

Perturbative and Exact Results on the Neumann Value for the Nonlinear Schrödinger on the Half-line

A. S. Fokas

Department of Applied Mathematics and Theoretical Physics, University of Cambridge, Cambridge CB3 0WA, UK, and Research Center of Mathematics, Academy of Athens, 11527, Greece.

E-mail: T.Fokas@damtp.cam.ac.uk

J. Lenells

Department of Mathematics and Center for Astrophysics, Space Physics & Engineering Research, Baylor University, One Bear Place #97328, Waco, TX 76798, USA.

E-mail: Jonatan.Lenells@baylor.edu

Abstract. The most challenging problem in the implementation of the so-called *unified transform* to the analysis of the nonlinear Schrödinger equation on the half-line is the characterization of the unknown boundary value in terms of the given initial and boundary conditions. For the so-called *linearizable* boundary conditions this problem can be solved explicitly. Furthermore, for non-linearizable boundary conditions which decay for large t , this problem can be largely bypassed in the sense that the unified transform yields useful asymptotic information for the large t behavior of the solution. However, for the physically important case of periodic boundary conditions it is necessary to characterize the unknown boundary value. Here, we first present a perturbative scheme which can be used to compute explicitly the asymptotic form of the Neumann boundary value in terms of the given τ -periodic Dirichlet datum to any given order in a perturbation expansion. We then discuss briefly an extension of the pioneering results of Boutet de Monvel and co-authors which suggests that if the Dirichlet datum belongs to a large class of particular τ -periodic functions, which includes $\{a \exp(i\omega t) \mid a > 0, \omega \geq a^2\}$, then the large t behavior of the Neumann value is given by a τ -periodic function which can be computed explicitly.

AMS Subject Classification (2000): 37K15, 35Q15.

Keywords: Initial-boundary value problem, long-time asymptotics, nonlinear Schrödinger equation.

1. Introduction

Let $q(x, t)$ satisfy the nonlinear Schrödinger (NLS) on the half-line with a given initial condition,

$$iq_t + q_{xx} - 2\lambda|q|^2q = 0, \quad 0 < x < \infty, \quad t > 0, \quad \lambda = \pm 1, \quad (1)$$

$$q(x, 0) = q_0(x), \quad 0 < x < \infty, \quad (2)$$

where $q_0(x)$ has sufficient decay as $x \rightarrow \infty$.



In the above setting, the so-called unified transform [8] can be used to analyze problems with either a linearizable boundary condition or with a non-linearizable boundary condition which decays for large t .

1.1. Linearizable boundary conditions

Suppose that equations (1) and (2) are supplemented with any of the following boundary conditions:

$$q(0, t) = 0, \tag{3a}$$

$$q_x(0, t) = 0, \tag{3b}$$

$$q_x(0, t) - \chi q(0, t) = 0, \quad \chi \text{ real constant.} \tag{3c}$$

Then, the unified transform yields a solution with the same level of efficiency as the celebrated inverse scattering transform [8, 9, 11].

The above problems can also be solved by the usual inverse scattering transform after mapping them to certain equivalent problems on the line [1]. However, the unified transform can also be implemented for linearizable boundary value problems for integrable nonlinear PDEs involving a third order spatial derivative—such as the KdV equation—which *cannot* be mapped to appropriate problems on the line.

1.2. Boundary conditions which decay as $t \rightarrow \infty$

Suppose that equations (1) and (2) are supplemented with the Dirichlet boundary condition

$$g_0(t) = q(0, t), \quad 0 < t < \infty, \tag{4}$$

where $g_0(t)$ vanishes as $t \rightarrow \infty$. In this case, the unified transform expresses the solution $q(x, t)$ in terms of the solution of a 2×2 matrix Riemann-Hilbert (RH) problem which is uniquely defined in terms of the *spectral functions* $\{a(k), b(k), A(k), B(k)\}$. The first two of these functions can be obtained in terms of the given initial datum $q_0(x)$, however $\{A(k), B(k)\}$ depend on both $g_0(t)$ and on $g_1(t) = q_x(0, t)$. Thus, in order to compute $q(x, t)$ it is necessary to first characterize the so-called Dirichlet to Neumann map, i.e. to express $g_1(t)$ in terms of $g_0(t)$ and $q_0(x)$. However, since $g_0(t)$ vanishes for large t , using the fact that the above spectral functions are t independent, it is still possible to compute the large t -asymptotics of the above RH problem, and hence the large t -asymptotics of $q(x, t)$ without characterizing fully the spectral functions $A(k)$ and $B(k)$.

1.3. The Dirichlet to Neumann map

Two different approaches for analyzing the Dirichlet to Neumann map were recently presented in [11] and [14]. The formulation in [11] is based on the analysis of the eigenfunctions involved in the definition of $\{A(k), B(k)\}$ (see also [7, 10]), whereas the formulation in [14] is based on an extension of the Gelfand-Levitan-Marchenko approach first introduced in [2].

1.4. τ -periodic boundary condition

Suppose that equations (1) and (2) are supplemented by equation (4), where $g_0(t)$ is a periodic function with period $\tau > 0$.

A perturbative scheme for computing the large t behavior of $g_1(t)$ is introduced in [14]. In particular, it is shown in [14] for the NLS, and in [13] for the mKdV, that if $g_0(t) = \alpha \sin t$, $\alpha \in \mathbb{R}$, then it is possible to obtain the function $g_1(t)$ for the NLS and the functions $\{g_1(t), q_{xx}(0, t)\}$ for the mKdV as a perturbation expansion in the parameter α , up to and including terms of $O(\alpha^3)$. Furthermore, these functions are 2π -periodic, at least up to this order. Unfortunately,

the perturbative approach of [14] is quite cumbersome and it is practically impossible to go beyond terms of $O(\alpha^3)$.

In the particular case that

$$g_0(t) = \alpha e^{i\omega t}, \quad \alpha > 0, \quad \omega \in \mathbb{R}, \quad (5)$$

pioneering results for the focusing NLS have been obtained in a series of papers of Boutet de Monvel and co-authors [3, 4, 5, 6] by employing the unified transform. In particular, it is shown in [5] that for the focusing NLS, $\lambda = -1$, there exists a solution $q(x, t)$, such that

$$g_1(t) \sim ce^{i\omega t}, \quad t \rightarrow \infty, \quad c \in \mathbb{C}, \quad (6)$$

if and only if the triple of constants $\{\alpha, \omega, c\}$ satisfy either

$$\left(c = \pm\alpha\sqrt{\omega - \alpha^2} \text{ and } \omega \geq \alpha^2\right) \text{ or } \left(c = i\alpha\sqrt{|\omega| + 2\alpha^2} \text{ and } \omega \leq -6\alpha^2\right). \quad (7)$$

1.5. Outline of the paper

Here we first present a new perturbative scheme for computing the large t asymptotic behavior of $g_1(t)$ in terms of the τ -periodic function $g_0(t)$. As illustrations, we consider the example of the single exponential (6), as well as the example of

$$g_0(t) = \alpha e^{i\omega t} + \beta e^{-i\omega t}, \quad \alpha, \beta \in \mathbb{C}, \quad \omega > 0. \quad (8)$$

The main difference between this new perturbative approach and the one used in [14] is that in the latter approach the analysis was first carried out for all t and then the limit $t \rightarrow \infty$ was computed, whereas in the new approach the analysis is carried out directly in the limit $t \rightarrow \infty$. This simplifies the relevant algorithm enormously, as illustrated by the fact that for the above two examples we give formulae up to and including terms of order eight.

We also discuss briefly an extension of the formalism introduced in [3, 4, 5, 6], which suggests that using the technique of finite gap integration, it is possible to obtain a large class of τ -periodic functions which have the crucial property that if they are assigned as the Dirichlet data for the NLS on the half line, then the associated function $g_1(t)$ has the property that it asymptotes as $t \rightarrow \infty$ to a τ -periodic function and furthermore this function can be computed explicitly.

2. A new perturbative approach

In the context of asymptotically t -periodic data, there are situations where we do not necessarily need to know the Neumann value $g_1(t)$ for all $t > 0$, but it is sufficient to know its asymptotic form $g_1^B(t)$. It is therefore natural to ask the following question: Given the asymptotic form $g_0^B(t)$ of the Dirichlet datum, can we find the asymptotic form $g_1^B(t)$ of the Neumann value? Theorem 2.2 below provides a constructive algorithm for computing the asymptotic form of the Neumann value from the asymptotic form of the Dirichlet data in a perturbative expansion. In other words, it provides an explicit construction of the Dirichlet to Neumann map for asymptotically t -periodic data in the limit of large t and small data.

2.1. Main result

We first define what we mean by a perturbative solution.

Definition 2.1 A *perturbative solution of the NLS equation (1) in the quarter plane* is a sequence of smooth functions $\{q_N(x, t)\}_1^\infty$ defined for $x > 0$ and $t > 0$ with the following properties:

(i) The formal power series

$$q = \epsilon q_1 + \epsilon^2 q_2 + \epsilon^3 q_3 + \dots \tag{9}$$

satisfies (1) in the quarter plane $\{x > 0, t > 0\}$ to all orders in a perturbative expansion, that is,

$$\begin{aligned} O(\epsilon) : \quad & i q_{1t} + q_{1xx} = 0, \\ O(\epsilon^2) : \quad & i q_{2t} + q_{2xx} = 0, \\ O(\epsilon^3) : \quad & i q_{3t} + q_{3xx} - 2\lambda |q_1|^2 q_1 = 0, \\ O(\epsilon^4) : \quad & i q_{4t} + q_{4xx} - 2\lambda (q_1^2 \bar{q}_2 + 2q_1 q_2 \bar{q}_1) = 0, \\ & \vdots \\ O(\epsilon^N) : \quad & i q_{Nt} + q_{Nxx} - 2\lambda \{|q|^2 q\}_N = 0, \end{aligned}$$

where $\{\dots\}_N$ denotes the coefficient of ϵ^N of the enclosed expression.

- (ii) For each N , $q_N(x, t)$ and all its partial derivatives have continuous extensions to $\{x \geq 0, t \geq 0\}$.
- (iii) For each N , $q_N(\cdot, 0)$ belongs to the Schwartz space $\mathcal{S}(\mathbb{R}_+)$.

We can now state our main result.

Theorem 2.2 *Let $\{g_{0N}^B(t), g_{1N}^B(t)\}_1^\infty$ be periodic functions of period $\tau = \frac{2\pi}{\omega} > 0$. Suppose $\{q_N(x, t)\}_1^\infty$ is a perturbative solution of NLS in the quarter plane such that, for each N , the Dirichlet and Neumann boundary values of q_N asymptote towards $g_{0N}^B(t)$ and $g_{1N}^B(t)$ respectively in the sense that*

$$\begin{aligned} q_N(0, \cdot) - g_{0N}^B &\in \mathcal{S}(\mathbb{R}_+), & N \geq 1, \\ q_{Nx}(0, t) - g_{1N}^B(t) &= O(t^{-3/2}), & t \rightarrow \infty, \quad N \geq 1. \end{aligned} \tag{10}$$

Suppose the set $\{q_N(\cdot, t) | t \geq 0\}$ is bounded in $L^1(\mathbb{R}_+)$ for each $N \geq 1$.

Then the asymptotic Neumann values $\{g_{1N}^B(t)\}_{N=1}^\infty$ can be constructed explicitly from the asymptotic Dirichlet values $\{g_{0N}^B(t)\}_{N=1}^\infty$ as follows. Let $\{a_{N,n}\}_{n=-\infty}^\infty$ denote the Fourier coefficients of g_{0N}^B :

$$g_{0N}^B(t) = \sum_{n=-\infty}^\infty a_{N,n} e^{in\omega t}, \quad N \geq 1. \tag{11}$$

Then the Fourier coefficients $\{c_{N,n}\}_{n=-\infty}^\infty$ of

$$g_{1N}^B(t) = \sum_{n=-\infty}^\infty c_{N,n} e^{in\omega t}, \quad N \geq 1, \tag{12}$$

are given by

$$\begin{aligned} c_{N,n} = \left\{ \left(-2i\lambda k \sum_{l,m=-\infty}^\infty \bar{a}_l d_m(k) d_{n+l-m}(k) - \lambda \sum_{l,m=-\infty}^\infty \bar{c}_l d_m(k) d_{n+l-m}(k) \right. \right. \\ \left. \left. + 2\lambda \sum_{l,m=-\infty}^\infty \bar{a}_l a_m d_{n+l-m}(k) + 2ika_n \right) \Big|_{k=k_1(n)} \right\}_N, \end{aligned} \tag{13}$$

where

- $a_n = \sum_{N=1}^{\infty} a_{N,n} \epsilon^N$, $c_n = \sum_{N=1}^{\infty} c_{N,n} \epsilon^N$, and $d_n(k) = \sum_{N=1}^{\infty} d_{N,n}(k) \epsilon^N$.
- The coefficients $d_{N,n}(k)$, $N \geq 1$, $n \in \mathbb{Z}$, satisfy

$$d_{N,n}(k) = \frac{1}{4ik^2 + in\omega} \left\{ -2\lambda k \sum_{l,m=-\infty}^{\infty} \bar{a}_l d_m(k) d_{n+l-m}(k) + \lambda i \sum_{l,m=-\infty}^{\infty} \bar{c}_l d_m(k) d_{n+l-m}(k) - 2i\lambda \sum_{l,m=-\infty}^{\infty} \bar{a}_l a_m d_{n+l-m}(k) + 2ka_n + ic_n \right\}_N, \quad n \in \mathbb{Z}, \quad N \geq 1, \quad k \in \partial D_1^0, \quad (14)$$

where $D_1^0 = \{\text{Re } k > 0\} \cap \{\text{Im } k > 0\}$ denotes the first quadrant of the complex k -plane.

- $k_1(n)$ denotes the unique root of $4k^2 + n\omega = 0$ in ∂D_1^0 , i.e.

$$k_1(n) = \begin{cases} \frac{i\sqrt{n\omega}}{2}, & n \geq 0, \\ \frac{\sqrt{-n\omega}}{2}, & n < 0. \end{cases}$$

We refer to [12] for a proof of Theorem 2.2. In the remainder of this section, we explain how the $c_{N,n}$'s are determined from equations (13)-(14) and provide several examples.

2.2. Construction of the $c_{N,n}$'s

Equation (13) with $N = 1$ yields

$$c_{1,n} = 2ik_1(n)a_{1,n}, \quad n \in \mathbb{Z}. \quad (15a)$$

Substituting this into equation (14) with $N = 1$, we find

$$d_{1,n}(k) = \frac{2ka_{1,n} + ic_{1,n}}{4ik^2 + in\omega} = \frac{a_{1,n}}{2i(k + k_1(n))}, \quad n \in \mathbb{Z}, \quad k \in \partial D_1^0. \quad (15b)$$

Similarly, equations (13) and (14) with $N = 2$ yield

$$c_{2,n} = 2ik_1(n)a_{2,n}, \quad n \in \mathbb{Z}, \quad (16a)$$

$$d_{2,n}(k) = \frac{a_{2,n}}{2i(k + k_1(n))}, \quad n \in \mathbb{Z}, \quad k \in \partial D_1^0. \quad (16b)$$

Continuing in this way, equations (13) and (14) with $N = 3$ yield

$$c_{3,n} = \left(-2i\lambda k \sum_{l,m=-\infty}^{\infty} \bar{a}_{1,l} d_{1,m}(k) d_{1,n+l-m}(k) - \lambda \sum_{l,m=-\infty}^{\infty} \bar{c}_{1,l} d_{1,m}(k) d_{1,n+l-m}(k) + 2\lambda \sum_{l,m=-\infty}^{\infty} \bar{a}_{1,l} a_{1,m} d_{1,n+l-m}(k) + 2ika_{3,n} \right) \Big|_{k=k_1(n)}, \quad n \in \mathbb{Z}, \quad (17a)$$

$$d_{3,n}(k) = \frac{1}{4ik^2 + in\omega} \left(-2\lambda k \sum_{l,m=-\infty}^{\infty} \bar{a}_{1,l} d_{1,m}(k) d_{1,n+l-m}(k) + \lambda i \sum_{l,m=-\infty}^{\infty} \bar{c}_{1,l} d_{1,m}(k) d_{1,n+l-m}(k) \right) \quad (17b)$$

$$- 2i\lambda \sum_{l,m=-\infty}^{\infty} \bar{a}_{1,l} a_{1,m} d_{1,n+l-m}(k) + 2ka_{3,n} + ic_{3,n} \Big), \quad n \in \mathbb{Z}, \quad k \in \partial D_1^0.$$

This process can be continued indefinitely. Indeed, suppose we have determined $\{c_{M,n}\}_{n=-\infty}^{\infty}$ and $\{d_{M,n}(k)\}_{n=-\infty}^{\infty}$ for $N \leq M - 1$. Then equation (13) with $N = M$ yields

$$c_{M,n} = \left(F_{Mn}(k) + 2ika_{M,n} \right) \Big|_{k=k_1(n)}, \quad n \in \mathbb{Z},$$

where the function $F_{Mn}(k)$ is given in terms of known lower order terms:

$$F_{Mn}(k) = \lambda \sum_{l,m=-\infty}^{\infty} \left\{ - 2ik\bar{a}_l d_m(k) d_{n+l-m}(k) - \bar{c}_l d_m(k) d_{n+l-m}(k) + 2\bar{a}_l a_m d_{n+l-m}(k) \right\}_M.$$

We can now use equation (14) with $N = M$ to determine $d_{M,n}(k)$:

$$d_{M,n}(k) = \frac{1}{4ik^2 + in\omega} (F_{Mn}(k) + 2ka_{M,n} + ic_{M,n}), \quad n \in \mathbb{Z}.$$

This determines $\{c_{M,n}\}_{n=-\infty}^{\infty}$ and $\{d_{M,n}(k)\}_{n=-\infty}^{\infty}$ for $N = M$ and completes the inductive step.

2.3. Examples

Example 2.3 (Single exponential) Suppose

$$g_0^B(t) = \epsilon e^{i\omega t}, \quad t \geq 0, \quad \omega > 0.$$

In this case, all coefficients $a_{N,n}$ are zero except for $a_{1,1} = 1$. Equations (15) imply that all the coefficients $c_{1,n}$ and $d_{1,n}(k)$ vanish except for

$$c_{1,1} = -\sqrt{\omega}, \quad d_{1,1}(k) = -\frac{i}{2k + i\sqrt{\omega}}.$$

Equations (16) yield $c_{2,n} = d_{2,n}(k) = 0$ for all n . Equations (17) imply that all the coefficients $c_{3,n}$ and $d_{3,n}(k)$ vanish except for

$$c_{3,1} = -\frac{\lambda}{2\sqrt{\omega}}, \quad d_{3,1}(k) = \frac{\lambda}{2(2ik - \sqrt{\omega})^2 \sqrt{\omega}}.$$

Continuing in this way, we find that the nonzero coefficients $c_{N,n}$ with $N \leq 8$ are

$$c_{1,1} = -\sqrt{\omega}, \quad c_{3,1} = -\frac{\lambda}{2\sqrt{\omega}}, \quad c_{5,1} = \frac{1}{8\omega^{3/2}}, \quad c_{7,1} = -\frac{\lambda}{16\omega^{5/2}}.$$

In summary, we have found that

$$g_1^B(t) = -\epsilon \left(\sqrt{\omega} + \frac{\epsilon^2 \lambda}{2\sqrt{\omega}} - \frac{\epsilon^4}{8\omega^{3/2}} + \frac{\epsilon^6 \lambda}{16\omega^{5/2}} + O(\epsilon^8) \right) e^{i\omega t}.$$

The summation of this perturbative expansion suggests

$$g_1^B(t) = -\epsilon \sqrt{\omega + \lambda \epsilon^2} e^{i\omega t}, \tag{18}$$

which, upon identifying α and ϵ , is in agreement with (7a) (note that the inequality $\omega \geq \alpha^2$ is automatically satisfied in the perturbative limit $\alpha \rightarrow 0$).

Example 2.4 (Single exponential) Suppose

$$g_{01}^B(t) = \epsilon e^{-i\omega t}, \quad t \geq 0, \quad \omega > 0.$$

In this case, all coefficients $a_{N,n}$ are zero except for $a_{1,-1} = 1$. Proceeding as in the previous example, we find that the nonzero coefficients $c_{N,n}$ with $N \leq 8$ are

$$c_{1,-1} = i\sqrt{\omega}, \quad c_{3,-1} = -\frac{i\lambda}{\sqrt{\omega}}, \quad c_{5,-1} = -\frac{i}{2\omega^{3/2}}, \quad c_{7,-1} = -\frac{i\lambda}{2\omega^{5/2}}.$$

In summary,

$$g_1^B(t) = i\epsilon \left(\sqrt{\omega} - \frac{\epsilon^2\lambda}{\sqrt{\omega}} - \frac{\epsilon^4}{2\omega^{3/2}} - \frac{\epsilon^6\lambda}{2\omega^{5/2}} + O(\epsilon^8) \right) e^{-i\omega t}.$$

The summation of this perturbative expansion suggests

$$g_1^B(t) = i\epsilon \sqrt{\omega - 2\lambda\epsilon^2} e^{-i\omega t},$$

which, upon identifying α and ϵ and letting $\omega \rightarrow -\omega$, is in agreement with (7b) (note that the inequality $\omega \leq -6\alpha^2$ is automatically satisfied in the perturbative limit $\alpha \rightarrow 0$).

Example 2.5 (Sum of exponentials) We consider the case of

$$g_0^B(t) = \epsilon(\alpha e^{i\omega t} + \beta e^{-i\omega t}), \quad t \geq 0, \quad (19)$$

where $\alpha, \beta \in \mathbb{C}$ and $\omega > 0$ are constants. In this case, all coefficients $a_{N,n}$ vanish except for

$$a_{1,1} = \alpha, \quad a_{1,-1} = \beta.$$

Equations (15) imply that all the $c_{1,n}$'s and $d_{1,n}(k)$'s vanish except for

$$c_{1,1} = -\alpha\sqrt{\omega}, \quad c_{1,-1} = i\beta\sqrt{\omega}, \quad d_{1,1}(k) = -\frac{i\alpha}{2k + i\sqrt{\omega}}, \quad d_{1,-1}(k) = -\frac{i\beta}{2k + \sqrt{\omega}}.$$

Equations (16) yield $c_{2,n} = d_{2,n}(k) = 0$ for all n . In general, $c_{N,n} = 0$ unless both N and n are odd and $N \geq n$. Continuing in this way, we find that the nonzero coefficients $c_{N,n}$ with $N \leq 8$ are

$$c_{1,1} = -\alpha\sqrt{\omega}, \quad c_{1,-1} = i\beta\sqrt{\omega}, \quad (20a)$$

$$\begin{aligned} c_{3,3} &= -\frac{i(\sqrt{3} + (-2 - i)\alpha^2\bar{\beta}\lambda)}{2\sqrt{\omega}}, & c_{3,1} &= -\frac{\alpha\lambda(|\alpha|^2 + 4|\beta|^2)}{2\sqrt{\omega}}, \\ c_{3,-1} &= -\frac{i\beta\lambda(|\beta|^2 + (1 - i)|\alpha|^2)}{\sqrt{\omega}}, & c_{3,-3} &= \frac{(\sqrt{3} - 2 - i)\beta^2\bar{\alpha}\lambda}{2\sqrt{\omega}}, \end{aligned} \quad (20b)$$

$$\begin{aligned} c_{5,5} &= \frac{((2 + i) - (5 + 2i)\sqrt{3} - (2 + 3i)\sqrt{5} + (3 + 2i)\sqrt{15})\alpha^3\bar{\beta}^2}{16\omega^{3/2}}, \\ c_{5,3} &= -\frac{i\alpha^2\bar{\beta}(((3 + 6i) + (2 + 3i)\sqrt{3})|\alpha|^2 + (5 + 2i)((3 + 2i) + (2 + i)\sqrt{3})|\beta|^2)}{2((12 + 3i) + (7 + 2i)\sqrt{3})\omega^{3/2}}, \\ c_{5,1} &= \frac{\alpha(|\alpha|^4 - 4(\sqrt{3} - 6)|\alpha|^2|\beta|^2 + 2(9 - i\sqrt{3})|\beta|^4)}{8\omega^{3/2}}, \end{aligned}$$

$$\begin{aligned}
 c_{5,-1} &= \frac{\beta((\sqrt{3} + (-2 + 3i))|\alpha|^4 - 2(\sqrt{3} + (2 - 5i))|\alpha|^2|\beta|^2 - 2i|\beta|^4)}{4\omega^{3/2}}, \\
 c_{5,-3} &= -\frac{\beta^2\bar{\alpha}(((48 + 9i) + (29 + 4i)\sqrt{3})|\alpha|^2 + 8((3 + 5i) + (2 + 3i)\sqrt{3})|\beta|^2)}{4((21 + 12i) + (12 + 7i)\sqrt{3})\omega^{3/2}}, \\
 c_{5,-5} &= \frac{((1 - 2i) - (2 - 5i)\sqrt{3} - (3 - 2i)\sqrt{5} + (2 - 3i)\sqrt{15})\beta^3\bar{\alpha}^2}{16\omega^{3/2}}, \tag{20c} \\
 c_{7,7} &= \frac{\alpha^4\bar{\beta}^3\lambda}{32(56 + 23\sqrt{7})\omega^{5/2}} \left\{ (-595 + 147i) + (525 - 147i)\sqrt{3} + (196 + 49i)\sqrt{5} \right. \\
 &\quad \left. - (223 - 30i)\sqrt{7} - (189 - 14i)\sqrt{15} + (192 - 39i)\sqrt{21} + (67 + 19i)\sqrt{35} \right. \\
 &\quad \left. - (63 + i)\sqrt{105} \right\}, \\
 c_{7,5} &= \frac{i\alpha^3\bar{\beta}^2\lambda}{8((26 + 97i) + (15 + 56i)\sqrt{3})((25 + 10i) + (11 + 4i)\sqrt{5})\omega^{5/2}} \\
 &\quad \times \left\{ ((1110 + 2345i) + (645 + 1350i)\sqrt{3} + (106 + 1011i)\sqrt{5} \right. \\
 &\quad \left. + (63 + 582i)\sqrt{15})|\alpha|^2 + (1 + i)((4542 + 903i) + (2623 + 524i)\sqrt{3} \right. \\
 &\quad \left. + (1612 + 1359i)\sqrt{5} + (931 + 786i)\sqrt{15})|\beta|^2 \right\}, \\
 c_{7,3} &= -\frac{\alpha^2\bar{\beta}\lambda}{96((627 + 2340i) + (362 + 1351i)\sqrt{3})\omega^{5/2}} \left\{ ((42588 + 1476i) \right. \\
 &\quad \left. + (24588 + 852i)\sqrt{3})|\alpha|^4 - (6 - 6i)((-25328 - 35261i) \right. \\
 &\quad \left. - (14623 + 20358i)\sqrt{3} + (3594 + 1545i)\sqrt{5} + (2075 + 892i)\sqrt{15})|\alpha|^2|\beta|^2 \right. \\
 &\quad \left. + ((395885 + 21770i) + (228564 + 12569i)\sqrt{3} - (9405 + 7020i)\sqrt{5} \right. \\
 &\quad \left. - (5430 + 4053i)\sqrt{15})|\beta|^4 - 96i((795 + 627i) + (459 + 362i)\sqrt{3})|\alpha|^2|\beta|^2 \right\}, \\
 c_{7,1} &= -\frac{\alpha\lambda}{48\omega^{5/2}} \left\{ 3|\alpha|^6 + 6(4 + 5\sqrt{3})|\alpha|^4|\beta|^2 + 2((162 + 51i) \right. \\
 &\quad \left. - (13 + 12i)\sqrt{3})|\alpha|^2|\beta|^4 - 6|\alpha|^2|\beta|^2((\sqrt{3} - 6)|\alpha|^2 \right. \\
 &\quad \left. + (2 + 2i)(\sqrt{3} + (2 + 5i))|\beta|^2) + 16(15 - 2i\sqrt{3})|\beta|^6 \right\}, \\
 c_{7,-1} &= \frac{\beta\lambda}{24\omega^{5/2}} \left\{ 3(3 - 2i\sqrt{3})|\alpha|^6 + ((39 + 42i) + (28 - 45i)\sqrt{3})|\alpha|^4|\beta|^2 \right. \\
 &\quad \left. + ((36 - 15i) + (23 + 36i)\sqrt{3})|\alpha|^2|\beta|^4 + 3|\alpha|^2|\beta|^2((-2 + 2i)(\sqrt{3} - 6)|\alpha|^2 \right. \\
 &\quad \left. - (\sqrt{3} + (2 + 5i))|\beta|^2) - 12i|\beta|^6 \right\}, \\
 c_{7,-3} &= \frac{(1 + i)\beta^2\bar{\alpha}\lambda}{96((2340 + 8733i) + (1351 + 5042i)\sqrt{3})\omega^{5/2}} \left\{ 3((32088 + 120795i) \right. \\
 &\quad \left. + (18526 + 69741i)\sqrt{3} + (8106 - 6393i)\sqrt{5} + (4680 - 3691i)\sqrt{15})|\alpha|^4 \right. \\
 &\quad \left. + 6i((168629 + 32048i) + (97358 + 18503i)\sqrt{3} + (8733 + 11700i)\sqrt{5} \right. \\
 &\quad \left. + (5042 + 6755i)\sqrt{15})|\alpha|^2|\beta|^2 - (4 - 4i)\beta(4((265 + 7718i) \right. \\
 &\quad \left. + (153 + 4456i)\sqrt{3})|\beta|^2\bar{\beta} - 9((892 + 3329i) + (515 + 1922i)\sqrt{3})|\alpha|^2\bar{\beta}) \right\}, \\
 c_{7,-5} &= -\frac{(1 - i)\beta^3\bar{\alpha}^2\lambda}{24((97 + 26i) + (56 + 15i)\sqrt{3})((15 + 40i) + (7 + 18i)\sqrt{5})\omega^{5/2}}
 \end{aligned}$$

$$\begin{aligned}
 & \times \left\{ 3((3820 + 3375i) + (2205 + 1950i)\sqrt{3} + (1464 + 1395i)\sqrt{5} \right. \\
 & + (845 + 806i)\sqrt{15})|\alpha|^2 + (1 + i)((7755 + 7962i) + (4466 + 4602i)\sqrt{3} \\
 & + (2742 + 2586i)\sqrt{5} + (1578 + 1495i)\sqrt{15})|\beta|^2 \left. \right\}, \\
 c_{7,-7} = & \frac{\beta^4 \bar{\alpha}^3 \lambda}{32(56 + 23\sqrt{7})\omega^{5/2}} \left\{ (-147 - 595i) + (147 + 525i)\sqrt{3} - (49 - 196i)\sqrt{5} \right. \\
 & - (30 + 223i)\sqrt{7} - (14 + 189i)\sqrt{15} + (39 + 192i)\sqrt{21} - (19 - 67i)\sqrt{35} \\
 & \left. + (1 - 63i)\sqrt{105} \right\}. \tag{20d}
 \end{aligned}$$

3. Finite-gap solutions and potentially asymptotically admissible pairs

Finally, we propose an approach for generating pairs $\{g_0^B, g_1^B\}$ of periodic functions which can arise as asymptotic Dirichlet and Neumann values of a solution of NLS. We make the following definition.

Definition 3.1 A pair of functions $\{g_0^B(t), g_1^B(t)\}$ is *asymptotically admissible* if there exists a smooth solution $q(x, t)$ of NLS in the quarter plane $\{x > 0, t > 0\}$, such that

- (i) $q(x, t)$ and all its partial derivatives have continuous extensions to $\{x \geq 0, t \geq 0\}$.
- (ii) The initial data decay for large x , that is, $q(\cdot, 0) \in \mathcal{S}(\mathbb{R}_+)$.
- (iii) As $t \rightarrow \infty$, the Dirichlet and Neumann boundary values of q asymptote towards $g_0^B(t)$ and $g_1^B(t)$ respectively, that is,

$$q(0, \cdot) - g_0^B \in \mathcal{S}(\mathbb{R}_+), \quad q_x(0, t) - g_1^B(t) = O(t^{-3/2}), \quad t \rightarrow \infty. \tag{21}$$

The spectral functions $\{a(k), b(k), A(k), B(k)\}$ associated with the initial and boundary values $\{q_0(x), g_0(t), g_1(t)\}$ of a solution $q(x, t)$ of (1) in the quarter plane are not independent, but satisfy an important relation called the *global relation*. If the Dirichlet and Neumann values satisfy (21) where $\{g_0^B(t), g_1^B(t)\}$ is a pair of smooth periodic functions of period $\tau > 0$, then the global relation requires that the quotient $B(k)/A(k)$ be continuous in a certain region of the complex k -plane. In situations where $B(k)/A(k)$ can be explicitly computed, it is possible to use this continuity condition to impose constraints on $\{g_0^B(t), g_1^B(t)\}$. This approach was employed in [5] to derive the characterization (7) of asymptotically admissible single exponential boundary values for the focusing NLS equation. In [5], the continuity requirement that the global relation imposes on $B(k)/A(k)$ characterizes $\{g_0^B(t), g_1^B(t)\}$ completely.

In general, the nonexplicit nature of the quotient $B(k)/A(k)$ makes it challenging to apply the above approach. However, there exists a large class of pairs $\{g_0^B, g_1^B\}$ for which the quotient $B(k)/A(k)$ is known explicitly. This is the class of finite-gap solutions generated by the Baker-Akhiezer formalism. We propose that potentially asymptotically admissible pairs for the NLS equation can be generated by taking finite-gap solutions for which the data $\{g_0^B, g_1^B\}$ is periodic (not just quasiperiodic) and then enforcing the continuity condition derived from the global relation on $B(k)/A(k)$.

Acknowledgments

The authors acknowledge support from the EPSRC, UK.

References

- [1] Biondini G, Fokas A S and Shepelsky D 2013 Comparison of two approaches to the IBVP for the NLS equation on the half-line with Robin boundary conditions, preprint.

- [2] Boutet de Monvel A, Fokas A S and Shepelsky D 2003 The analysis of the global relation for the nonlinear Schrödinger equation on the half-line, *Lett. Math. Phys.* **65** 199–212
- [3] Boutet de Monvel A, Its A and Kotlyarov V 2007 Long-time asymptotics for the focusing NLS equation with time-periodic boundary condition, *C. R. Math. Acad. Sci. Paris* **345** 615–620.
- [4] Boutet de Monvel A, Its A and Kotlyarov V 2009 Long-time asymptotics for the focusing NLS equation with time-periodic boundary condition on the half-line, *Comm. Math. Phys.* **290** 479–522
- [5] Boutet de Monvel A, Kotlyarov V and Shepelsky D 2009 Decaying long-time asymptotics for the focusing NLS equation with periodic boundary condition, *Int. Math. Res. Not. IMRN* 547–577
- [6] Boutet de Monvel A, Kotlyarov V, Shepelsky D and Zheng C 2010 Initial boundary value problems for integrable systems: towards the long time asymptotics, *Nonlinearity* **23** 2483.
- [7] Degasperis A, Manakov S V and Santini P M 2001 On the initial-boundary value problems for soliton equations, *JETP Letters* **74** 481–485.
- [8] Fokas A S 1997 A unified transform method for solving linear and certain nonlinear PDEs, *Proc. Roy. Soc. Lond. A* **453** 1411–1443.
- [9] Fokas A S 2002 Integrable nonlinear evolution equations on the half-line, *Comm. Math. Phys.* **230** 1–39.
- [10] Fokas A S 2005 A generalised Dirichlet to Neumann map for certain nonlinear evolution PDEs, *Comm. Pure Appl. Math.* **LVIII** 639–670.
- [11] Fokas A S and Lenells J 2012 The unified method: I Non-linearizable problems on the half-line, *J. Phys. A: Math. Theor.* **45**, 195201.
- [12] Fokas A S and Lenells J 2013 The asymptotics of the Neumann value for the nonlinear Schrödinger equation with t-periodic data, preprint.
- [13] Hwang G and Fokas A S 2013 The modified Korteweg-de Vries equation on the half-line with a sine-wave as Dirichlet datum, *J. Nonl. Math. Phys.*, to appear.
- [14] Lenells J and Fokas, A S 2012 The unified method: II NLS on the half-line with t-periodic boundary conditions, *J. Phys. A: Math. Theor.* **45**, 195202.