

About linear superpositions of special exact solutions of Nizhnik-Veselov-Novikov equation

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Abstract. General scheme for calculations via Zakharov and Manakov $\bar{\partial}$ -dressing method of exact solutions, nonstationary and stationary, of Nizhnik-Veselov-Novikov (NVN) equation in the forms of simple nonlinear and linear superpositions of arbitrary number N of exact special solutions $u^{(n)}, n = 1, \dots, N$ is presented. Simple nonlinear superpositions are given up to a constant by the sums of solutions $u^{(n)}$ and calculated by $\bar{\partial}$ -dressing of the first auxiliary linear problem with nonzero asymptotic values of potential at infinity. It is remarkable that in the zero limit of asymptotic values of potential simple nonlinear superpositions convert in to linear ones in the form of the sums of special solutions $u^{(n)}$. It is shown that the sums $u = u^{(k_1)} + \dots + u^{(k_m)}, 1 \leq k_1 < k_2 < \dots < k_m \leq N$ of arbitrary subsets of these N solutions are also exact solutions of NVN equation. The obtained results are illustrated in detail by hyperbolic version of NVN equation, i. e. by NVN-II equation. The presented exact solutions include as superpositions of special line solitons and also superpositions of plane wave type singular periodic solutions. PACS numbers: 02.30.Ik, 02.30.Jr, 02.30.Zz, 05.45.Yv

1. Introduction

This paper is dedicated to the memory of famous russian scientist **Sergey Manakov**. The discovery of $\bar{\partial}$ -dressing method which we use here for constructions of exact solutions of NVN equation is one of the most great achievements of Vladimir Zakharov and **Sergey Manakov** in developing of IST.

Among different (2+1)-dimensional integrable nonlinear equations [1, 2, 3, 4, 5, 6, 7, 8, 9] the famous Nizhnik-Veselov-Novikov (NVN) equation is well known and has long history. This equation discovered in the papers of Nizhnik (1980) [10] and Veselov, Novikov (1984) [11] has the following form:

$$u_t + \kappa_1 u_{\xi\xi\xi} + \kappa_2 u_{\eta\eta\eta} + 3\kappa_1(u\partial_\eta^{-1}u_\xi)_\xi + 3\kappa_2(u\partial_\xi^{-1}u_\eta)_\eta = 0 \quad (1)$$

where $u(\xi, \eta, t)$ is scalar function, κ_1, κ_2 are some constants; $\xi = x + \sigma y, \eta = x - \sigma y, \sigma^2 = \pm 1$; ∂_ξ^{-1} and ∂_η^{-1} are operators inverse to ∂_ξ and ∂_η , $\partial_\eta^{-1}\partial_\eta = \partial_\xi^{-1}\partial_\xi = 1$. In elliptic case $\sigma = i$ and for constants $\kappa_1 = \bar{\kappa}_2 = \kappa$ equation (1) is known as NVN-I equation or simply as Veselov-Novikov (VN) equation; in hyperbolic case $\sigma = 1$ and for real constants κ_1, κ_2 equation (1) is known as NVN-II equation; below, in calculations valid for both types of equations, we will use common notation - NVN-equation.

NVN equation (1) can be represented as compatibility condition in the form of **Manakov's** triad [12]:

$$[L_1, L_2] = BL_1, \quad B = 3(\kappa_1\partial_\eta^{-1}u_{\xi\xi} + \kappa_2\partial_\xi^{-1}u_{\eta\eta}) \quad (2)$$

of two linear auxiliary problems

$$L_1\psi = (\partial_{\xi\eta}^2 + u)\psi = 0, \quad (3)$$

$$L_2\psi = (\partial_t + \kappa_1\partial_\xi^3 + \kappa_2\partial_\eta^3 + 3\kappa_1(\partial_\eta^{-1}u_\xi)\partial_\xi + 3\kappa_2(\partial_\xi^{-1}u_\eta)\partial_\eta)\psi = 0. \quad (4)$$

The invention by **Manakov** [12] such kind of triad representation was the prominent step for exact integration of some (2+1)-dimensional integrable equations.

Several classes of exact solutions of NVN equation (1) have been constructed in last three decades (1980 – 2010) via different methods [10, 11, 13, 14, 15, 18, 16, 17, 19, 20, 21, 22, 23, 24], see also the books [3, 4]. These solutions include finite-zone type solutions (Veselov, Novikov (1984)) [11]; rationally localized solutions or lumps (Grinevich, **Manakov** (1986, 1988)) [13, 14], (Grinevich, Novikov R. (1988)) [15], (Grinevich (1999,2000)) [16, 17], (Dubrovsky, Formusatik, (2001,2003)) [20, 21]; particular examples of solutions with functional parameters (Nizhnik (1980) [10], Matveev and Salle (1991) [18], Dubrovsky, Topovsky and Basalaev (2010,2011) [22, 23]); multi-line soliton solutions (Dubrovsky, Topovsky and Basalaev (2010,2011)) [22, 24] and so on.

Recently in the paper [23] general class of exact solutions with functional parameters and constant asymptotic values $-\epsilon$ at infinity

$$u(\xi, \eta, t) = \tilde{u}(\xi, \eta, t) - \epsilon, \quad \tilde{u}(\xi, \eta, t)|_{\xi^2+\eta^2 \rightarrow \infty} \rightarrow 0 \quad (5)$$

of NVN equation (1) via $\bar{\partial}$ -dressing method of Zakharov and **Manakov** [6, 7, 8, 9] has been calculated and subclass of multi-line soliton solutions has been presented [23].

In another paper [24] (see also [22]) it was established that for some special solutions $u^{(1)}$ and $u^{(2)}$ with zero asymptotic values $-\epsilon = 0$ at infinity, for special linear (plane) solitons or for special plane wave type singular periodic solutions, their sum $u^{(1)} + u^{(2)}$ is also exact solution of NVN equation.

In present paper this result [22, 24] is generalized to the case of linear superpositions of arbitrary number N of special line solitons (or special plane wave type singular periodic solutions) $u^{(n)}$, $n = 1, \dots, N$ in such a way, that the sums of arbitrary subsets of these solutions

$$u = u^{(k_1)} + \dots + u^{(k_m)}, \quad 1 \leq k_1 < k_2 < \dots < k_m \leq N \quad (6)$$

are also exact solutions of NVN equation (1).

For convenience here some useful formulas of $\bar{\partial}$ -dressing method for NVN equation (1) [4, 20, 21, 23, 22, 24] are presented. Central object of this method is the scalar wave function χ

$$\chi(\lambda; \xi, \eta, t) = e^{-F(\lambda; \xi, \eta, t)}\psi(\xi, \eta, t), \quad F(\lambda; \xi, \eta, t) = i[\lambda\xi - \frac{\epsilon}{\lambda}\eta + (\kappa_1\lambda^3 - \kappa_2\frac{\epsilon^3}{\lambda^3})t] \quad (7)$$

which satisfies to corresponding $\bar{\partial}$ -problem or equivalently to following singular integral equation:

$$\chi(\lambda) = 1 + \int \int_C \frac{d\lambda' \wedge d\bar{\lambda}'}{2\pi i(\lambda' - \lambda)} \int \int_C \chi(\mu, \bar{\mu}) R(\mu, \bar{\mu}; \lambda', \bar{\lambda}') d\mu \wedge d\bar{\mu}, \quad (8)$$

here canonical normalization $\chi \rightarrow \chi_\infty = 1$, as $\lambda \rightarrow \infty$, of wave function is assumed and the kernel R is given by the formula [4, 23, 22, 24]

$$R(\mu, \bar{\mu}; \lambda, \bar{\lambda}; \xi, \eta, t) = R_0(\mu, \bar{\mu}; \lambda, \bar{\lambda}) e^{F(\mu; \xi, \eta, t) - F(\lambda; \xi, \eta, t)}. \quad (9)$$

Solutions $u(\xi, \eta, t)$ of NVN equation are expressed via reconstruction formulas

$$u = -\epsilon - i\chi_{-1}\eta = -\epsilon + i\epsilon\chi_1\xi \quad (10)$$

through the coefficients χ_{-1} and/or χ_1 of Taylor's series expansions: $\chi = \chi_0 + \chi_1\lambda + \chi_2\lambda^2 + \dots$ and $\chi = \chi_\infty + \frac{\chi_{-1}}{\lambda} + \frac{\chi_{-2}}{\lambda^2} + \dots$ in the neighborhoods of points $\lambda = 0$ and $\lambda = \infty$ of complex plane \mathbb{C} .

In constructing of exact solutions u of NVN equation (1) two conditions must be satisfied [4, 20, 21, 23, 22, 24]: the condition of potentiality of operator L_1 , or the absence in the first

auxiliary linear problem (3) of the terms with first derivatives $u_1 \partial_\xi$ and $u_2 \partial_\eta$, and the condition of reality $\bar{u} = u$ of solutions.

The potentiality condition of operator L_1 in (3), or equivalently in terms of wave function χ , the condition $\chi_0 = 1$, imposes severe restrictions on the kernel R_0 of $\bar{\partial}$ -problem. These restrictions were successfully satisfied for broad classes of exact solutions of NVN equation (1), such as lumps [20, 21], solutions with functional parameters, multi-line solitons and plane wave type singular periodic solutions [23, 22, 24].

The condition of reality of solutions $\bar{u} = u$ leads to another following restrictions on the kernel R_0 [23, 22, 24]:

$$R_0(\mu, \bar{\mu}; \lambda, \bar{\lambda}) = \frac{\epsilon^3}{|\mu|^2 |\lambda|^2 \bar{\mu} \bar{\lambda}} \overline{R_0\left(-\frac{\epsilon}{\bar{\lambda}}, -\frac{\epsilon}{\bar{\lambda}}, -\frac{\epsilon}{\bar{\mu}} - \frac{\epsilon}{\mu}\right)} \quad (11)$$

for NVN-I equation and

$$R_0(\mu, \bar{\mu}; \lambda, \bar{\lambda}) = \overline{R_0(-\bar{\mu}, -\mu; -\bar{\lambda}, -\lambda)} \quad (12)$$

for NVN-II equation; these restrictions are obtained in the limit of "weak" fields and lead to broad classes of exact solutions, such as lumps [20, 21], solutions with functional parameters [23] and multi-line soliton solutions [23, 22, 24].

But one can do not use the limit of weak fields and impose the reality condition $u = \bar{u}$ directly to calculated complex solutions of NVN equation satisfying only to potentiality condition. This approach makes it possible to receive besides multi line soliton solutions also plane wave type singular periodic solutions [22, 24].

2. Simple nonlinear superpositions of complex solutions of NVN equation

The choice of delta-functional kernel

$$R_0(\mu, \bar{\mu}; \lambda, \bar{\lambda}) = \pi \sum_n A_n \delta(\mu - M_n) \delta(\lambda - \Lambda_n) \quad (13)$$

with constant coefficients A_n and discrete spectral parameters M_n, Λ_n leads to simple determinant formula [24, 22]

$$u = -\epsilon + \frac{\partial^2}{\partial \xi \partial \eta} \ln \det A, \quad A_{lk} = \delta_{lk} + \frac{2i A_k}{M_l - \Lambda_k} e^{F(M_l) - F(\Lambda_k)} \quad (14)$$

for exact multi-line soliton and plane wave type singular periodic solutions of NVN equation. The main problem in using this last formula is satisfaction to reality and potentiality conditions.

It was shown in [22, 24] that the choice of kernel R_0 in the form

$$R_0(\mu, \bar{\mu}; \lambda, \bar{\lambda}) = \pi \sum_{m=1}^N \left[a_m \lambda_m \delta(\mu - \mu_m) \delta(\lambda - \lambda_m) + a_m \mu_m \delta(\mu + \lambda_m) \delta(\lambda + \mu_m) \right] \quad (15)$$

of N paired terms with discrete spectral parameters (μ_m, λ_m) allows to satisfy the potentiality condition $\chi_0 = 1$.

In the simplest cases $N = 1, 2$ of one or two paired terms in (15) one obtains due to (14) the following expressions for $\det A$ [22, 24]:

$$N = 1 : \det A = \left(1 + s_1 e^{\Delta F(\mu_1, \lambda_1)} \right)^2, \quad (16)$$

$$N = 2 : \det A = \left(1 + s_1 e^{\Delta F(\mu_1, \lambda_1)} + s_n e^{\Delta F(\mu_n, \lambda_n)} + w e^{\Delta F(\mu_1, \lambda_1) + \Delta F(\mu_n, \lambda_n)} \right)^2. \quad (17)$$

Here, in (15) – (17) a_n, μ_n, λ_n ($n = 1, \dots, N$) are some complex constants; μ_n and λ_n also known as discrete spectral parameters which give spectral characterization for corresponding exact solutions. The quantities s_n, w and $\Delta F(\mu_n, \lambda_n)$ are given by the formulas:

$$s_n := i a_n \frac{\mu_n + \lambda_n}{\mu_n - \lambda_n}; \quad \Delta F(\mu_n, \lambda_n) := F(\mu_n) - F(\lambda_n), \quad (18)$$

$$w := -s_1 s_n \cdot \frac{(\lambda_1 - \lambda_n)(\lambda_n + \mu_1)(\mu_1 - \mu_n)(\lambda_1 + \mu_n)}{(\lambda_1 + \lambda_n)(\lambda_n - \mu_1)(\mu_1 + \mu_n)(\lambda_1 - \mu_n)}. \quad (19)$$

The expression for $\det A$ in the case $N = 2$ (17) is generated by two pairs of terms in (15) with discrete spectral variables (μ_1, λ_1) and (μ_n, λ_n) .

The formula for generally complex solution corresponding to one arbitrary pair (μ_n, λ_n) , $(n = 1, \dots, N)$ of discrete spectral variables due to (14), (16) and (18) has the form:

$$u^{(n)}(\xi, \eta, t) = -\epsilon + \tilde{u}^{(n)}(\xi, \eta, t) = -\epsilon - \epsilon \frac{2s_n(\mu_n - \lambda_n)^2}{\mu_n \lambda_n} \frac{e^{\Delta F(\mu_n, \lambda_n)}}{(1 + s_n e^{\Delta F(\mu_n, \lambda_n)})^2}. \quad (20)$$

It is remarkable that for the case $N = 2$ of two paired terms in (15) with spectral variables (μ_1, λ_1) , (μ_n, λ_n) , $(n = 2, \dots, N)$ the choice $w = s_1 s_n$ in (17) i. e.

$$\frac{(\lambda_1 - \lambda_n)(\lambda_n + \mu_1)(\mu_1 - \mu_n)(\lambda_1 + \mu_n)}{(\lambda_1 + \lambda_n)(\lambda_n - \mu_1)(\mu_1 + \mu_n)(\lambda_1 - \mu_n)} = -1, \quad (21)$$

which is equivalent to relation

$$(\lambda_1 - \mu_1)(\lambda_n - \mu_n)(\lambda_1 \mu_1 + \lambda_n \mu_n) = 0, \quad n \neq 1, \quad (22)$$

the expression for $\det A$ (17) greatly simplifies

$$\det A = \left(1 + s_1 e^{\Delta F(\mu_1, \lambda_1)}\right)^2 \left(1 + s_n e^{\Delta F(\mu_n, \lambda_n)}\right)^2 \quad (23)$$

and via (14) leads to very simple formula for complex solution of NVN equation

$$u(\xi, \eta, t) = -\epsilon - 2\epsilon \sum_{m=1, n} \frac{s_m(\mu_m - \lambda_m)^2}{\mu_m \lambda_m} \frac{e^{\Delta F(\mu_m, \lambda_m)}}{(1 + s_m e^{\Delta F(\mu_m, \lambda_m)})^2}. \quad (24)$$

This solution modulo ϵ is the sum $u = \epsilon + u^{(1)} + u^{(n)}$ of two solutions $u^{(1)}$ and $u^{(n)}$ (20).

The solutions $\mu_1 = \lambda_1$ and $\mu_n = \lambda_n$ of (22) correspond to lumps (rationally decreasing at infinity exact solutions u [20, 21]) and will not be considered here. Below it is assumed that $\mu_n \neq \lambda_n$ for all terms $n = 1, \dots, N$ with discrete spectral variables μ_n, λ_n in (15). Under this requirement the relations (21), (22) reduce to more simple one:

$$\lambda_n \mu_n + \lambda_1 \mu_1 = 0, \quad n = 2, \dots, N. \quad (25)$$

The relations (25) with arbitrary $n \neq 1$ guarantee that $u(\xi, \eta, t) = -\epsilon + \tilde{u}^{(1)} + \tilde{u}^{(n)}$, as nonlinear superpositions (24) of two solutions $u^{(1)}(\xi, \eta, t) = -\epsilon + \tilde{u}^{(1)}$ and $u^{(n)}(\xi, \eta, t) = -\epsilon + \tilde{u}^{(n)}$, $(n = 2, \dots, N)$ of type (20), are exact solutions of NVN equation. This result can be extended to arbitrary number $N > 2$ of paired terms in the sum (15). More exactly the following lemma was proved.

Lemma. The nonlinear superposition

$$u(\xi, \eta, t) = -\epsilon + \tilde{u}^{(1)}(\xi, \eta, t) + \sum_{m=2}^N \tilde{u}^{(m)}(\xi, \eta) = -\epsilon - 2\epsilon \frac{s_1(\mu_1 - \lambda_1)^2}{\mu_1 \lambda_1} \frac{e^{\Delta F(\mu_1, \lambda_1)}}{(1 + s_1 e^{\Delta F(\mu_1, \lambda_1)})^2} - \\ - 2\epsilon \sum_{m=2}^N \frac{s_m(\mu_m - \lambda_m)^2}{\mu_m \lambda_m} \frac{e^{\Delta F(\mu_m, \lambda_m)}}{(1 + s_m e^{\Delta F(\mu_m, \lambda_m)})^2} = (N-1)\epsilon + u^{(1)}(\xi, \eta, t) + \sum_{m=2}^N u^{(m)}(\xi, \eta) \quad (26)$$

of arbitrary number $N > 2$ of complex solutions: nonstationary solution $u^{(1)}(\xi, \eta, t) = -\epsilon + \tilde{u}^{(1)}(\xi, \eta, t)$ of the type (20) with nonstationary phase

$$\varphi_1(\xi, \eta, t) := \Delta F(\mu_1, \lambda_1)|_{\kappa_1 \lambda_1^3 = \kappa_2 \frac{\epsilon^3}{\mu_1^3}} = i \left[(\mu_1 - \lambda_1)\xi - \left(\frac{\epsilon}{\mu_1} - \frac{\epsilon}{\lambda_1} \right) \eta - 2 \left(\kappa_1 \lambda_1^3 - \kappa_2 \frac{\epsilon^3}{\lambda_1^3} \right) t \right], \quad (27)$$

and of $N - 1$ stationary solutions $u^{(m)}(\xi, \eta) = -\epsilon + \tilde{u}^{(m)}(\xi, \eta)$, $m = 2, \dots, N$ of the type (20) with stationary phases

$$\varphi_m(\xi, \eta) := \Delta F(\mu_m, \lambda_m)|_{\kappa_1 \lambda_1^3 = \kappa_2 \frac{\epsilon^3}{\mu_1^3}} = i \left[(\mu_m - \lambda_m) \xi - \left(\frac{\epsilon}{\mu_m} - \frac{\epsilon}{\lambda_m} \right) \eta \right], \quad (m = 2, \dots, N) \quad (28)$$

is also exact solution of NVN equation, when conditions (25) ($\lambda_n \mu_n + \lambda_1 \mu_1 = 0$, $n = 2, \dots, N$) on parameters λ_n , μ_n , λ_1 , μ_1 and additional restriction

$$\kappa_1 \lambda_1^3 - \kappa_2 \frac{\epsilon^3}{\mu_1^3} = 0 \quad (29)$$

on the parameters λ_1 and μ_1 are imposed.

Proof. Note at first that according to definitions (7), (18) time-dependent part of phase φ_1 in (27) arises due to restriction (29) on parameters λ_1 , μ_1 ; time-dependent parts of phases φ_m in (28) are absent due to both restrictions (25), (29) on parameters λ_n , μ_n , λ_1 , μ_1 .

The NVN equation (1) can be represented in the form:

$$N(u) = L(u) + \text{Bil}(u, u) = 0 \quad (30)$$

of the sum of linear $L(u)$ and nonlinear $N(u)$ parts

$$L(u) = u_t + \kappa_1 u_{\xi\xi\xi} + \kappa_2 u_{\eta\eta\eta}; \quad \text{Bil}(u, u) = 3\kappa_1 (u \partial_\eta^{-1} u_\xi)_\xi + 3\kappa_2 (u \partial_\xi^{-1} u_\eta)_\eta \quad (31)$$

with bilinear form

$$\text{Bil}(u, v) = 3\kappa_1 (u \partial_\eta^{-1} v_\xi)_\xi + 3\kappa_2 (u \partial_\xi^{-1} v_\eta)_\eta. \quad (32)$$

The proof of the fact that (26) is solution of NVN equation can be done by direct substitution (26) into (30) and regrouping arising terms:

$$N \left(-\epsilon + \sum_{k=1}^N \tilde{u}^{(k)} \right) = \sum_{k=1}^N N(u^{(k)}) + \sum_{k \neq j} \text{Bil}(\tilde{u}^{(k)}, \tilde{u}^{(j)}). \quad (33)$$

Here due to (26) $u^{(1)}(\xi, \eta, t) = -\epsilon + \tilde{u}^{(1)}(\xi, \eta, t)$ and $u^{(k)}(\xi, \eta) = -\epsilon + \tilde{u}^{(k)}(\xi, \eta)$ are correspondingly nonstationary and stationary ($k = 2, \dots, N$) solutions of NVN equation with phases (27) and (28).

Using formula (20), taking into account (27) and (28) and the relations between derivatives of phases

$$\frac{\partial \varphi_k}{\partial \eta} = \frac{\epsilon}{\lambda_k \mu_k} \frac{\partial \varphi_k}{\partial \xi}, \quad k = 1, \dots, N \quad (34)$$

one obtains due to (25)

$$\text{Bil}(\tilde{u}^{(1)}, \tilde{u}^{(k)}) = \frac{3\kappa_1}{\epsilon} (\lambda_k \mu_k + \lambda_1 \mu_1) (\tilde{u}^{(1)} \tilde{u}^{(k)})_\xi + 3\kappa_2 \epsilon \left(\frac{1}{\lambda_k \mu_k} + \frac{1}{\lambda_1 \mu_1} \right) (\tilde{u}^{(1)} \tilde{u}^{(k)})_\eta = 0, \quad (35)$$

here $k = 1, \dots, N$. Quite analogously one calculates for $k \neq m$ ($k, m = 2, \dots, N$)

$$\begin{aligned} \text{Bil}(\tilde{u}^{(k)}, \tilde{u}^{(m)}) &= 3 \left(\frac{\kappa_2 \epsilon^2}{\lambda_k^2 \mu_k^2} + \frac{\kappa_2 \epsilon^2}{\lambda_k \mu_k \lambda_m \mu_m} + \frac{\kappa_1 \lambda_k \mu_k}{\epsilon} + \frac{\kappa_1 \lambda_m \mu_m}{\epsilon} \right) (\tilde{u}^{(m)} \tilde{u}_\xi^{(k)}) + \\ &3 \left(\frac{\kappa_2 \epsilon^2}{\lambda_m^2 \mu_m^2} + \frac{\kappa_2 \epsilon^2}{\lambda_k \mu_k \lambda_m \mu_m} + \frac{\kappa_1 \lambda_k \mu_k}{\epsilon} + \frac{\kappa_1 \lambda_m \mu_m}{\epsilon} \right) (\tilde{u}_\xi^{(m)} \tilde{u}^{(k)}) = 6 \left(\frac{\kappa_2 \epsilon^2}{\lambda_1^2 \mu_1^2} - \frac{\kappa_1 \lambda_1 \mu_1}{\epsilon} \right) (\tilde{u}^{(k)} \tilde{u}^{(m)})_\xi = 0 \end{aligned} \quad (36)$$

here in derivation of (35) and (36) the role of constraints (25) and (29) on parameters (λ_1, μ_1) and (λ_k, μ_k), $k = 2, \dots, N$ appears clearly.

Due to (33), (35) and (36) one derives the required result of **Lemma** for nonlinear superposition (26).

Similar to the arguments used in proof of basic **Lemma** of present section one can prove also that every subsum of arbitrary terms $1 \leq i < i+1 < \dots < j-1 < j \leq N$ of sum (26)

$$u = -\epsilon + \sum_{n=i}^j \tilde{u}^{(n)} \quad (37)$$

under conditions (25) and (29) is exact solution of NVN equation. Simple nonlinear superpositions given by (37) of complex solutions $u^{(n)}$ of NVN equation due to (26) and (27), (28) can be divided on two classes: the class of nonstationary solutions with $i \geq 1$ and class of stationary solutions with $i \geq 2$.

In the limit $u(\xi, \eta, t)|_{\xi^2+\eta^2 \rightarrow \infty} = -\epsilon \rightarrow 0$ of zero asymptotic values for potential of the first auxiliary problem (3) nonlinear superposition (26) converts to linear one, in the form of the sum of exact solutions. Underline that basic **Lemma** of present section is proved in general position and can be easily specialized, by corresponding choices of parameters, to calculations of real solutions for both cases of $\bar{\partial}$ -dressing for nonzero and zero asymptotic values of potential.

In recent paper [25] the case of $\bar{\partial}$ -dressing for zero asymptotic values for potential ($\bar{\partial}$ -dressing on zero energy level) by special limiting procedure is treated in detail and corresponding exact real solutions of elliptic version of Nizhnik-Veselov-Novikov (NVN-I or Veselov-Novikov) equation in the form of sums of arbitrary number of simple line solitons and in the form of sums of simple plane wave type singular periodic solutions with specially chosen parameters are calculated.

In following sections 3 and 4 analogous results for hyperbolic version of Nizhnik-Veselov-Novikov (or for NVN-II equation) are presented and corresponding exact solutions are calculated. The constructed exact solutions of NVN-II equation are also the exact potentials of one-dimensional Klein-Gordon equation and/or exact potentials of perturbed string equation.

3. Nonlinear and linear superpositions of line soliton solutions for hyperbolic version of the Nizhnik-Veselov-Novikov equation

For construction of real multi-line solitons via (14) besides potentiality condition satisfied by the kernel R_0 of the type (15) the reality condition $u = \bar{u}$ for solutions u must be fulfilled. This can be done choosing appropriately complex constants a_n and complex discrete spectral parameters (μ_n, λ_n) in (15) – (37) by several ways [22, 24]. For example, by imposing reality condition $u = \bar{u}$ on complex solutions (20), (24), (26) and (37) with additional assumption of real phases $\Delta F(\mu_n, \lambda_n) = \overline{\Delta F(\mu_n, \lambda_n)}$ (18) we have calculated real multi-line soliton solutions.

It was shown in the papers [22, 24] that to such real solutions u leads the following choice of parameters

$$a_n = -\bar{a}_n := -ia_{n0}, \quad \mu_n = -\bar{\mu}_n := i\mu_{n0}, \quad \lambda_n = -\bar{\lambda}_n := i\lambda_{n0}, \quad n = 1, \dots, N. \quad (38)$$

with real constants a_{n0} , μ_{n0} , λ_{n0} . Due to (38) and under additional assumption of positive values of real constants s_n given by (18)

$$s_n = a_{n0} \frac{\mu_{n0} + \lambda_{n0}}{\mu_{n0} - \lambda_{n0}} = e^{\phi_{0n}} > 0, \quad n = 1, \dots, N, \quad (39)$$

the solution (20) corresponding to one arbitrary pair (μ_n, λ_n) , $(n = 1, \dots, N)$ of discrete spectral variables takes the form of real nonsingular one-line soliton solution:

$$u^{(n)}(x, y, t) = -\epsilon + \tilde{u}^{(n)}(x, y, t) = -\epsilon - \frac{\epsilon(\lambda_{n0} - \mu_{n0})^2}{2\lambda_{n0}\mu_{n0}} \frac{1}{\cosh^2 \frac{\varphi_n(x, y, t) + \phi_{0n}}{2}} \quad (40)$$

with real phases

$$\varphi_n(x, y, t) := \Delta F(\mu_n, \lambda_n) = (\lambda_{n0} - \mu_{n0}) \sqrt{2 + \frac{2\epsilon^2}{\mu_{n0}^2 \lambda_{n0}^2}} (\vec{N}_n \vec{r} - V_n t) \quad (41)$$

where $\vec{r} = (x, y)$ and \vec{N}_n unit vectors of normals to lines of constant values of phases $\tilde{\varphi}_n(x, y, t)$, V_n are corresponding velocities of one-line solitons

$$\vec{N}_n = \frac{\left(1 - \frac{\epsilon}{\mu_{n0}\lambda_{n0}}, 1 + \frac{\epsilon}{\mu_{n0}\lambda_{n0}}\right)}{\sqrt{2 + \frac{2\epsilon^2}{\mu_{n0}^2\lambda_{n0}^2}}}, \quad V_n = \frac{(\lambda_{n0}^3 - \mu_{n0}^3) \left(\kappa_1 - \kappa_2 \frac{\epsilon^3}{\lambda_{n0}^3\mu_{n0}^3}\right)}{(\lambda_{n0} - \mu_{n0}) \sqrt{2 + \frac{2\epsilon^2}{\mu_{n0}^2\lambda_{n0}^2}}}, \quad n = 1, \dots, N. \quad (42)$$

For the cases of nonlinear superpositions (24) ($N = 2$), (26) and (37) ($N \geq 2$) of exact solutions of the type (40) the conditions (25) for discrete spectral parameters (μ_n, λ_n) , ($n > 1$) due to (38) lead to following relation

$$\mu_{10}\lambda_{10} + \mu_{n0}\lambda_{n0} = 0, \quad n = 2, \dots, N. \quad (43)$$

Nonsingular two-line soliton solution characterized by two pairs of discrete spectral variables (μ_1, λ_1) and (μ_2, λ_2) due to (24), (38), (39) and (43) takes the form

$$u(x, y, t) = -\epsilon + \sum_{n=1}^2 \tilde{u}^{(n)}(x, y, t) = -\epsilon - \sum_{n=1}^2 \frac{\epsilon(\lambda_{n0} - \mu_{n0})^2}{2\lambda_{n0}\mu_{n0}} \frac{1}{\cosh^2 \frac{\varphi_n(x, y, t) + \phi_{0n}}{2}}, \quad (44)$$

where $u^{(n)}(x, y, t) = -\epsilon + \tilde{u}^{(n)}(x, y, t)$, ($n = 1, 2$) are one-line soliton solutions of the type (40), the phases $\varphi_n(x, y, t)$, ($n = 1, 2$) under conditions (43) are given by (41).

Due to expressions for vectors of normals (42) and condition (43) it is evident that solitons $u^{(1)}$ and $u^{(2)}$ of superposition (44) move in the plane (x, y) perpendicularly to each other.

One of two one-line solitons $u^{(1)}$ or $u^{(2)}$ (not both) of superposition (44) due to (42) and (43) can be "stopped", i. e. by special choice of spectral parameter λ_1 one can take ones of velocities $V_1 = 0$ or $V_2 = 0$ equal to zero. For example one can choose $V_2 = 0$, this achieves by the use of (42) and (43) for λ_1 satisfying to condition

$$\lambda_{10}\mu_{10} = -\epsilon \sqrt[3]{\frac{\kappa_2}{\kappa_1}}, \quad (45)$$

The results of second section for simple nonlinear superpositions of $N > 2$ complex solutions $u^{(n)}$, ($n = 2, \dots, N$) can be easily specialized to the case of simple nonlinear superpositions of real one-line solitons (40), such superpositions are given by (26) with parameters a_n , (μ_n, λ_n) and s_n satisfying to (38), (39), the conditions (25) or (43) and (45) also must be fulfilled. Basic **Lemma** of previous section works fully also for real multi-line soliton solutions because basic relations (34), (35) and (36) of **Lemma** remain valid for special choice of parameters (μ_1, λ_1) , (μ_n, λ_n) given by the formulas (39), (43) and (45) of present section. The condition (29) due to (38) leads to relation (45), so the simple nonlinear superposition (26) takes in considered case the following real form

$$u(x, y, t) = -\epsilon + \tilde{u}^{(1)}(x, y, t) + \sum_{n=2}^N \tilde{u}^{(n)}(x, y) = -\epsilon - \frac{\epsilon(\lambda_{10} - \mu_{10})^2}{2\lambda_{10}\mu_{10}} \frac{1}{\cosh^2 \frac{\varphi_1(x, y, t) + \phi_{01}}{2}} - \sum_{n=2}^N \frac{\epsilon(\lambda_{n0} - \mu_{n0})^2}{2\lambda_{n0}\mu_{n0}} \frac{1}{\cosh^2 \frac{\varphi_n(x, y) + \phi_{0n}}{2}}, \quad (46)$$

with phases φ_n (27), (28) given due to (38), (39) and (43), (45) by expressions

$$\varphi_1(x, y, t) = (\lambda_{10} - \mu_{10}) \sqrt{2 + \frac{2\epsilon^2}{\mu_{10}^2\lambda_{10}^2}} (\vec{N}_1 \vec{r} - V_1 t), \quad \varphi_n(x, y) = (\lambda_{n0} - \mu_{n0}) \sqrt{2 + \frac{2\epsilon^2}{\mu_{10}^2\lambda_{10}^2}} (\vec{N}_n \vec{r}), \quad (47)$$

In formulas (47) $n = 2, \dots, N$, $\vec{r} = (x, y)$; the unit vectors of normals \vec{N}_1 , \vec{N}_2 and velocity V_1 are given by following formulas

$$\vec{N}_n = \frac{\left(1 - \frac{\epsilon}{\mu_{n0}\lambda_{n0}}, 1 + \frac{\epsilon}{\mu_{n0}\lambda_{n0}}\right)}{\sqrt{2 + \frac{2\epsilon^2}{\mu_{10}^2\lambda_{10}^2}}}, \quad V_1 = \frac{2\kappa_1 (\lambda_{10}^3 - \mu_{10}^3)}{(\lambda_{10} - \mu_{10}) \sqrt{2 + \frac{2\epsilon^2}{\mu_{10}^2\lambda_{10}^2}}}. \quad (48)$$

One-line soliton $u^{(1)}(x, y, t) = -\epsilon + \tilde{u}^{(1)}(x, y, t)$ of nonlinear superposition (46) due to (47) and (48) moves in the plane (x, y) perpendicularly to others stationary solitons $u^{(n)}(x, y) = -\epsilon + \tilde{u}^{(n)}(x, y)$, $(n = 2, \dots, N)$ of this superposition with parallel lines of constant values of phases $\varphi_n(x, y)$. Evidently particular case of (44) with $V_2 = 0$ due to (45) coincides with two-line soliton (for $N = 2$) nonlinear superposition (46).

It was shown in the papers [24, 22] that the limiting procedure for calculation of exact solutions u of NVN equation with zero asymptotic values at infinity $u|_{\xi^2 + \eta^2 \rightarrow \infty} = -\epsilon \rightarrow 0$ can be defined by the following way: it is supposed at first that $\epsilon := c_n \mu_{n0}$ with some real constants c_n , and due to (43) one have

$$\mu_{n0} \rightarrow 0, \quad \frac{\mu_{n0}}{\mu_{m0}} = \frac{c_m}{c_n} \neq 0, \quad \lambda_{n0} = -\frac{\lambda_{10} c_n}{c_1}. \quad (49)$$

It is assumed that under procedure (49) the relations (43) remain to be valid.

In the limit (49) two-line soliton solution (44) converts to linear superposition

$$u(x, y, t) = u_{\epsilon=0}^{(1)} + u_{\epsilon=0}^{(2)} = -\sum_{n=1}^2 \frac{c_n \lambda_{n0}}{2 \cosh^2 \frac{\tilde{\varphi}_n(x, y, t) + \phi_{0n}}{2}}, \quad (50)$$

of two one-line solitons $u_{\epsilon=0}^{(1)}$ and $u_{\epsilon=0}^{(2)}$

$$u_{\epsilon=0}^{(n)}(x, y, t) = \frac{c_n \lambda_{n0}}{2 \cosh^2 \frac{\tilde{\varphi}_n(x, y, t) + \phi_{0n}}{2}}, \quad n = 1, 2, \quad (51)$$

with phases $\tilde{\varphi}_n(x, y, t)$ given due to (41) and (49) by formulas

$$\tilde{\varphi}_n(x, y, t) = \frac{\lambda_{n0}}{|\lambda_{10}|} \sqrt{2\lambda_{10}^2 + 2c_1^2} (\vec{N}_n \vec{r} - V_n t), \quad (52)$$

Here $\vec{r} = (x, y)$, \vec{N}_n are unit vectors of normals to lines of constant values of phases $\tilde{\varphi}_n(x, y, t)$; V_n are corresponding velocities of one-line solitons

$$\vec{N}_n = \frac{|\lambda_{10}| \left(1 - \frac{c_n}{\lambda_{n0}}, 1 + \frac{c_n}{\lambda_{n0}}\right)}{\sqrt{2\lambda_{10}^2 + 2c_1^2}}, \quad V_n = \frac{|\lambda_{10}| (\kappa_1 \lambda_{n0}^3 - \kappa_2 c_n^3)}{\lambda_{n0} \sqrt{2\lambda_{10}^2 + 2c_1^2}}. \quad (53)$$

derived by the use of (42), (43) and (49). By special choice of spectral parameter λ_1 one of two one-line solitons $u_{\epsilon=0}^{(1)}$ or $u_{\epsilon=0}^{(2)}$ (not both) in linear superposition (50) can be "stopped".

Basic **Lemma** of previous section works fully also for considered limit because basic relations (34), (35) and (36) of **Lemma** remain valid for special choice of parameters (μ_1, λ_1) , (μ_n, λ_n) given by the formulas (38), (39), (43) and (45) and the rules (49) of zero limit $\epsilon \rightarrow 0$. In the limit $\epsilon \rightarrow 0$ following to the rules (49), under requirement that relations (43) and (45) remain to be valid, we obtain from (46) linear superposition of N one-line solitons

$$u = u_{\epsilon=0}^{(1)}(x, y, t) + \sum_{n=2}^N u_{\epsilon=0}^{(n)}(x, y) = -\frac{c_1 \lambda_{10}}{2 \cosh^2 \frac{\tilde{\varphi}_1(x, y, t) + \phi_{01}}{2}} - \sum_{n=2}^N \frac{c_n \lambda_{n0}}{2 \cosh^2 \frac{\tilde{\varphi}_n(x, y, t) + \phi_{0n}}{2}}, \quad (54)$$

here the phases $\tilde{\varphi}_n$ given by expressions

$$\tilde{\varphi}_1(x, y, t) = \frac{\lambda_{10}}{|\lambda_{10}|} \sqrt{2\lambda_{10}^2 + 2c_1^2} (\vec{N}_1 \vec{r} - V_1 t), \quad \tilde{\varphi}_n(x, y) = \frac{\lambda_{n0}}{|\lambda_{10}|} \sqrt{2\lambda_{10}^2 + 2c_1^2} (\vec{N}_n \vec{r}). \quad (55)$$

are obtained from phases φ_n (47) in limit $\epsilon \rightarrow 0$ (49). The first one-line soliton $u_{\epsilon=0}^{(1)}(x, y, t)$ from linear superposition (54) moves with velocity $V_1 = \frac{2\kappa_1 |\lambda_{10}| \lambda_{10}^2}{\sqrt{2\lambda_{10}^2 + 2c_1^2}}$ in the plane (x, y) perpendicularly to other $(N - 1)$ stationary one-line solitons $u_{\epsilon=0}^{(n)}(x, y)$.

Following to ideas used in proof of basic **Lemma** of second section one can show that the subsums of arbitrary numbers of solitons $u_{\epsilon=0}^{(n)}$, $(n = 1, \dots, N)$ from (54) are also solutions of NVN equation. So the set of such solutions can be divided in two subsets: subset of nonstationary linear superpositions (with the first moving line soliton $u_{\epsilon=0}^{(1)}(x, y, t)$ in the sum) of line solitons and subset of stationary linear superpositions (without moving line soliton $u_{\epsilon=0}^{(1)}(x, y, t)$ in the sum) of stationary line solitons.

4. Nonlinear and linear superpositions of plane wave type periodic solutions for hyperbolic version of the Nizhnik-Veselov-Novikov equation

Multi-line soliton solutions are calculated in preceding section by imposing reality condition $u = \bar{u}$ on complex solutions (20), (24), (26) and (37) with additional assumption of real phases $\Delta F(\mu_n, \lambda_n) = \overline{\Delta F(\mu_n, \lambda_n)}$ (18). In contrast the application of reality condition $u = \bar{u}$ with assumption of pure imaginary phases $\Delta F(\mu_n, \lambda_n) = -\overline{\Delta F(\mu_n, \lambda_n)}$ (18) leads to plane wave type periodic solutions and their's superpositions.

Plane wave type solutions can be obtained by this way for example by the following choice of parameters $a_n, (\mu_n, \lambda_n)$ and s_n in (15) – (37) [24, 22]:

$$a_n = \left| \frac{\lambda_{n0} - \mu_{n0}}{\lambda_{n0} + \mu_{n0}} \right| e^{i \arg a_n}, \quad \mu_n = \bar{\mu}_n := \mu_{n0}, \quad \lambda_n = \bar{\lambda}_n := \lambda_{n0}, \quad s_n = -ie^{i \arg a_n} \text{sign} \left(\frac{\lambda_n - \mu_n}{\lambda_n + \mu_n} \right) \quad (56)$$

where $n = 1, \dots, N$. Simple plane wave type periodic solution corresponding to one arbitrary pair of spectral variables (μ_n, λ_n) , $(n = 1, \dots, N)$, due to (20) and (56) takes the form

$$u^{(n)}(x, y, t) = -\epsilon + \tilde{u}^{(n)}(x, y, t) = -\epsilon - \frac{\epsilon(\lambda_{n0} - \mu_{n0})^2}{2\lambda_{n0}\mu_{n0}} \frac{1}{\cos^2 \left(\frac{\varphi_n(x, y, t) + \arg a_n}{2} \mp \frac{\pi}{4} \right)} \quad (57)$$

where sign "–" correspond to case of $\frac{\lambda_{n0} - \mu_{n0}}{\lambda_{n0} + \mu_{n0}} > 0$ and sign "+" – case of $\frac{\lambda_{n0} - \mu_{n0}}{\lambda_{n0} + \mu_{n0}} < 0$. The real phases $\varphi_n(x, y, t) := -i\Delta F(\mu_n, \lambda_n)$ in (57) due to (18) and (56) are given by expressions

$$\varphi_n(x, y, t) = (\lambda_{n0} - \mu_{n0}) \sqrt{2 + \frac{2\epsilon^2}{\mu_{n0}^2 \lambda_{n0}^2}} (\vec{N}_n \vec{r} - V_n t), \quad n = 1, \dots, N. \quad (58)$$

In (57), (58) $\vec{r} = (x, y)$; \vec{N}_n are unit vectors of normals to lines of constant values of phases $\varphi_n(x, y, t)$ and velocities V_n of periodic solutions are given by expressions:

$$\vec{N}_n = \frac{\left(1 + \frac{\epsilon}{\mu_{n0}\lambda_{n0}}, 1 - \frac{\epsilon}{\mu_{n0}\lambda_{n0}} \right)}{\sqrt{2 + \frac{2\epsilon^2}{\mu_{n0}^2 \lambda_{n0}^2}}}, \quad V_n = -\frac{(\lambda_{n0}^3 - \mu_{n0}^3) \left(\kappa_1 + \kappa_2 \frac{\epsilon^3}{\lambda_{n0}^3 \mu_{n0}^3} \right)}{(\lambda_{n0} - \mu_{n0}) \sqrt{2 + \frac{2\epsilon^2}{\mu_{n0}^2 \lambda_{n0}^2}}}, \quad n = 1, \dots, N. \quad (59)$$

Using general formulas (24) and (26) we also construct nonlinear superpositions of simple wave type periodic solutions of the type (57). The conditions (25) for discrete spectral parameters (μ_n, λ_n) , $(n > 1)$ in nonlinear superpositions (24) and (26) due to (56) in considered case lead to following conditions

$$\mu_{10}\lambda_{10} + \mu_{n0}\lambda_{n0} = 0, \quad n = 2, \dots, N. \quad (60)$$

Nonlinear superposition (24) of two simple plane wave type periodic solutions of the type (57) due to (56) and (60) takes the form

$$u(x, y, t) = -\epsilon + \sum_{n=1}^2 \tilde{u}^{(n)}(x, y, t) = -\epsilon - \sum_{n=1}^2 \frac{\epsilon(\lambda_{n0} - \mu_{n0})^2}{2\lambda_{n0}\mu_{n0}} \frac{1}{\cos^2 \left(\frac{\varphi_n(x, y, t) + \arg a_n}{2} \mp \frac{\pi}{4} \right)}, \quad (61)$$

where sign "–" correspond to case of $\frac{\lambda_{n0} - \mu_{n0}}{\lambda_{n0} + \mu_{n0}} > 0$ and sign "+" – case of $\frac{\lambda_{n0} - \mu_{n0}}{\lambda_{n0} + \mu_{n0}} < 0$. The phases $\varphi_n(x, y, t)$ in solution (61) are given by (61) under conditions (60). Due to expressions for vectors of normals (59) and to parametrization (60) it is evident that lines of constant values of phases $\varphi_n(x, y, t)$, $(n = 1, 2)$ for solutions $u^{(1)} = -\epsilon + \tilde{u}^{(1)}$ and $u^{(2)} = -\epsilon + \tilde{u}^{(2)}$ of superposition (61) move perpendicularly to each other.

One of two simple plane wave type periodic solutions $u^{(1)}$ or $u^{(2)}$ (not both) of superposition (61) due to (59) and (60) can be made stationary, i. e. by special choice of spectral parameter λ_1 one can take ones of velocities $V_1 = 0$ or $V_2 = 0$ equal to zero. For example one can choose $V_2 = 0$, this achieves due to (59) and (60) for λ_1 satisfying to condition

$$\lambda_{10}\mu_{10} = -\epsilon \sqrt[3]{\frac{\kappa_2}{\kappa_1}}, \quad (62)$$

The results of second section for simple nonlinear superpositions of $N > 2$ complex solutions $u^{(n)}$, ($n = 2, \dots, N$) can be easily specialized to the case of simple nonlinear superpositions of real plane wave type periodic solutions (57) with parameters a_n , (μ_n, λ_n) and s_n satisfying to (56), the conditions (25) or (60) and (62) also must be fulfilled. Basic **Lemma** of second section works fully also for real solutions of considered plane wave periodic type because basic relations (34), (35) and (36) of **Lemma** remain valid for special choice of parameters (μ_1, λ_1) , (μ_n, λ_n) given by the formulas (56), (60) and (62) of present section. The condition (29) due to (56) transforms into (62), so the solution (26) takes the form

$$u(x, y, t) = -\epsilon - \frac{\epsilon(\lambda_{10} - \mu_{10})^2}{2\lambda_{10}\mu_{10}} \frac{1}{\cos^2\left(\frac{\varphi_1(x, y, t) + \arg a_1}{2} \mp \frac{\pi}{4}\right)} - \sum_{n=2}^N \frac{\epsilon(\lambda_{n0} - \mu_{n0})^2}{2\lambda_{n0}\mu_{n0}} \frac{1}{\cos^2\left(\frac{\varphi_n(x, y) + \arg a_n}{2} \mp \frac{\pi}{4}\right)}, \quad (63)$$

where sign "−" correspond to case of $\frac{\lambda_{n0} - \mu_{n0}}{\lambda_{n0} + \mu_{n0}} > 0$ and sign "+" – case of $\frac{\lambda_{n0} - \mu_{n0}}{\lambda_{n0} + \mu_{n0}} < 0$. The phases φ_n (27), (28) in superposition (63) due to (56) and (60), (62) are given by expressions

$$\varphi_1(x, y, t) = (\lambda_{10} - \mu_{10}) \sqrt{2 + \frac{2\epsilon^2}{\mu_{10}^2 \lambda_{10}^2}} (\vec{N}_1 \vec{r} - V_1 t), \quad \varphi_n(x, y) = (\lambda_{n0} - \mu_{n0}) \sqrt{2 + \frac{2\epsilon^2}{\mu_{10}^2 \lambda_{10}^2}} (\vec{N}_n \vec{r}), \quad (64)$$

where $n = 2, \dots, N$, $\vec{r} = (x, y)$; unit vectors \vec{N}_n and velocity V_1 are given by following formulas

$$\vec{N}_n = \frac{\left(1 + \frac{\epsilon}{\mu_{n0}\lambda_{n0}}, 1 - \frac{\epsilon}{\mu_{n0}\lambda_{n0}}\right)}{\sqrt{2 + \frac{2\epsilon^2}{\mu_{10}^2 \lambda_{10}^2}}}, \quad V_1 = \frac{-2\kappa_1 (\lambda_{10}^3 - \mu_{10}^3)}{(\lambda_{10} - \mu_{10}) \sqrt{2 + \frac{2\epsilon^2}{\mu_{10}^2 \lambda_{10}^2}}}. \quad (65)$$

The lines of constant values of phase $\varphi_1(x, y, t)$ of simple periodic solution $u^{(1)}(x, y, t) = -\epsilon + \tilde{u}^{(1)}(x, y, t)$ of superposition (63) move in plane (x, y) perpendicularly to parallel lines of constant phases $\varphi_n(x, y)$ of others stationary periodic solutions $u^{(n)}(x, y, t) = -\epsilon + \tilde{u}^{(n)}(x, y, t)$, ($n = 2, \dots, N$) of this superposition. Evidently particular case of (61) with $V_2 = 0$ coincides due to (62) with the case $N = 2$ of linear superposition (63).

It was shown in the papers [24, 22] that the limiting procedure of calculation of exact solutions u of NVN equation with zero values of parameter $\epsilon = 0$ ($\bar{\partial}$ -dressing for solutions with zero asymptotic values at infinity) can be defined by the following way

$$\epsilon = c_n \mu_{n0} \rightarrow 0, \quad \frac{\mu_{n0}}{\mu_{m0}} = \frac{c_m}{c_n} \neq 0, \quad \lambda_{n0} = -\frac{\lambda_{10} c_n}{c_1}. \quad (66)$$

Such procedure is applicable also in considered case of plane wave type solutions and their's superpositions. It is assumed that under procedure (66) the relations (60) remain to be valid.

In the limit (66) nonlinear superposition (61) of two plane wave type periodic solutions converts to linear superposition

$$u(x, y, t) = u_{\epsilon=0}^{(1)} + u_{\epsilon=0}^{(2)} = - \sum_{n=1}^2 \frac{c_n \lambda_{n0}}{2 \cos^2\left(\frac{\tilde{\varphi}_n(x, y, t) + \arg a_n}{2} - \frac{\pi}{4}\right)}, \quad (67)$$

of two periodic solitons $u_{\epsilon=0}^{(1)}$ and $u_{\epsilon=0}^{(2)}$

$$u_{\epsilon=0}^{(n)}(x, y, t) = - \frac{c_n \lambda_{n0}}{2 \cos^2\left(\frac{\tilde{\varphi}_n(x, y, t) + \arg a_n}{2} - \frac{\pi}{4}\right)}, \quad n = 1, 2, \quad (68)$$

the phases $\tilde{\varphi}_n(x, y, t)$ in (67), (68) are given due to (58) and (66) by formulas

$$\varphi_n(x, y, t) = \frac{\lambda_{n0}}{|\lambda_{10}|} \sqrt{2\lambda_{10}^2 + 2c_1^2} (\vec{N}_n \vec{r} - V_n t), \quad (69)$$

here $\vec{r} = (x, y)$. Unit vectors of normals to lines of constant values of phases $\tilde{\varphi}_n(x, y, t) - \vec{N}_n$ and velocities V_n of simple plane wave type periodic solutions (68) are derived by the use of (59), (60) and (66) and are given by formulas

$$\vec{N}_n = \frac{|\lambda_{10}| \left(1 + \frac{c_n}{\lambda_{n0}}, 1 - \frac{c_n}{\lambda_{n0}}\right)}{\sqrt{2\lambda_{10}^2 + 2c_1^2}}, \quad V_n = -\frac{|\lambda_{10}| (\kappa_1 \lambda_{n0}^3 + \kappa_2 c_n^3)}{\lambda_{n0} \sqrt{2\lambda_{10}^2 + 2c_1^2}}. \quad (70)$$

By special choice of spectral parameter λ_1 one of two of these periodic solutions, $u_{\epsilon=0}^{(1)}$ or $u_{\epsilon=0}^{(2)}$ (not both), in linear superposition (67) can be made stationary.

Basic **Lemma** of second section works fully also for limit of zero ϵ of considered in present section real plane wave periodic solutions because basic relations (34), (35) and (36) of **Lemma** remain valid for special choice of parameters (μ_1, λ_1) , (μ_n, λ_n) given by the formulas (56), (60), (62) and the rules (66) of zero limit $\epsilon \rightarrow 0$. In the limit $\epsilon \rightarrow 0$ ($\bar{\partial}$ -dressing for solutions with zero asymptotic values at infinity) following to the rules (66), with assumption that relations (60) and (62) remain to be valid, we obtain from (63) linear superposition of N simple plane wave type periodic solutions

$$u = u_{\epsilon=0}^{(1)}(x, y, t) + \sum_{n=2}^N u_{\epsilon=0}^{(n)}(x, y) = -\frac{c_1 \lambda_{10}}{2 \cos^2 \left(\frac{\tilde{\varphi}_1(x, y, t) + \arg a_1}{2} - \frac{\pi}{4} \right)} - \sum_{n=2}^N \frac{c_n \lambda_{n0}}{2 \cos^2 \left(\frac{\tilde{\varphi}_n(x, y) + \arg a_n}{2} - \frac{\pi}{4} \right)}, \quad (71)$$

here phases $\tilde{\varphi}_n$ are obtained from phases φ_n (64) by setting $\epsilon = 0$ in accordance with (66), these phases have the forms:

$$\tilde{\varphi}_1(x, y, t) = \frac{\lambda_{10}}{|\lambda_{10}|} \sqrt{2\lambda_{10}^2 + 2c_1^2} (\vec{N}_1 \vec{r} - V_1 t), \quad \tilde{\varphi}_n(x, y) = \frac{\lambda_{n0}}{|\lambda_{10}|} \sqrt{2\lambda_{10}^2 + 2c_1^2} (\vec{N}_n \vec{r}), \quad n = 2, \dots, N. \quad (72)$$

The lines of constant values of phase $\varphi_1(x, y, t)$ of the first periodic solution $u_{\epsilon=0}^{(1)}$ from linear superposition (71) move with velocity $V_1 = -\frac{2\kappa_1 |\lambda_{10}|^3}{\sqrt{2\lambda_{10}^2 + 2c_1^2}}$ perpendicularly to lines of constant phases $\varphi_n(x, y)$ of $(N - 1)$ other stationary periodic solutions $u_{\epsilon=0}^{(n)}$, ($n = 2, \dots, N$) of this superposition.

Following to ideas used in proof of basic **Lemma** of second section one can show that the subsums of arbitrary numbers of solutions $u_{\epsilon=0}^{(n)}$, ($n = 1, \dots, N$) from (71) are also solutions of NVN-II equation. So the set of constructed in present section solutions can be divided in two subsets: subset of nonstationary linear superpositions (with the first moving line soliton $u_{\epsilon=0}^{(1)}(x, y, t)$ in the sum) and subset of stationary linear superpositions (without moving line soliton $u_{\epsilon=0}^{(1)}(x, y, t)$ in the sum).

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