leq T\}$. It is convenient to take the inner subregion delimited by a level surface of the Lyapunov function

$$\mathcal{D}_r^T = \{(\mathbf{x}, t) \in D^T :\ U(\mathbf{x}, T-t) < r^2\},\ r = \text{const} > \text{o}.$$

The outer subregion is the complement $D^T \setminus \mathcal{D}_r^T$. Under condition (5) the boundary $\partial\mathcal{D}_r^T$ is localized in the cylinder layer $\{r/\sqrt{M_0} \leq |\mathbf{x}| \leq r,\ T > 0\}$.

The first part of the barrier $V_0(\mathbf{x}, t; T, \mu)$ is determined by formula:

$$V_0(\mathbf{x}, t; T, \mu) = U(\mathbf{x}, T-t)\exp(-\alpha_0 t) + \frac{\mu^2 M_2}{\alpha_0}[1 - \exp(-\alpha_0 t)],\ \alpha_0 = \text{const} > 0.$$

If we take $\alpha_0 \leq \gamma$, then under conditions (4),(5) the barrier property $L_{T,\mu}V_0(\mathbf{x}, t; T, \mu) \geq 0$ holds in the subregion $\mathcal{D}_r^T$.

If $r = \infty$ occurs (which means that $U(\mathbf{x}, T)$ is a global Lyapunov function), then the function $V_0(\mathbf{x}, t; T, \mu)$ is a desirable barrier in D^T. This case corresponds to [8], p.249.

In general case there is not any global Lyapunov function. It means $r < \infty$ and the barrier property for the function $V_0(\mathbf{x}, t; T, \mu)$ can violate for large $|\mathbf{x}| > r$. Then we add the exponentially growing function

$$V_2(\mathbf{x}, t; T, \mu) = P(U)\exp(\alpha t), \quad U = U(\mathbf{x}, T-t),\ (\mathbf{x}, t) \in D^T \setminus \mathcal{D}_\rho^T,\ 0 < \rho < r.$$

The factor $P(U)$ is determined by the formulas:

$$P(U) = v + \mu^2\big(m - \frac{\lambda\, v}{\mu^2 + \nu\, v}\big), \quad v = [U - \rho^2\chi(U)]\frac{\gamma}{M_1}; \quad \alpha, m, \lambda, \nu = \text{const} > 0.$$

Here the function $\chi(U) \in C^\infty(\mathbb{R})$ is a smooth steady decreasing patch function with property:

$$\chi(U) \equiv 1,\ U \leq \rho^2;\ \chi(U) \equiv 0,\ U \geq r^2.$$

In order to avoid exponentially grows in the neighborhood of the point $\mathbf{x} = 0$, the second part of the barrier is continued in $\mathcal{D}_\rho^T$ by the function

$$V_1(\mathbf{x}, t; T, \mu) = m\,\mu^2\ \exp(\alpha t + v/\mu^2), \quad (\mathbf{x}, t) \in \overline{\mathcal{D}_\rho^T}.$$

Since the function $v = [U(\mathbf{x}, t; T) - \rho^2]\gamma/M_1 < 0$ is negative in $\mathcal{D}_\rho^T$, hence the $V_1(\mathbf{x}, t; T, \mu)$ is exponentially small as $\mu \to 0$. In the more narrow domain, where $U(\mathbf{x}, t; T) < \rho^2/2$, the estimate $V_1(\mathbf{x}, t; T, \mu) = \mathcal{O}(\mu^2)$ is uniform with $(\mathbf{x}, t) \in \mathcal{D}_\rho^T$ under $\forall\, T \leq \mathcal{O}(\mu^{-2})$. The last property is the main purpose of the construction.

Theorem 1. *Let the Lyapunov function $U(\mathbf{x}, T)$ for the equation* (1) *enjoy the properties* (4)-(7). *Then for all sufficiently small $|\mu| < \mu_0$ there is a barrier function $W(\mathbf{x}, t; T, \mu)$ for the equation* (3), *which is expressed by the formula*

$$W = V_0(\mathbf{x}, t; T, \mu) + \begin{cases} V_1(\mathbf{x}, t; T, \mu), \ (\mathbf{x}, t) \in \overline{\mathcal{D}^T_\rho}, \\ V_2(\mathbf{x}, t; T, \mu), \ (\mathbf{x}, t) \in D^T \setminus \overline{\mathcal{D}^T_\rho} \end{cases} \tag{8}$$

under appropriate constants $\alpha_0, \alpha, m, \lambda, \nu, \mu_0$ which are not depending on the parameters μ, T.

Proof. The proof of the theorem consists in the fitting of the barrier parameters $\alpha_0, \alpha, m, \lambda, \nu, \mu_0$. There are two requirements which must be satisfied. The first corresponds to smoothness of the function $W(\mathbf{x}, t; T, \mu)$ on the matching surface $\partial\mathcal{D}^T_\rho$ where $v = 0$ and the second is the barrier property $L_{T,\mu}W \geq 0$ in D^T.

The relations $\lambda = 1 - m, \quad \nu = m/2(1 - m), \quad 0 < m < 1$ provide a sufficient smooth $W \in C_2(D^T)$. The value m is determined in what follows.

If we take into account the conditions (4),(5), then the estimates

$$L_{T,\mu}V_0 \geq 0, \ L_{T,\mu}V_1 \geq 0$$

in the inner domain $\mathcal{D}^T_\rho$ is derived[1] under appropriate choice of parameters: $\alpha_0 \leq \gamma, \ \alpha \geq \gamma M_2/M_1$. Hence the barrier property is proved in the inner domain $\mathcal{D}^T_\rho$.

In the outer domain $D^T \setminus \mathcal{D}^T_\rho$ the second part of the barrier is taken. It gives

$$L_{T,\mu}V_2 = \Big[\alpha P(U) + \big(\partial_t U - \mathbf{a}\,\partial_{\mathbf{x}} U\big)P'(U) -$$

$$-\mu^2 \sum_{i,j=1}^{n} a_{i,j}(\mathbf{x}, t; T)\big[P'(U)\partial_{x_i}\partial_{x_j}U + P''(U)\partial_{x_i}U\partial_{x_j}U\big]\Big] \exp(\alpha t) + qV_2.$$

In the far subregion $D^T \setminus \overline{\mathcal{D}^T_r}$ the condition (7) is applied. Here $v \equiv U$, and the leading term of the expression $L_{T,\mu}V_2$ is as follows: $\alpha P(U)\exp(\alpha t)$. Since

$$P(U) = U + \mu^2\big(m - \frac{\lambda U}{\mu^2 + \nu U}\big) \geq U + (m - \lambda/\nu) = U + \mu^2(4m - m^2 - 2)/m,$$

then the value m is chosen between root of the quadratic polynomial $4m - m^2 - 2$ under condition $m < 1$. It means $2 - \sqrt{2} < m < 1$. Under this m we have $4m - m^2 - 2 > 0$. Hence the first term in the expression $L_{T,\mu}V_2$ is minorized as follows

$$\alpha P(U)\exp(\alpha t) \geq \alpha U \exp(\alpha t).$$

Due to (7) all remainder terms in the expression

$$L_{T,\mu}W = L_{T,\mu}V_0 + L_{T,\mu}V_2, \quad (\mathbf{x}, t) \in D^T \setminus \overline{\mathcal{D}^T_r}.$$

are minorized by $-\tilde{M}U\exp(\alpha t)$ with a constant $\tilde{M} > 0$ which is not depending on μ, T. Hence the choice $\alpha \geq \tilde{M}$ provides the inequality $L_{T,\mu}V \geq 0$ in the domain $D^T \setminus \overline{\mathcal{D}^T_r}$.

In the intermediate domain $\mathcal{D}^T_r \setminus \overline{\mathcal{D}^T_\rho}$ the properties $v > 0$ and $\chi'(U) \leq 0$ are used. Due to them the inequalities:

$$P(U) = v - \mu^2\frac{\lambda v}{\mu^2 + \nu v} + m\mu^2 \geq (1 - \lambda)v + m\mu^2 = m\,(v + \mu^2) > 0,$$

[1] The inequality $L_{T,\mu}V_0 \geq 0$ takes place in the larger domain $\mathcal{D}^T_r$.

$$P'(U) = \Big[1 - \frac{\mu^4\lambda}{(\mu^2+\nu v)^2}\Big]\big(1-\chi'(U)\big) \geq 1-\lambda = m$$

are derived. Since in this domain both the inequalities $\rho^2 < U(\mathbf{x},t;T) < r^2$ and the condition (4) take place, hence

$$\big(\partial_t U - \mathbf{a}\partial_{\mathbf{x}}U\big)P'(U) \geq \gamma m \rho^2 = \text{const} > 0, \quad (\mathbf{x},t) \in D_r^T \setminus \overline{\mathcal{D}_\rho^T}.$$

Due to this inequality and conditions (5), (6) the value $L_{T,\mu}V_2 \geq 0$ turns out to be nonnegative for all sufficiently small $|\mu| < \mu_0$. In this step the boundary of the parameter μ_0 is chosen. Remind that $L_{T,\mu}V_0 \geq 0$ in the domain $\mathcal{D}_r^T$. Hence

$$L_{T,\mu}W = L_{T,\mu}V_0 + L_{T,\mu}V_2 \geq 0, \quad (\mathbf{x},t) \in D_r^T \setminus \overline{\mathcal{D}_\rho^T}.$$

So the barrier property is proved in the whole domain D^T.

Corollary 1. *If there exists the solution of the problem* (3) *with initial data* $\mathbb{E}[|\mathbf{x}|^2]$, *then it is majorized by the barrier constructed above:*

$$0 \leq u(\mathbf{x},t;T,\mu) \leq W(\mathbf{x},t;T,\mu), \; 0 < t < T, \; \forall T > 0, \; |\mu| \leq \mu_0.$$

Proof. The property obtained above

$$V_0|_{t=0} = U|_{t=0}, \quad V_1|_{t=0} > 0, \quad V_2|_{t=0} = P(U)|_{t=0} \geq 0$$

provides $W|_{t=0} \geq U|_{t=0} \geq |\mathbf{x}|^2$. Then the desired estimate follows from maximum principle [11].

Note the same barrier is suitable for a set of equations (3) with different matrices $A = \{a_{i,j}\}$, under fixed constants M_2, M_3 in the conditions (6),(7).

8. Stability

Now we came back to the problem of stability with respect to white noise perturbation. In order to discuss stability one has to be sure that the perturbed equation (2) has a global solution i.e. the solution exists on the infinite interval $T \in [0,\infty)$. To ensure this property we have to require additional restrictions on the coefficients. In the most simple approach it is enough to demand a bounded grows of the coefficients with $\mathbf{y}$ at infinity [8, 3]:

$$|\mathbf{a}(\mathbf{y},T)| \leq \text{const}\cdot(1+|\mathbf{y}|), \; \forall\, \mathbf{y} \in \mathbb{R}^n, \; T > 0, \tag{9}$$

$$||B(\mathbf{y},T)|| =^{def} \max_{i,j} |b_{i,j}(\mathbf{x},t;T)| \leq M\cdot(1+|\mathbf{y}|), \; \forall\, \mathbf{y} \in \mathbb{R}^n, \; T > 0. \tag{10}$$

Moreover in view of application of the parabolic equation we impose an additional condition: $\mathbf{a}(\mathbf{y},T), B(\mathbf{y},T) \in C_{2,0}(D)$. It ensures smoothness of the expectation $\mathbb{E}[(\mathbf{y}_\mu(T;\mathbf{x})^2]$ with respect to $\mathbf{x}$ on a random trajectory $\mathbf{y}_\mu(T;\mathbf{x})$, [12]. Let the derivatives $\partial_{\mathbf{y}}B(\mathbf{y},T), \partial^2_{\mathbf{y}}B(\mathbf{y},T)$ satisfy the conditions (9), (10) as well. To identify a set of matrices of that type we introduce the ball in the linear space $\mathcal{M}^n$ of matrix-fuctions:

$$S_M^{2,0} = \{B(\mathbf{y},T) \in \mathcal{M}^n : \max_{0\leq k\leq 2} \sup_{\mathbf{y},T} ||(1+|\mathbf{y}|)^{-1}\partial^k_{\mathbf{y}}B(\mathbf{y},T)|| \leq M\}. \tag{11}$$

Theorem 2. *Let the deterministic system* (1) *with the smooth coefficient* $\mathbf{a}(\mathbf{y}, T) \in C_{2,0}(D)$ *under condition* (9) *have the Lyapunov function with the properties* (4)-(7). *Then for each* $M > 0$ *there exists* $\delta_M, \Delta_M > 0$ *such that the equilibrium* $\mathbf{y} = 0$ *is weak stable with respect to white noise under given estimates* $\delta(\varepsilon) = \delta_M\sqrt{\varepsilon}$, $\Delta(\varepsilon) = \delta_M\sqrt{\varepsilon}$ *on a large interval* $T \in (0, \mathcal{O}(\mu^{-2}))$ *uniformly with matrices* $B(\mathbf{y}, T)$ *from the ball* $S_M^{2,0}$.

Proof. The expectation $\mathbb{E}[(\mathbf{y}_\mu(T; \mathbf{x})^2]$ along the random trajectory is a function on $(\mathbf{x}, T)$. Due to the condition (11) this function is a solution of the parabolic problem (3) taken on the upper boundary

$$\mathbb{E}[\big(\mathbf{y}_\mu(T; \mathbf{x})\big)^2] = u(\mathbf{x}, t; T, \mu)|_{t=T}.$$

Hence it is majorized by barrier constructed above. Due to the properties of the barrier the expectation has the estimate:

$$\mathbb{E}[\big(\mathbf{y}_\mu(T; \mathbf{x})\big)^2] \le M_0|\mathbf{x}|^2 + \mu^2\big[M_3/\alpha_0 + m\big]. \tag{12}$$

From here the property of weak stability follows: $\mathbb{E}[\big(\mathbf{y}_\mu(T; \mathbf{x})\big)^2] < \varepsilon$ under $|\mathbf{x}| < \delta(\varepsilon) = \sqrt{\varepsilon/2M_0}$, $|\mu| < \Delta(\varepsilon) = \sqrt{\varepsilon/2\big[M_3/\alpha_0 + m\big]}$, $0 < T \le R^2/4\alpha\mu^2$.

9. Conclusion

Stability of equilibrium with respect to white noise perturbation on a long time interval was proved in this paper. The length of the interval μ^{-2} corresponds to reciprocal quantity of the noise intensity μ. There is the hypothesis of exponentially large interval $\mathcal{O}\big(\exp(S\mu^{-2})\big)$, $\mu \to 0$, $S = \text{const} > 0$. This conjecture is derived by analogy the formula for the first passage time, which is known in the case of autonomous systems [1]; the nonautonomous systems were not considered. In the approach discussed above the hypothesis is reduced to find an appropriate barrier for the parabolic equation (3).

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