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Riemann-Hilbert Problems, families of commuting operators and soliton equations

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Abstract. We use Riemann-Hilbert Problems with canonical normalization to develop technique for constructing families of commuting operators. As a result we are able to derive new hierarchies of integrable nonlinear evolution equations.

1. Introduction

Let Γ be a contour in the complex λ-plane splitting $\mathbb{C}$ into two parts $\mathbb{C} \equiv \Gamma_+ \cup \Gamma_-$. By multiplicative Riemann-Hilbert problem (RHP)[4, 18] we mean the problem of constructing two functions $\xi^+(\lambda)$ and $\xi^-(\lambda)$ analytic for $\lambda \in \Gamma_+$ and $\lambda \in \Gamma_-$ respectively such that:

$$\xi^+(\lambda) = \xi^-(\lambda) G(\lambda), \qquad \text{for } \lambda \in \Gamma. \tag{1}$$

If the functions $\xi^\pm(\lambda)$ are scalar and have no zeroes in their regions of analyticity we can solve the multiplicative RHP by the Plemelj-Sokhotzky formulae [4, 18].

We will consider more general and special RHP for functions $\xi^\pm(\vec{x}, t, \lambda)$ taking values in a simple Lie group $\mathcal{G}$ with simple Lie algebra $\mathfrak{g}$.

Let us outline the special properties of our RHP. First, as contour Γ we will choose either the real axis $\mathbb{R}$, or a set of straight lines intersecting at the origin of $\mathbb{C}$. Second, we will assume that the solutions and the sewing function depend on two or more auxiliary variables one of which we will call the time t and the others $\vec{x} = (x_1, \dots, x_p)^T$ will be spatial variables. It is natural that their number p is smaller than the rank of the algebra $r = \operatorname{rank} \mathfrak{g}$. The third special property is that we will specify explicitly the dependence of the sewing function $G(\vec{x}, t, \lambda)$ by:

$$i\frac{\partial G}{\partial t} - \lambda[K, G(\vec{x}, t, \lambda)] = 0, \qquad i\frac{\partial G}{\partial x_s} - \lambda[J_s, G(\vec{x}, t, \lambda)] = 0, \qquad \lambda \in \Gamma, \tag{2}$$

where K and J_s belong to the Cartan subalgebra $\mathfrak{h} \in \mathfrak{g}$.

We shall say that the functions $\xi^\pm(\lambda)$ are regular solution of the multiplicative RHP if:

(i) on the contour Γ they satisfy the equation:

$$\xi^+(\vec{x}, t, \lambda) = \xi^-(\vec{x}, t, \lambda) G(\vec{x}, t, \lambda), \qquad \lambda \in \Gamma; \tag{3}$$

(ii) the functions $\xi^{\pm}(\lambda)$ have no singularities or degeneracies for $\lambda \in \mathbb{C}\backslash\Gamma$;

The RHP (3) will have unique solution only after imposing a normalization condition. This in our case is the canonical normalization condition:

$$\lim_{\lambda\to\infty} \xi^{\pm}(\vec{x}, t, \lambda) = \mathbb{1}. \tag{4}$$

In what follows we will start with an RHP whose sewing functions have special dependence on the auxiliary variables t and x_s and will demonstrate that their solutions are simultaneous fundamental analytic solutions to a set of commuting operators M, L_s. In fact for the simplest nontrivial case when J_s, K are real and $\Gamma \equiv \mathbb{R}$ this has been known for long time [21, 22, 6]. However the RHP was used just for deriving the soliton solutions through the Zakharov-Shabat dressing method [21, 22], see also [14, 10].

The family of operators M, L_s commute provided the first nontrivial coefficient $Q_1(\vec{x}, t)$ in the asymptotic expansion of $\xi^{\pm}(\vec{x}, t, \lambda)$ (see eq. (6)) satisfies a certain set of nonlinear evolution equations (NLEE). We will demonstrate on several nontrivial examples that this method allows one to derive new types of integrable interactions. Thus we show that the formal approach of Gel'fand and Dickey [3] can be made more precise.

In Section 2 we outline the main idea of constructing the families of commuting operators [7, 8]. The next Section 3 contains several new examples of N-wave type interactions extending the ones found in [12] and their reductions [11, 10, 9]. The last Section contains discussion and conclusions.

2. RHP with canonical normalization

2.1. RHP and Generalized Zakharov-Shabat operators

Consider the simplest nontrivial RHP when $\Gamma \equiv \mathbb{R}$ and the dependence of $G(\vec{x}, t, \lambda)$ is provided by eq. (2) with *real* K and $J_s \in \mathfrak{h} \subset \mathfrak{g}$.

Remark 1. *With this choice of K and J_s it is easy to see that*

$$G(\vec{x}, t, \lambda) = \exp\left(-iKt - i\sum_{s=1}^{p} x_s J_s\right) G(\vec{0}, 0, \lambda) \exp\left(iKt + i\sum_{s=1}^{p} x_s J_s\right). \tag{5}$$

We assume that $G(\vec{0}, 0, \lambda)$ is a smooth bounded function of $\lambda \in \mathbb{R}$ and as a consequence it follows that $G(\vec{x}, t, \lambda)$ is a smooth bounded function of $\lambda \in \mathbb{R}$ for all $\vec{x}$ and t.

Remark 2. *The canonical normalization (4) along with the analyticity properties of the solutions ensure that $\xi^{\pm}(\vec{x}, \lambda)$ allow asymptotic expansions of the form:*

$$\xi^{\pm}(\vec{x}, \lambda) = \exp\left(\mathcal{Q}(\vec{x}, t, \lambda)\right), \qquad \mathcal{Q}(\vec{x}, t, \lambda) = \sum_{k=1}^{\infty} Q_k(\vec{x}, t)\lambda^{-k}. \tag{6}$$

where $Q_k(\vec{x}, t)$ are smooth functions of $\vec{x}$ and t vanishing fast enough for $\vec{x} \to \infty$. The rigorous proof of this fact is out of the scope of the present paper.

It is obvious that

$$\mathcal{J}_s(\vec{x}, t, \lambda) = \xi^{\pm}(\vec{x}, t, \lambda) J_s \hat{\xi}^{\pm}(\vec{x}, t, \lambda), \qquad \mathcal{K}(\vec{x}, t, \lambda) = \xi^{\pm}(\vec{x}, t, \lambda) K \hat{\xi}^{\pm}(\vec{x}, t, \lambda), \tag{7}$$

belong to the algebra $\mathfrak{g}$ for any J_s and K from $\mathfrak{g}$ and allow analytic extensions for $\lambda \in \mathbb{C}_{\pm}$; here $\hat{\xi} \equiv \xi^{-1}$. Since K and J_s belong to the Cartan subalgebra $\mathfrak{h}$, then

$$[\mathcal{J}_s(\vec{x}, t, \lambda), \mathcal{K}(\vec{x}, t, \lambda)] = 0. \tag{8}$$

Theorem 1 (Zakharov-Shabat, 1974). *Let $\xi^{\pm}(x,t,\lambda)$ be solutions to the RHP (3) whose sewing function depends on the auxiliary variables $\vec{x}$ and t as above. Then $\xi^{\pm}(x,t,\lambda)$ are fundamental analytic solutions of the set of differential operators*

$$\begin{aligned} L_s\xi^{\pm} &\equiv i\frac{\partial \xi^{\pm}}{\partial x_s} + [J_s, Q_1(\vec{x},t,\lambda)]\xi^{\pm}(\vec{x},t,\lambda) - \lambda[J_s, \xi^{\pm}(\vec{x},t,\lambda)] = 0, \\ M\xi^{\pm} &\equiv i\frac{\partial \xi^{\pm}}{\partial t} + [K, Q_1(\vec{x},t,\lambda)]\xi^{\pm}(\vec{x},t,\lambda) - \lambda[K, \xi^{\pm}(\vec{x},t,\lambda)] = 0. \end{aligned} \tag{9}$$

where $Q_1(\vec{x},\lambda)$ is the first nontrivial coefficient in the asymptotic expansion (6) of $\xi^{\pm}(\vec{x},t,\lambda)$.

Proof. Introduce the functions:

$$\begin{aligned} g_s^{\pm}(\vec{x},t,\lambda) &= i\frac{\partial \xi^{\pm}}{\partial x_s}\hat{\xi}^{\pm}(\vec{x},t,\lambda) + \lambda\xi^{\pm}(\vec{x},t,\lambda)J_s\hat{\xi}^{\pm}(\vec{x},t,\lambda), \\ g^{\pm}(\vec{x},t,\lambda) &= i\frac{\partial \xi^{\pm}}{\partial t}\hat{\xi}^{\pm}(\vec{x},t,\lambda) + \lambda\xi^{\pm}(\vec{x},t,\lambda)K\hat{\xi}^{\pm}(\vec{x},t,\lambda), \end{aligned} \tag{10}$$

and using (2) prove that

$$g_s^{+}(\vec{x},t,\lambda) = g_s^{-}(\vec{x},t,\lambda), \qquad g^{+}(\vec{x},t,\lambda) = g^{-}(\vec{x},t,\lambda), \qquad \lambda \in \Gamma, \tag{11}$$

which means that these functions are analytic functions of λ in the whole complex λ-plane. Next we find that for $\lambda \to \infty$:

$$\lim_{\lambda\to\infty}(g_s^{\pm}(\vec{x},t,\lambda) - \lambda J_s) = -[J_s, Q(\vec{x},t)], \qquad \lim_{\lambda\to\infty} g^{\pm}(\vec{x},t,\lambda) - \lambda K) = -[J_s, Q(\vec{x},t)]. \tag{12}$$

Then we make use of Liouville theorem to get

$$\begin{aligned} g_s^{+}(\vec{x},t,\lambda) = g_s^{-}(\vec{x},t,\lambda) = \lambda J_s - [J_s, Q_1(\vec{x},t)], \\ g^{+}(\vec{x},t,\lambda) = g^{-}(\vec{x},t,\lambda) = \lambda K - [K, Q_1(\vec{x},t)]. \end{aligned} \tag{13}$$

□

Lemma 1. *The set of operators L_s and M have a common FAS, i.e. they all must commute, that is $Q_1(\vec{x},t)$ satisfies the following NLEE:*

$$i\left[J_k, \frac{\partial Q_1}{\partial x_s}\right] - i\left[J_s, \frac{\partial Q_1}{\partial x_k}\right] + [[J_s, Q_1(\vec{x},t)], [J_k, Q_1(\vec{x},t)]] = 0, \tag{14a}$$

$$i\left[J_s, \frac{\partial Q_1}{\partial t}\right] - i\left[K, \frac{\partial Q_1}{\partial x_s}\right] + [[J_s, Q_1(\vec{x},t)], [K, Q_1(\vec{x},t)]] = 0, \tag{14b}$$

Proof. Follows naturally from the definitions of the operators L_s and M (9).

□

2.2. RHP and Operators of Caudry-Beals-Coifman type

In this Subsection we consider more complicated RHP which is formulated as follows.

- First we introduce the complex valued elements K and J_s of the Cartan subalgebra. The conditions [13]

$$\operatorname{Im}\lambda\alpha(J) = 0, \qquad \alpha \in \Delta, \tag{15}$$

where Δ is the root system of $\mathfrak{g}$, gives a set of M straight lines, or equivalently, a set of $2M$ rays l_ν starting from the origin. We then define the contour as $\Gamma \equiv \bigcup_{\nu=1}^{2M} l_\nu$.

- To each ray l_ν (15) one can associate the subset of roots $\delta_\nu \subset \Delta$ and the corresponding subalgebra $\mathfrak{g}_\nu \subset \mathfrak{g}$. Then the corresponding sewing function takes values in the corresponding subgroup $G_\nu(\vec{x},t,\lambda) \in \mathcal{G}_\nu$ and are bounded functions for $\lambda \in l_\nu$, see remark 1.

The rest of the details are the same as in the previous Subsection. Quite similarly is formulated the generalization of the Zakharov-Shabat theorem. The family of commuting operators formally coincides with the ones in eqs. (9); the difference is that now the Cartan subalgebra elements are complex. As a result however, the operator L in (9) takes the form of a CBC system [2, 1, 13].

2.3. Jets of order k

Another natural generalization consists in formulating the RHP on the complex plane of λ.

$$\xi^+(\vec{x},t,\lambda) = \xi^-(\vec{x},t,\lambda) G(\vec{x},t,\lambda), \qquad \text{for } \lambda^k \in \mathbb{R}, \tag{16}$$

with the canonical normalization (4).

The Zaharov-Shabat method can easily be generalized also for the RHP (16). The result is formulated as

Theorem 2. *Let $\xi^\pm(x,t,\lambda)$ be solutions to the RHP (16) whose sewing function depends on the auxiliary variables $\vec{x}$ and t as follows:*

$$i\frac{\partial G}{\partial t} - \lambda^k [K, G(\vec{x},t,\lambda)] = 0, \qquad i\frac{\partial G}{\partial x_s} - \lambda^k [J_s, G(\vec{x},t,\lambda)] = 0, \qquad \lambda \in \Gamma, \tag{17}$$

where K and J_s belong to the Cartan subalgebra $\mathfrak{h} \in \mathfrak{g}$. Then $\xi^\pm(x,t,\lambda)$ are fundamental analytic solutions of the set of differential operators

$$\begin{aligned} L_s \xi^\pm \equiv & i\frac{\partial \xi^\pm}{\partial x_s} + U_s(\vec{x},t,\lambda)\xi^\pm(\vec{x},t,\lambda) - \lambda^k [J_s, \xi^\pm(\vec{x},t,\lambda)] = 0, \\ M\xi^\pm \equiv & i\frac{\partial \xi^\pm}{\partial t} + V(\vec{x},t,\lambda)\xi^\pm(\vec{x},t,\lambda) - \lambda^k [K, \xi^\pm(\vec{x},t,\lambda)] = 0. \end{aligned} \tag{18}$$

Here $U_s(\vec{x},t,\lambda)$ and $V(\vec{x},t,\lambda)$ are the jets of order k of $\mathcal{J}_s(x,\lambda)$ and $\mathcal{K}(x,\lambda)$, i.e.:

$$\begin{aligned} \lambda^k J_s - U_s(\vec{x},t,\lambda) &= \left(\lambda^k \xi^\pm(\vec{x},t,\lambda) J_l \hat{\xi}^\pm(\vec{x},t,\lambda) \right)_+, \\ \lambda^k K - V(\vec{x},t,\lambda) &= \left(\lambda^k \xi^\pm(\vec{x},t,\lambda) K \hat{\xi}^\pm(\vec{x},t,\lambda) \right)_+. \end{aligned} \tag{19}$$

where the subscript + means that we retain only the nonnegative powers of λ.

Proof. Introduce the functions:

$$\begin{aligned} g_s^\pm(\vec{x},t,\lambda) &= i\frac{\partial \xi^\pm}{\partial x_s} \hat{\xi}^\pm(\vec{x},t,\lambda) + \lambda^k \xi^\pm(\vec{x},t,\lambda) J_s \hat{\xi}^\pm(\vec{x},t,\lambda), \\ g^\pm(\vec{x},t,\lambda) &= i\frac{\partial \xi^\pm}{\partial t} \hat{\xi}^\pm(\vec{x},t,\lambda) + \lambda^k \xi^\pm(\vec{x},t,\lambda) K \hat{\xi}^\pm(\vec{x},t,\lambda), \end{aligned} \tag{20}$$

and using (16) and (17) prove that

$$g_s^+(\vec{x},t,\lambda) = g_s^-(\vec{x},t,\lambda), \qquad g^+(\vec{x},t,\lambda) = g^-(\vec{x},t,\lambda). \tag{21}$$

Thus we find that these functions are analytic functions of λ in the whole complex λ^k-plane. Next we find that:

$$\lim_{\lambda\to\infty} g_s^+(\vec{x},t,\lambda) = \lambda^k J_s, \qquad \lim_{\lambda\to\infty} g^+(\vec{x},t,\lambda) = \lambda^k K. \tag{22}$$

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and make use of Liouville theorem to get

$$
\begin{aligned}
g_s^+(\vec{x},t,\lambda) &= g_s^-(\vec{x},t,\lambda) = \lambda^k J_s - \sum_{l=1}^{k} U_{s;l}(\vec{x},t)\lambda^{k-l}, \\
g^+(\vec{x},t,\lambda) &= g^-(\vec{x},t,\lambda) = \lambda^k K - \sum_{l=1}^{k} V_l(\vec{x},t)\lambda^{k-l}.
\end{aligned}
\tag{23}
$$

□

Let us now express $U_s(x) \in \mathfrak{g}$ in terms of the asymptotic coefficients Q_s in eq. (6).

$$
\mathcal{J}_s(\vec{x},t,\lambda) = J_s + \sum_{k=1}^{\infty} \frac{1}{k!} \mathrm{ad}\,_Q^k J_s, \qquad \mathcal{K}(\vec{x},t,\lambda) = K + \sum_{k=1}^{\infty} \frac{1}{k!} \mathrm{ad}\,_Q^k K. \tag{24}
$$

Thus for the first three coefficients of $U_s(\vec{x},t,\lambda)$ we get:

$$
\begin{aligned}
&U_{s;1}(\vec{x},t) = -\mathrm{ad}\,_{Q_1} J_s, \qquad U_{s;2}(\vec{x},t) = -\mathrm{ad}\,_{Q_2} J_s - \frac{1}{2}\mathrm{ad}\,_{Q_1}^2 J_s \\
&U_{s;3}(\vec{x},t) = -\mathrm{ad}\,_{Q_3} J_s - \frac{1}{2}\left(\mathrm{ad}\,_{Q_2}\mathrm{ad}\,_{Q_1} + \mathrm{ad}\,_{Q_1}\mathrm{ad}\,_{Q_2}\right) J_s - \frac{1}{6}\mathrm{ad}\,_{Q_1}^3 J_s,
\end{aligned}
\tag{25}
$$

and similar expressions for $V(\vec{x},t,\lambda)$ with J_s replaced by K.

Lemma 2. *The set of operators L_s and M (18) have a common FAS, i.e. they all must commute, that is, the set of functions $Q_1(\vec{x},t)$, ..., $Q_k(\vec{x},t)$ satisfy the following NLEE:*

$$
\begin{aligned}
&i\frac{\partial U_s}{\partial x_j} - i\frac{\partial U_j}{\partial x_s} + [U_s(\vec{x},t,\lambda) - \lambda^k J_s, U_j(\vec{x},t,\lambda) - \lambda^k J_j] = 0, \\
&i\frac{\partial U_s}{\partial t} - i\frac{\partial V}{\partial x_s} + [U_s(\vec{x},t,\lambda) - \lambda^k J_s, V(\vec{x},t,\lambda) - \lambda^k K] = 0.
\end{aligned}
\tag{26}
$$

Proof. Follows naturally from the definitions of the operators L_s and M (18). □

Remark 3. *Obviously one can use families of operators with different maximal powers of λ. Using them one can derive not only the relevant N-wave type equations, but also the higher order NLEE of the hierarchy.*

Remark 4. *Considering RHP of the form (16) one must take special care of the behavior of the solutions when $\lambda \to 0$. We are not going to discuss here the conditions that are necessary to impose so that the RHP will be properly defined leving it for the future.*

3. Examples of new types of N-wave interactions

The integrability of the well known N-wave equations in two-dimensional space-time was discovered by Zakharov and Manakov [19]. To this end they used the Lax pair

$$
\begin{aligned}
L\chi &\equiv i\frac{\partial \chi}{\partial x} + ([J, Q(x,t)] - \lambda J)\chi(x,t,\lambda) = 0, \\
M\chi &\equiv i\frac{\partial \chi}{\partial t} + ([K, Q(x,t)] - \lambda K)\chi(x,t,\lambda) = -\lambda\chi(x,t,\lambda)K, \\
J &= \mathrm{diag}\,(a_1, a_2, \ldots, a_n), \qquad K = \mathrm{diag}\,(b_1, b_2, \ldots, b_n),
\end{aligned}
\tag{27}
$$

where the potential $Q(x,t\Delta)$ is a $n\times n$ matrix with $Q_{kk}=0$. The compatibility of this pair is

$$i\left[J,\frac{\partial Q}{\partial t}\right]-i\left[K,\frac{\partial Q}{\partial x}\right]+[[J,Q(x,t)],[K,Q(x,t)]]=0. \tag{28}$$

which is a system of $n(n-1)$ equations for the off-diagonal elements of $Q(x,t)$. This system admits the natural reduction $C_0QC_0=q^{\dagger}$ where $C_0=\operatorname{diag}(1,\epsilon_1,\dots\epsilon_{n-1})$ and $\epsilon_k=\pm 1$. After the N-wave system (28) reduces to $n(n-1)/2$ equations for $Q_{km}(x,t)$, $k<m$.

The N-wave are easily generalized to any other simple Lie algebra, see [5]. Indeed, let us consider the simple Lie algebra $\mathfrak{g}$ of rank r with root system Δ and Cartan-Weyl basis H_s, E_α, $\alpha\in\Delta$; assume also that the Cartan generators H_s satisfy $\langle H_s,H_k\rangle=\delta_{sk}$. Then consider the Lax pair

$$\begin{aligned}
&L\chi\equiv i\frac{\partial\chi}{\partial x}+([J,Q(x,t)]-\lambda J)\chi(x,t,\lambda)=0,\\
&M\chi\equiv i\frac{\partial\chi}{\partial t}+([K,Q(x,t)]-\lambda K)\chi(x,t,\lambda)=-\lambda\chi(x,t,\lambda)K,\\
&J=\sum_{s=1}^{r}a_sH_s,\qquad K=\sum_{s=1}^{r}b_sH_s,\qquad Q(x,t)=\sum_{\alpha\in\Delta_+}(q_\alpha(x,t)E_\alpha+p_\alpha(x,t)E_{-\alpha}),
\end{aligned}\tag{29}$$

where $\Delta_+\subset\Delta$ is the subset of positive roots.

The N-wave equations have as Hamiltonian

$$H_{\mathrm{Nw}}=\frac{1}{2i}\int_{x=-\infty}^{\infty}dx\ \left\langle Q,\left[K,\frac{\partial Q}{\partial x}\right]\right\rangle+\frac{1}{3}\int_{x=-\infty}^{\infty}dx\ \langle[J,Q],[Q,[K,Q]]\rangle, \tag{30}$$

where $\langle X,Y\rangle$ is the Killing form between the elements $X,Y\in\mathfrak{g}$. So typically one may say that a given set of NLEE are of N-wave type if: i) they are contain first order derivatives with respect to x and t; ii) the coefficients a_k and b_k are real which physically means that the group velocity of each of these waves is real; and iii) the nonlinearities in the equations are quadratic in the fields q_α.

These types of N-wave equations were known for a long time. A number of their inequivalent $\mathbb{Z}_2$-reductions for the low-rank Lie algebras were described in [11]. Obviously, using the ISM one can show, that in terms of the scattering data of L these equations become *linear* evolution equations. Note that the scattering data of L are determined through the scattering matrix $T(\lambda,t)$ for $\lambda\in\mathbb{R}$ – the continuous spectrum of L, and some additional data characterizing the discrete spectrum of L. The inverse scattering problem for the operator L is best of all reduced to a RHP (3) on the real line.

Our aim in this Section is to demonstrate new examples of qualitatively different N-wave equations which are also integrable.

3.1. CBC systems and 4-wave equations with complex group velocities, $k=1$.

The 4-wave equations below can be solved by the ISM applied to two operators of CBC type related to the $\mathfrak{g}\simeq so(5)$ algebra. So we consider the Lax pair (29) with

$$J=\operatorname{diag}(a_1,-a_1^*,0,a_1^*,-a_1),\qquad K=\operatorname{diag}(a_1^*,-a_1,0,a_1,-a_1^*),$$

$$Q(x,t)=\sum_{\alpha\in\Delta_{1,+}}(q_\alpha E_\alpha+p_\alpha E_{-\alpha})=\begin{pmatrix} 0 & q_1 & q_2 & q_4 & 0\\ p_1 & 0 & q_3 & 0 & q_4\\ p_2 & p_3 & 0 & q_3 & -q_2\\ p_2 & p_3 & 0 & q_3 & -q_2\\ p_4 & 0 & p_3 & 0 & q_1\\ 0 & p_4 & -p_2 & p_1 & 0\end{pmatrix}. \tag{31}$$

Here a_1 is a complex number that can be assumed to be $a_1 = e^{i\varphi_0}$ and $\Delta_{1,+} \equiv \{e_1 \pm e_2, e_1, e_2\}$ is the set of positive roots of the algebra $so(5)$.

The important difference between (31) and the Lax pair (27) for $\mathfrak{g} \simeq so(5)$ is that now J and K have complex eigenvalues. As a consequence of this the continuous spectrum of L_1 and M_1 fills up the real axis $\mathbb{R}$ and two lines $\mathbb{R}e^{\pm i\varphi_0}$ closing angles $\pm\phi_0$ with $\mathbb{R}$. Again the ISP for the operator L_1 reduces to a RHP, whose solution consists of 6 fundamental analytic solutions $\chi^\nu(x,t,\lambda)$, $\nu = 1,\dots,6$ – one for each of the 6 sectors Ω_ν into which the lines $\mathbb{R}$ and $\mathbb{R}e^{\pm i\varphi_0}$ split the complex λ-plane.

We need in addition a $\mathbb{Z}_2$-reduction which must relate q_j and p_j; several types of such reductions have been described in [11]. However the reduction we will use below

$$C_1 Q(x,t)^\dagger C_1 = Q(x,t), \quad C_1 J^\dagger C_1 = -J, \quad C_1 K^\dagger C_1 = -K, \tag{32}$$

where $C_1 = E_{12} + E_{21} + E_{33} - E_{45} - E_{54}$ corresponds to the Weyl reflection with respect to the root $e_1 - e_2$. This involution leads to a set of algebraic constraints on $q_j(x,t)$ and $p_j(x,t)$. For convenience we introduce

$$\begin{aligned} &p_1(x,t) = w_0(x,t), && q_1(x,t) = w_1(x,t), && q_2(x,t) = w_2(x,t), && q_3(x,t) = w_1(x,t), \\ &q_4(x,t) = w_4(x,t), && p_2(x,t) = w_3^*(x,t), && p_3(x,t) = w_2^*(x,t), && p_4(x,t) = -w_4^*(x,t), \end{aligned} \tag{33}$$

where $w_0(x,t)$ and $w_1(x,t)$ are real-valued functions. Thus instead of the standard 4 complex-valued functions for the 4-wave system (see eqs. (36) and (37) below) we have 2 real and 3 complex-valued functions. The corresponding new 4-wave eqs are:

$$\begin{aligned} &\frac{\partial w_0}{\partial t} - c_0 \frac{\partial w_0}{\partial x} + 2c_0 \sin(\varphi_0)|w_3|^2 = 0, \\ &\frac{\partial w_1}{\partial t} - c_0 \frac{\partial w_1}{\partial x} + 2c_0 \sin(\varphi_0)|w_2|^2 = 0, \\ &\frac{\partial w_2}{\partial t} - c_0 e^{-2i\varphi_0} \frac{\partial w_2}{\partial x} + 2c_0 e^{-2i\varphi_0} \sin(2\varphi_0)(w_2^* w_4 - w_1 w_3) = 0, \\ &\frac{\partial w_3}{\partial t} - c_0 e^{2i\varphi_0} \frac{\partial w_3}{\partial x} - 2c_0 e^{2i\varphi_0} \sin(2\varphi_0)(w_0 w_2 + w_3^* w_4) = 0, \\ &\frac{\partial w_4}{\partial t} + ic_0 \frac{\partial w_4}{\partial x} - 2c_0 \cos(\varphi_0) w_2 w_3 = 0. \end{aligned} \tag{34}$$

The Hamiltonian is given by:

$$\begin{aligned} H_{\rm 4w-1} &= \frac{c_0}{2} \int_{x=-\infty}^{\infty} dx \, \left\langle Q, \left[K, \frac{\partial Q}{\partial x}\right] \right\rangle - \frac{ic_0}{3} \int_{x=-\infty}^{\infty} dx \, \langle [J,Q], [Q,[K,Q]] \rangle \\ &= c_0 \int_{x=-\infty}^{\infty} dx \, \left(2\cos(\varphi_0) \left(w_1 \frac{\partial w_0}{\partial x} - w_0 \frac{\partial w_1}{\partial x} \right) - i\sin(\varphi_0) \left(w_4^* \frac{\partial w_4}{\partial x} - w_4 \frac{\partial w_4^*}{\partial x} \right) \right. \\ &\left. -e^{-i\varphi_0} w_3^* \frac{\partial w_2}{\partial x} - e^{i\varphi_0} w_3 \frac{\partial w_2^*}{\partial x} - 4\sin(2\varphi_0) \left(w_0|w_2|^2 - w_1|w_3|^2 + w_2 w_3 w_4^* + w_2^* w_3^* w_4 \right) \right). \end{aligned} \tag{35}$$

Let us briefly compare the new 4-wave system (34) with the well known 4-wave system (see Chapter 3 of [20]) which are obtained with the generalized Zakharov-Shabat system with real valued $J = a_1 H_1 + a_2 H_2$ and $K = b_1 H_1 + b_2 H_2$ and with the standard reduction $p_k = q_k^*$:

$$\begin{aligned} &i\frac{\partial q_1}{\partial t} - i\frac{b_1 - b_2}{a_1 - a_2}\frac{\partial q_1}{\partial x} - \kappa q_2 q_3^* = 0, && i\frac{\partial q_4}{\partial t} - i\frac{b_1 + b_2}{a_1 + a_2}\frac{\partial q_4}{\partial x} + \kappa q_2 q_3 = 0, \\ &i\frac{\partial q_2}{\partial t} - i\frac{b_1}{a_1}\frac{\partial q_2}{\partial x} + \kappa(q_1 q_3 - q_3^* q_4) = 0, && i\frac{\partial q_1}{\partial t} - i\frac{b_2}{a_2}\frac{\partial q_1}{\partial x} - \kappa(q_2^* q_4 - q_2 q_1^*) = 0, \end{aligned} \tag{36}$$

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where $\kappa = a_1 b_2 - a_2 b_1$. The Hamiltonian is

$$
\begin{aligned}
H_{4\mathrm{w}-2} = i \int_{x=-\infty}^{\infty} dx \; \Bigg((b_1 - b_2) \left(q_1 \frac{\partial q_1^*}{\partial x} - q_1^* \frac{\partial q_1}{\partial x} \right) + (b_1 + b_2) \left(q_4^* \frac{\partial q_4}{\partial x} - q_4 \frac{\partial q_4^*}{\partial x} \right) \\
+ b_1 \left(q_2 \frac{\partial q_2^*}{\partial x} - q_2^* \frac{\partial q_2}{\partial x} \right) + b_2 \left(q_3 \frac{\partial q_3^*}{\partial x} - q_3^* \frac{\partial q_3}{\partial x} \right) \Bigg) \\
- 2\kappa \int_{x=-\infty}^{\infty} dx \; \left(q_1 q_2^* q_3 - q_1^* q_2 q_3^* + q_2 q_3 q_4^* - q_2^* q_3 q_4^* \right).
\end{aligned}
\tag{37}
$$

The obvious differences are: i) the new 4-wave equation is a system of equations for 3 complex fields w_2, w_3, w_4 and 2 real fields $q_1/a_1, a_1 p_1$ (instead of 4 complex fields q_α); ii) the group velocities are now complex (instead of real); iii) the interaction 'strength' between the different waves is different (instead of being equal to κ).

Of course there will be differences between the soliton solutions, but these will be given elsewhere.

3.2. New types of 3*-wave interactions,* $k = 2$*.*

Let $\mathfrak{g} = sl(3)$ and

$$
Q_1(\vec{x}, t) = \begin{pmatrix} 0 & u_1 & u_3 \\ -v_1 & 0 & u_2 \\ -v_3 & -v_2 & 0 \end{pmatrix}, \qquad Q_2(\vec{x}, t) = \begin{pmatrix} q_{11} & w_1 & w_3 \\ -z_1 & q_{22} & w_2 \\ -z_3 & -z_2 & q_{33} \end{pmatrix}, \tag{38}
$$

Fix up $k = 2$. Then the Lax pair becomes

$$
\begin{aligned}
L\xi^{\pm} \equiv i \frac{\partial \xi^{\pm}}{\partial x} + U(x,t,\lambda) \xi^{\pm}(x,t,\lambda) - \lambda^2 [J, \xi^{\pm}(x,t,\lambda)] = 0, \\
M\xi^{\pm} \equiv i \frac{\partial \xi^{\pm}}{\partial t} + V(x,t,\lambda) \xi^{\pm}(x,t,\lambda) - \lambda^2 [K, \xi^{\pm}(x,t,\lambda)] = 0,
\end{aligned}
\tag{39}
$$

where

$$
\begin{aligned}
U \equiv U_2 + \lambda U_1 = \left([J, Q_2(x)] - \frac{1}{2} [[J, Q_1], Q_1(x)] \right) + \lambda [J, Q_1], \\
V \equiv V_2 + \lambda V_1 = \left([K, Q_2(x)] - \frac{1}{2} [[K, Q_1], Q_1(x)] \right) + \lambda [K, Q_1].
\end{aligned}
\tag{40}
$$

Impose a $\mathbb{Z}_2$-reduction of type a) with $A = \mathrm{diag}\,(1, \epsilon, 1)$, $\epsilon^2 = 1$. Thus Q_1 and Q_2 get reduced into:

$$
Q_1 = \begin{pmatrix} 0 & u_1 & 0 \\ \epsilon u_1^* & 0 & u_2 \\ 0 & \epsilon u_2^* & 0 \end{pmatrix}, \qquad Q_2 = \begin{pmatrix} 0 & 0 & w_3 \\ 0 & 0 & 0 \\ w_3^* & 0 & 0 \end{pmatrix}, \tag{41}
$$

New type of integrable 3-wave equations:

$$
\begin{aligned}
i(a_1 - a_2) \frac{\partial u_1}{\partial t} - i(b_1 - b_2) \frac{\partial u_1}{\partial x} + \epsilon \kappa u_2^* u_3 + \epsilon \frac{\kappa (a_1 - a_2)}{(a_1 - a_3)} u_1 |u_2|^2 = 0, \\
i(a_2 - a_3) \frac{\partial u_2}{\partial t} - i(b_2 - b_3) \frac{\partial u_2}{\partial x} + \epsilon \kappa u_1^* u_3 - \epsilon \frac{\kappa (a_2 - a_3)}{(a_1 - a_3)} |u_1|^2 u_2 = 0, \\
i(a_1 - a_3) \frac{\partial u_3}{\partial t} - i(b_1 - b_3) \frac{\partial u_3}{\partial x} - \frac{i\kappa}{a_1 - a_3} \frac{\partial (u_1 u_2)}{\partial x} \\
+ \epsilon \kappa \left(\frac{a_1 - a_2}{a_1 - a_3} |u_1|^2 + \frac{a_2 - a_3}{a_1 - a_3} |u_2|^2 \right) u_1 u_2 + \epsilon \kappa u_3 (|u_1|^2 - |u_2|^2) = 0,
\end{aligned}
\tag{42}
$$

where the interaction constant κ and u_3 are given by:

$$\kappa = \frac{1}{2}\left(a_1b_3 - a_2b_3 - a_1b_2 - b_1a_3 + b_1a_2 + b_2a_3\right), \qquad u_3 = w_3 + \frac{2a_2 - a_1 - a_3}{2(a_1 - a_3)}u_1u_2. \tag{43}$$

The diagonal terms in the Lax representation are λ-independent. Two of them read:

$$\begin{aligned} i(a_1 - a_2)\frac{\partial |u_1|^2}{\partial t} - i(b_1 - b_2)\frac{\partial |u_1|^2}{\partial x} - \epsilon\kappa(u_1u_2u_3^* - u_1^*u_2^*u_3) = 0, \\ i(a_2 - a_3)\frac{\partial |u_2|^2}{\partial t} - i(b_2 - b_3)\frac{\partial |u_2|^2}{\partial x} - \epsilon\kappa(u_1u_2u_3^* - u_1^*u_2^*u_3) = 0, \end{aligned} \tag{44}$$

These relations are satisfied identically as a consequence of the NLEE.

3.3. New types of 3-wave interactions, $k = 3$.

Our last example of 3-wave interactions is similar to the one, reported in [12] and involves Lax pair, which is cubic in λ:

$$U(x,t,\lambda) = \lambda U_2(x,t) + \lambda^2 U_1(x,t) - \lambda^3 J, \qquad V(x,t,\lambda) = \lambda V_2(x,t) + \lambda^2 V_1(x,t) - \lambda^3 K, \tag{45}$$

where

$$\begin{aligned} &J = \operatorname{diag}(a_1, a_2, a_3), && K = \operatorname{diag}(b_1, b_2, b_3), \\ &U_1(x,t) = \begin{pmatrix} 0 & 0 & u_{13} \\ u_{21} & 0 & 0 \\ 0 & u_{32} & 0 \end{pmatrix}, && U_2(x,t) = \begin{pmatrix} 0 & u_{12} & 0 \\ 0 & 0 & u_{23} \\ u_{31} & 0 & 0 \end{pmatrix}, \\ &V_1(x,t) = \begin{pmatrix} 0 & 0 & v_{13} \\ v_{21} & 0 & 0 \\ 0 & v_{32} & 0 \end{pmatrix}, && V_1(x,t) = \begin{pmatrix} 0 & v_{12} & 0 \\ 0 & 0 & v_{23} \\ v_{31} & 0 & 0 \end{pmatrix}, \end{aligned} \tag{46}$$

With this choice U and V automatically satisfy the reduction condition [17]

$$C_3 U_k(x,t) C_3^{-1} = \omega^k U_k(x,t), \qquad C_3 V_k(x,t) C_3^{-1} = \omega^k V_k(x,t), \qquad C_3 = \operatorname{diag}(1, \omega, \omega^2), \tag{47}$$

with $\omega = \exp(2\pi i/3)$. We can also impose the involution

$$C_2 U_k^\dagger(x,t) C_2^{-1} = \epsilon^{k+1} U_k(x,t), \qquad C_2 V_k^\dagger(x,t) C_2^{-1} = \epsilon^{k+1} V_k(x,t), \qquad C_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & \epsilon_1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \tag{48}$$

with $\epsilon = \pm 1, \epsilon_1 = \pm 1$ which gives:

$$\begin{aligned} &u_{12} = \epsilon_1 \epsilon u_{23}^*, && u_{31}^* = \epsilon u_{31}, && u_{13}^* = u_{13}, && u_{21} = \epsilon_1 u_{32}^*, \\ &a_1^* = \epsilon a_3, && a_2 = \epsilon a_2^*, && b_3^* = \epsilon b_1, && b_2^* = \epsilon b_2. \end{aligned} \tag{49}$$

and analogous relations for v_{ij}. The compatibility condition now gives:

Introduce also the coefficients $Q_{1,2}$ which automatically satisfy the above reductions:

$$Q_1(x,t) = \begin{pmatrix} 0 & 0 & R_1 \\ w_1 & 0 & 0 \\ 0 & -\epsilon\epsilon_1 w_1^* & 0 \end{pmatrix}, \qquad Q_2(x,t) = \begin{pmatrix} 0 & w_2 & 0 \\ 0 & 0 & -\epsilon_1 w_2^* \\ R_2 & 0 & 0 \end{pmatrix}, \quad R_1 = -\epsilon R_1^*, \quad R_2 = -R_2^* \tag{50}$$

Thus we get the following parametrization for U_k and V_k

$$\begin{aligned} U_1(x,t) &= [Q_1(x,t), J], & U_2(x,t) &= [Q_2(x,t), J] + \frac{1}{2}[Q_1(x,t), [Q_1(x,t), J]], \\ V_1(x,t) &= [Q_1(x,t), K], & V_2(x,t) &= [Q_2(x,t), K] + \frac{1}{2}[Q_1(x,t), [Q_1(x,t), K]], \end{aligned} \tag{51}$$

or in components:

$$\begin{aligned} u_{13} &= (a_3 - a_1)R_1, & u_{12} &= (a_2 - a_1)w_2 - \frac{1}{2}\epsilon\epsilon_1(a_1 + a_2 - 2a_3)w_1^* R_1, \\ u_{21} &= (a_1 - a_2)w_1, & u_{23} &= \epsilon_1(a_2 - a_3)w_2^* + \frac{1}{2}(a_2 + a_3 - 2a_1)w_1 R_1, \\ u_{32} &= \epsilon\epsilon_1(a_3 - a_2)w_1^*, & u_{31} &= (a_1 - a_3)R_2 - \frac{1}{2}\epsilon\epsilon_1(a_1 + a_3 - 2a_2)|w_1|^2. \end{aligned} \tag{52}$$

The expressions for v_{jk} differ from (52) only by replacing a_k by b_k.

The corresponding NLEE take the form:

$$\begin{aligned} &\frac{\partial R_1}{\partial t} - \frac{b_1 - b_3}{a_1 - a_3}\frac{\partial R_1}{\partial x} + \frac{\epsilon\epsilon_1}{a_1 - a_3}(u_{12}v_{12}^* - v_{12}u_{12}^*) = 0, \qquad \frac{\partial u_{12}}{\partial t} - \frac{\partial v_{12}}{\partial x} = 0, \\ &i\frac{\partial w_1}{\partial t} - i\frac{b_1 - b_2}{a_1 - a_2}\frac{\partial w_1}{\partial x} + \frac{\epsilon_1\epsilon}{a_1 - a_2}(u_{12}^* v_{31} - v_{12}^* u_{31}) = 0, \qquad \frac{\partial u_{31}}{\partial t} - \frac{\partial v_{31}}{\partial x} = 0. \end{aligned} \tag{53}$$

Inserting here the notations from eq. (52) we obtain the four NLEE for the four independent functions w_1, w_2, R_1 and R_2.

4. Conclusions and open questions

We proposed a method for constructing new integrable NLEE based on the use of the RHP with canonical normalization combined with the Mikhailov's reduction group [16, 17]. Obviously we can derive many new classes of such equations making appropriate choices of the: i) the order k of the jets, ii) the simple Lie algebra $\mathfrak{g}$ and iii) the reduction group and its realization as a subgroup of the group of automorphisms of $\mathfrak{g}$.

Since the method is based on the RHP one can apply Zakharov-Shabat dressing method for constructing their explicit (N-soliton) solutions.

The new NLEE are expected to be Hamiltonian. So one must show that the jets $U(\vec{x}, t, \lambda)$ can be viewed as elements of more complicated co-adjoint orbits of the relevant Kac-Moody algebra, generated by the chosen grading of $\mathfrak{f}$. The corresponding Poisson brackets can be derived using the results of Kulish and Reyman [15] and imposing the reduction conditions as constraints.

The last but not least important problem concerns the possible physical applications of these equations.

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