

# Projective differential geometry of multidimensional dispersionless integrable hierarchies

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*In the memory of S.V. Manakov*

**Abstract.** We introduce a general setting for multidimensional dispersionless integrable hierarchy in terms of differential  $m$ -form  $\Omega_m$  with the coefficients satisfying the Plücker relations, which is gauge-invariantly closed and its gauge-invariant coordinates (ratios of coefficients) are (locally) holomorphic with respect to one of the variables (the spectral variable). We demonstrate that this form defines a hierarchy of dispersionless integrable equations in terms of commuting vector fields locally holomorphic in the spectral variable. The equations of the hierarchy are given by the gauge-invariant closedness equations.

## 1. Introduction

In this work we develop further the ideas of the work [1] and introduce a general setting for multidimensional dispersionless integrable hierarchy in terms of some differential  $m$ -form  $\Omega_m$ , in the spirit of construction of universal Whitham hierarchy given in [2], which corresponds to the case of  $\Omega_2$  with a Hamiltonian reduction. The coefficients of this form should satisfy the Plücker relations (we call it a Plücker form), it should be gauge-invariantly closed and its gauge-invariant coordinates (ratios of coefficients) should be (locally) holomorphic with respect to one of the variables (the spectral variable). We demonstrate that this form defines a hierarchy of dispersionless integrable equations in terms of commuting vector fields locally holomorphic in the spectral variable. The equations of the hierarchy are given by the gauge-invariant closedness equations.

First we consider the correspondence between Plücker forms and distributions and demonstrate that involutivity of the distribution is equivalent to the gauge-invariant closedness equations for Plücker form.

Then we introduce a spectral variable, suggesting that gauge-independent coordinates of the form (ratios of coefficients) are holomorphic with respect to one of the variables and consider some simple examples of integrable systems arising from gauge-invariant closedness equations.



We demonstrate that nonlinear vector Riemann-Hilbert problem (or  $\bar{\partial}$ -problem) is a natural tool to construct gauge-invariantly closed Plücker forms holomorphic in the complex plane, and show how to obtain polynomial  $\Omega_m$  and the corresponding multidimensional dispersionless hierarchy.

## 2. Integrable distributions and closed Plücker forms

Let us consider a domain with a set of local coordinates  $\mathbf{x} = (x_0, x_1, \dots, x_N)$ . Distribution is a  $K$ -dimensional subspace of the tangent space  $\Delta_x \subset T_x$ , depending smoothly on  $\mathbf{x}$  (there exists a basis of smooth vector fields). Involutive distribution is defined by the relation  $[\Delta, \Delta] \subset \Delta$ , where the standard commutator of vector fields is used. According to Frobenius theorem, the distribution is integrable (corresponds to a foliation)  $\Leftrightarrow$  the distribution is in involution.

There are several dual formulations of Frobenius theorem in terms of differential forms, where the subspace in cotangent space dual to the distribution (codistribution) is considered. Here we will consider decomposable differential  $m = (N + 1 - K)$  forms (we will call them Plücker forms) which are in one-to-one correspondence (up to a gauge) with the codistribution, and will formulate the property of these forms which is equivalent to the involutivity of original  $K$ -dimensional distribution.

Let us define a Plücker form as an  $m$ -form

$$\Omega_m = \sum_{0 \leq i_0 \leq \dots \leq i_{m-1} \leq N} \pi_{i_0 i_1 \dots i_{m-1}}(x) dx_{i_0} \wedge dx_{i_1} \wedge \dots \wedge dx_{i_{m-1}}$$

with coefficients satisfying Plücker relations (see e.g. [3])

$$\sum_{l=0}^m (-1)^l \pi_{i_0 \dots i_{m-2} j_l} \pi_{j_0 \dots \check{j}_l \dots j_m} = 0,$$

where indices  $i_p$  and  $j_p$  range over all possible values from zero to  $N$  and the notation  $\check{j}_p$  means the omission of the index. Due to Plücker relations, the form  $\Omega_m$  is decomposable

$$\Omega_m = \omega_0 \wedge \dots \wedge \omega_{m-1},$$

defines a vector subspace in cotangent space and a distribution as a dual object. It is also easy to construct a Plücker form for a given distribution, it is defined up to a gauge. We will call the Plücker forms which differ only by a gauge (multiplication by some function) equivalent.

Having the correspondence between Plücker forms and distributions, it is natural to ask a question what property of Plücker form corresponds to the involutivity of the distribution. To have a geometrical meaning, this property should be gauge-invariant. The answer to this question can be found using the gauge-invariant closedness conditions for Plücker forms introduced in [1].

Let us consider standard closedness equations

$$\left[ \frac{\partial \pi_{i_0 i_1 \dots i_{m-1}}}{\partial x_{i_m}} \right] = 0, \quad (1)$$

where the bracket [...] means antisymmetrization over all indices. We consider a pair of equations with the choice of indices  $(0, 1, \dots, m-1, q)$ ,  $(0, 1, \dots, m-1, r)$ ,  $q, r \in \{1, \dots, K = N - m + 1\}$ ,  $q \neq r$ , introducing the basic set of gauge independent affine coordinates

$$a_{qk} = (-1)^k J^{-1} \pi_{0 \dots k-1 k+1 \dots m-1 m-1+q},$$

$$J = \pi_{0 1 \dots m-1}$$

where  $k \in \{0, \dots, m-1\}$ ,  $q \in \{1, \dots, K = N - m + 1\}$ . All the affine coordinates are expressed through the basic set using Plücker relations, and the closedness equations take the form

$$\frac{\partial J}{\partial x_{q+(m-1)}} + \sum_{l=0}^{m-1} \frac{\partial(Ja_{ql})}{\partial x_l} = 0, \quad \frac{\partial J}{\partial x_{r+(m-1)}} + \sum_{l=0}^{m-1} \frac{\partial(Ja_{rl})}{\partial x_l} = 0, \quad (2)$$

$$\frac{\partial a_{qk}}{\partial x_{r+(m-1)}} - \frac{\partial a_{rk}}{\partial x_{q+(m-1)}} + \sum_{l=0}^{m-1} \left( a_{rl} \frac{\partial a_{qk}}{\partial x_l} - a_{ql} \frac{\partial a_{rk}}{\partial x_l} \right) = 0. \quad (3)$$

Subsystem (3), invariant under the gauge transformations mentioned above, is the gauge invariant part of the system (1) for Plücker form [1]; we will call it gauge-invariant closedness equations for Plücker form. Equations (2) can be viewed as the equations for the gauge variable  $J$  which transforms as  $J \rightarrow \rho J$  under the gauge transformations. It is a straightforward check that equations (2) are compatible due to the subsystem (3). We note that the system (2), (3) can be rewritten in the form

$$D_q \ln J + \sum_{n=0}^{m-1} \frac{\partial a_{qn}}{\partial x_n} = 0, \quad D_r \ln J + \sum_{n=0}^{m-1} \frac{\partial a_{rn}}{\partial x_n} = 0, \quad (4)$$

$$D_r a_{qk} - D_q a_{rk} = 0, \quad k = 0, \dots, m-1 \quad (5)$$

where  $D_q$  and  $D_r$ ,  $q, r \in \{1, \dots, K\}$ , are vector fields

$$D_q = \frac{\partial}{\partial x_{q+(m-1)}} + \sum_{n=0}^{m-1} a_{qn} \frac{\partial}{\partial x_n}, \quad D_r = \frac{\partial}{\partial x_{r+(m-1)}} + \sum_{n=0}^{m-1} a_{rn} \frac{\partial}{\partial x_n}. \quad (6)$$

Equations of the type (4) were considered in [4], [5] as equations for the Jacobian of solutions of linear equations defined by vector fields. So the subsystem (3) is equivalent to the commutativity  $[D_q, D_r] = 0$  of vector fields  $D_q$  and  $D_r$ .

Considering the set of gauge-invariant closedness equations (3), we come to the following conclusions:

**Proposition 1** *The set of gauge-invariant closedness equations (3) is equivalent to the existence of a gauge, in which the Plücker form is closed in the standard sense.*

The gauge variable  $J$  corresponding to closed form is defined by linear equations (2).

**Proposition 2** *The Plücker form is gauge-invariantly closed  $\Leftrightarrow$  the corresponding distribution is in involution*

The Proposition 2 can be easily proved by purely geometrical means, using the Frobenius theorem and some well-known properties of differential forms. It is known that closed decomposable differential form has a decomposition in terms of exact 1-forms (see e.g. [6, 7]). The fact that the Plücker form is decomposable implies that (gauge invariantly) closed Plücker form possesses a decomposition into exact 1-forms (up to a gauge).

**Proposition 3** *Gauge-invariantly closed Plücker form can be represented as*

$$\Omega_m = g df_0 \wedge \dots \wedge df_{m-1}, \quad (7)$$

where  $g, f_0, \dots, f_{m-1}$  are some functions.

### Examples of closedness equations

Let us consider some simple examples of closedness equations (2), (3) for Plücker forms, for more detail see [1]. We will use the representation

$$\Omega_m = J\tilde{\Omega}_m = J(dx_0 \wedge \cdots \wedge dx_{m-1} + \dots), \quad (8)$$

where  $J = \pi_{01\dots m-1}$  is a gauge-dependent variable, and the Plücker form  $\tilde{\Omega}_m$  is gauge-independent, it may be considered as the initial form in affine gauge.

The simplest case corresponds to  $m = 1$ , let us take  $N = 2$ :

$$\Omega_1 = J(dx_0 - a_{10}dx_1 - a_{20}dx_2).$$

In this case we have no Plücker relations, and the closedness equations are

$$\frac{\partial J}{\partial x_1} + \frac{\partial(Ja_{10})}{\partial x_0} = 0, \quad \frac{\partial J}{\partial x_2} + \frac{\partial(Ja_{20})}{\partial x_0} = 0, \quad (9)$$

$$\frac{\partial a_{10}}{\partial x_2} - \frac{\partial a_{20}}{\partial x_1} + a_{20} \frac{\partial a_{10}}{\partial x_0} - a_{10} \frac{\partial a_{20}}{\partial x_0} = 0. \quad (10)$$

For the case  $m = 2$ ,  $N = 3$  we have

$$\begin{aligned} \Omega_2 = & J(dx_0 \wedge dx_1 - a_{11}dx_0 \wedge dx_2 - a_{21}dx_0 \wedge dx_3 + a_{10}dx_1 \wedge dx_2 + \\ & + a_{20}dx_1 \wedge dx_3 - (a_{11}a_{20} - a_{10}a_{21})dx_2 \wedge dx_3), \end{aligned} \quad (11)$$

where the Plücker relations are taken into account in the last term. The closedness equations read

$$\frac{\partial J}{\partial x_2} + \sum_{m=0}^1 \frac{\partial(Ja_{1m})}{\partial x_m} = 0, \quad \frac{\partial J}{\partial x_3} + \sum_{m=0}^1 \frac{\partial(Ja_{2m})}{\partial x_m} = 0, \quad (12)$$

$$\frac{\partial a_{1k}}{\partial x_3} - \frac{\partial a_{2k}}{\partial x_2} + \sum_{l=0}^1 \left( a_{2l} \frac{\partial a_{1k}}{\partial x_l} - a_{1l} \frac{\partial a_{2k}}{\partial x_l} \right) = 0, \quad k = 0, 1. \quad (13)$$

### 3. The spectral variable. Dispersionless integrable systems

To introduce dispersionless integrable systems, we need the solutions of gauge-invariant closedness equations (3) holomorphic with respect to one variable (the spectral variable). Let us consider (gauge invariantly) closed Plücker form  $\Omega_m$  with affine coordinates (ratios of coefficients) holomorphic with respect to  $\lambda = x_0$  in some complex domain. This form defines a hierarchy of dispersionless integrable equations in terms of commuting vector fields locally holomorphic in  $\lambda$ . The equations of the hierarchy are given by the gauge-invariant closedness equations.

More specifically, we consider the forms meromorphic in the complex plane (in the affine gauge).

This setting for  $m = 2$  can be reduced to Whitham hierarchy (Krichever [2]), for  $m = 3$  to heavenly equation hierarchy (Takasaki [8], [9]) and connected Dunajski equation hierarchy [10], [11].

Important reductions are volume (or area) conservation corresponding to closedness in the affine gauge ( $J = 1$ ) and hyper CR (Cauchy-Riemann) reduction  $\Omega_m \wedge d\lambda = 0$ .

Most known examples correspond to the case when there exists polynomial (or Laurent polynomial) set of affine coordinates (affine gauge). Below we will restrict ourselves to the polynomial case. Examples corresponding to Laurent polynomials were considered in [1], [12].

A closely related geometric picture of dispersionless integrable systems in terms of coisotropic deformations was introduced in [13], [14], and connection between this picture and the setting of the present work was discussed in [1].

*Polynomial  $\Omega_1$*  The case  $m = 1$  is non-generic, in this case there is no hierarchy of commuting systems, and it is not clear how to solve it in general. The equations corresponding to this case were considered in the framework of inverse scattering method with variable spectral parameter [15]. The gauge-invariant closedness condition for the form

$$\Omega_1 = J(d\lambda + udx - (1 + v\lambda + \lambda^2)dy),$$

which is given by equation (10), leads to the Liouville equation

$$\varphi_{xy} = e^\varphi, \quad (14)$$

where  $u = \frac{1}{2}e^\varphi$ , the Lax pair for this equation reads

$$\begin{aligned} \partial_x \psi &= u \partial_\lambda \psi \\ \partial_y \psi &= -(1 + v\lambda + \lambda^2) \partial_\lambda \psi. \end{aligned}$$

The case of third order polynomial

$$\Omega_1 = J(d\lambda + udx - (1 + w\lambda + w'\lambda^2 + \lambda^3)dz)$$

leads to ‘higher Liouville equation’ introduced in [15],

$$\varphi_{xxz} - \varphi_{xz}\varphi_x = \frac{3}{2}e^\varphi, \quad (15)$$

where  $w = \varphi_z$ ,  $w' = e^{-\varphi}\varphi_{xz}$ , the Lax pair for this equation is

$$\begin{aligned} \partial_x \psi &= u \partial_\lambda \psi \\ \partial_z \psi &= -(1 + w\lambda + w'\lambda^2 + \lambda^3) \partial_\lambda \psi. \end{aligned}$$

We should emphasize that equations (14), (15) are not commuting and they do not belong to a hierarchy. It is also possible to introduce ‘higher Liouville equations’ of arbitrary order.

*Polynomial  $\Omega_2$*  For the case  $m = 2$ ,  $N = 3$  we consider the form (11) with polynomial coefficients

$$\begin{aligned} a_{10} &= u_0(x), & a_{11} &= u_1(x) + \lambda, \\ a_{20} &= v_0(x) + \lambda v_1(x), & a_{21} &= v_2(x) + \lambda v_3(x) + \lambda^2. \end{aligned}$$

Considering gauge-invariant closedness equations (13) and denoting  $x = x_1$ ,  $y = x_2$ ,  $t = x_3$  (see [1] for more detail), one gets the Manakov-Santini system [16], [17]

$$\begin{aligned} u_{xt} + u_{yy} + (uu_x)_x + v_x u_{xy} - v_y u_{xx} &= 0, \\ v_{xt} + v_{yy} + uv_{xx} + v_x v_{xy} - v_y v_{xx} &= 0. \end{aligned} \quad (16)$$

Considering more variables and higher order polynomials, it is possible to introduce the form  $\Omega_2$  corresponding to the Manakov-Santini hierarchy [21].

*Reductions* There are two important classes of reductions of dispersionless systems (hierarchies) defined in terms of the form  $\Omega_m$ .

- (i) The form  $\Omega_m$  is closed in standard sense in affine gauge ( $J=1$ ). In general, the closedness in the affine gauge leads to representation of the hierarchy in terms of volume-preserving (divergence-free) vector fields. In this case the equations of gauge-invariant closedness (3) are complemented by equations, implied by linear subsystem (2) for  $J = 1$ ,

$$\sum_{l=0}^{m-1} \frac{\partial a_{ql}}{\partial x_l} = 0, \quad q \in \{1, \dots, K\}.$$

For the Manakov-Santini system this reduction leads to the condition  $v = 0$ , defining the dKP equation.

- (ii) Reduction  $\Omega_m \wedge d\lambda = 0$ . In this case it is possible to consider  $\Omega_m = d\lambda \wedge \Omega'_{m-1}$  with  $\Omega'_{m-1}$  not containing  $d\lambda$  and gauge-invariantly closed, and  $\lambda$  plays a role of a parameter. Vector fields do not contain a derivative over spectral variable. In the case of Manakov-Santini system (16) this reduction leads to the condition  $u = 0$  and the equation

$$v_{xt} + v_{yy} + v_x v_{xy} - v_y v_{xx} = 0$$

considered in [18], [19], [20]. For general  $\Omega_m$  this reduction defines the hyper CR (Cauchy-Riemann) hierarchies.

#### 4. Nonlinear Riemann-Hilbert problem

Let us consider a form

$$\Omega_m = d\Psi^0 \wedge d\Psi^1 \wedge \dots \wedge d\Psi^{m-1} = J\tilde{\Omega}_m,$$

where  $\Psi^k$  are some functions (series in  $\lambda$ ),  $J$  is some coefficient of the form in coordinates  $\mathbf{x}$ ,  $\tilde{\Omega}_m$  is a gauge-invariant (affine) factor. It is easy to see that  $\Omega_m$  is a closed Plücker form, and the only thing we need to construct a solution of some dispersionless integrable system is to provide definite analytic properties of the affine factor.

**Question** *How to provide some simple analytic properties of the affine factor? What kind of functions  $\Psi^k$  correspond to a polynomial affine factor?*

It is easy to see that  $\tilde{\Omega}_m$  is invariant under diffeomorphism

$$(\Psi^0, \Psi^1, \dots, \Psi^{m-1}) \rightarrow \mathbf{F}(\Psi^0, \Psi^1, \dots, \Psi^{m-1})$$

Let the functions  $\Psi^k$  be holomorphic (meromorphic) inside and outside the unit circle (or some curve in the complex plane), having a discontinuity on it. If they satisfy a nonlinear vector Riemann-Hilbert problem on the unit circle

$$(\Psi^0, \Psi^1, \dots, \Psi^{m-1})_{\text{in}} = \mathbf{F}(\Psi^0, \Psi^1, \dots, \Psi^{m-1})_{\text{out}}, \quad (17)$$

then the affine factor  $\tilde{\Omega}_m$  is holomorphic (meromorphic) in the complex plane.

Thus nonlinear vector Riemann-Hilbert problem gives a tool to construct closed Plücker forms with holomorphic (meromorphic) affine factor, generating commuting vector fields with holomorphic (meromorphic) coefficients.

Equivalently, it is possible to use a nonlinear vector  $\bar{\partial}$ -problem in some domain of the complex plane

$$\bar{\partial}\Psi^k = F^k(\Psi^0, \Psi^1, \dots, \Psi^{m-1}), \quad 0 \leq k \leq m-1.$$

It provides the analyticity of affine factor  $\tilde{\Omega}_m$  in the domain.

### General hierarchy for the polynomial case

We will demonstrate how for a special choice of structure of functions  $\Psi^k$ , using nonlocal Riemann-Hilbert problem, it is possible to obtain polynomial affine factor and the corresponding hierarchy. We consider the formal series

$$\Psi^0 = \lambda + \sum_{n=1}^{\infty} \Psi_n^0(\mathbf{t}^1, \dots, \mathbf{t}^{m-1}) \lambda^{-n}, \quad (18)$$

$$\Psi^k = \sum_{n=0}^{\infty} t_n^k (\Psi^0)^n + \sum_{n=1}^{\infty} \Psi_n^k(\mathbf{t}^1, \dots, \mathbf{t}^{m-1}) (\Psi^0)^{-n}, \quad (19)$$

where  $1 \leq k \leq m-1$ , depending on  $m-1$  infinite sequences of independent variables  $\mathbf{t}^k = (t_0^k, \dots, t_n^k, \dots)$ ,  $t_0^k = x_k$ ,  $\lambda = x_0$ .

Let us consider the Riemann-Hilbert problem (17) on the unit circle, where the functions  $\Psi^k$  are analytic inside and outside the circle and in the neighborhood of infinity are given by the series of the form (18), (19). Then the form  $\tilde{\Omega}_m$  is analytic in the complex plane, moreover, due to the structure of the series it is polynomial. Thus it corresponds to dispersionless hierarchy for the polynomial case. Equations of the hierarchy are generated by the relation (see [11], [21])

$$(J^{-1} d\Psi^0 \wedge d\Psi^1 \wedge \dots \wedge d\Psi^{m-1})_- = 0$$

where  $(\dots)_-$  denotes the projection on the part of  $(\dots)$  with negative powers in  $\lambda$  and  $J = \pi_{01\dots m-1} = \det(\partial_l \Psi^k)_{k,l=0,\dots,m-1}$ . Generating relation represents analyticity condition for affine factor of the closed Plücker form, which can be provided by the Riemann-Hilbert problem (17).

Using the Jacobian matrix

$$(\text{Jac}_0) = \left( \frac{D(\Psi^0, \dots, \Psi^{m-1})}{D(x_0, \dots, x_{m-1})} \right), \quad \det(\text{Jac}_0) = J,$$

it is possible to write Lax-Sato equation of the hierarchy in the form

$$\partial_n^k \Psi = \sum_{i=0}^{m-1} ((\text{Jac}_0)^{-1})_{ik} (\Psi^0)^n \partial_i \Psi, \quad 1 \leq k \leq m-1, \quad (20)$$

where  $1 \leq n < \infty$ ,  $\Psi = (\Psi^0, \dots, \Psi^{m-1})$ . First flows of the hierarchy read

$$\partial_1^k \Psi = (\lambda \partial_k - \sum_{p=1}^{m-1} (\partial_k u_p) \partial_p - (\partial_k u_0) \partial_\lambda) \Psi, \quad 1 \leq k \leq m-1, \quad (21)$$

where  $u_0 = \Psi_1^0$ ,  $u_k = \Psi_1^k$ ,  $1 \leq k \leq m-1$ .

A compatibility condition for any pair of linear equations (e.g., with  $\partial_1^k$  and  $\partial_1^q$ ,  $k \neq q$ ) implies closed nonlinear N-dimensional system of PDEs for the set of functions  $u_k$ ,  $u_0$ , which can be written in the form

$$\begin{aligned} \partial_1^k \partial_q \hat{u} - \partial_1^q \partial_k \hat{u} + [\partial_k \hat{u}, \partial_q \hat{u}] &= (\partial_k u_0) \partial_q - (\partial_q u_0) \partial_k, \\ \partial_1^k \partial_q u_0 - \partial_1^q \partial_k u_0 + (\partial_k \hat{u}) \partial_q u_0 - (\partial_q \hat{u}) \partial_k u_0 &= 0, \end{aligned} \quad (22)$$

where  $\hat{u}$  is a vector field,  $\hat{u} = \sum_{p=1}^{m-1} u_p \partial_p$ . For  $m=3$  this system after volume-preservation reduction corresponds to the Dunajski system (generalizing heavenly equation).

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