Towards the Dressing Phase in the AdS_3/CFT_2 Duality

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Abstract. In this paper we describe some recent progresses in the study of the leading quantum correction at strong coupling of the dressing phase appearing in the Bethe Ansatz for string theory on the $AdS_3 \times S^3 \times T^4$ background. Using the $SU(2)$ rigid circular string as guiding example, we find that the phase is different than in the $AdS_{5,4}$ cases. We discuss in detail the determination of the phase using both a Word-Sheet approach and the Algebraic Curve formalism.

1. Introduction

The discovery of the integrability in the context of the AdS/CFT correspondence opened the possibility for spectacular developments in the past years: in particular the spectral problem for the most studied example of the duality - $\mathcal{N} = 4$ SYM - II B string theory on $AdS_5 \times S^5$ - has been completely solved in terms of a system of Thermodinamic Bethe Equations. Many other interesting objects in the theory have been investigated exploiting the integrability of the model, ranging from correlation functions, Wilson loops, scattering amplitudes etc. The situation is similar for the second main example of integrable theories in the AdS/CFT duality: ABJM - II A string theory on $AdS_4 \times CP^3$ (for a review see [1]).

In both these cases the first step in the solution of the spectral problem has been its reformulation in terms of a spin chain description, mapping the calculation of the anomalous dimensions for (long) gauge invariant operators in the solutions of some sets of Bethe Equations (BE).

It is well known that other backgrounds with an AdS_n factor allow for integrable string theory, at least at the classical level (see for example [2]): recently the backgrounds $AdS_3 \times S^3 \times S^3 \times S^1$ and $AdS_3 \times S^3 \times T^4$ received much attention, and a first conjecture for the the Bethe Equations describing the spectrum of these models has been proposed in [3], followed by the second proposal of [4]. As it happens in the previous $AdS_{5,4}$ cases, these BE contains a dressing factor, which is not fixed by the symmetry of the model.

The main goal of this paper is to report some recent results [5, 6] in the study of the dressing phase appearing in the aforementioned set of BE for the $AdS_3 \times S^3 \times T^4$ background: technically this background can be thought as a special case of the more general $AdS_3 \times S^3 \times S^3 \times S^1$: setting the radius of AdS_3 to 1, then the radii of the two 3-spheres can be parametrized as $R_+^2 =$

α^{-1}, $R_-^2 = (1-\alpha)^{-1}$, and the $AdS_3 \times S^3 \times T^4$ model with $R_2 = \infty$ (after recompactification) corresponds to $\alpha = 1$; strings on $AdS_3 \times S^3 \times T^4$ and $AdS_3 \times S^3 \times S^3 \times S^1$ are described respectively by the GS superstring action on the supercosets $PSU(1,1|2) \times PSU(1,1|2)/SU(1,1) \times SU(2)$ and $D(2,1;\alpha) \times D(2,1;\alpha)/SU(1,1) \times SU(2) \times SU(2)$: It is important to note that, contrary to the $AdS_{5,4}$ case, the supercoset description of the background, which is the core of the integrabilty of the model, is only partial for the AdS_3 cases, the S^1 and T^4 factors being missing. Due to the difficulty to describe the complete background in the integrability setup, and the fact that the $\alpha \to 1$ limit is not smooth, it is simpler to consider separately the two cases; in this paper we concentrate on the $AdS_3 \times S^3 \times T^4$ case only.

In more detail the aim of the paper is to investigate the next to leading correction at strong coupling of the dressing phase: the phase for the scattering of two magnons with momenta p_j and p_k can be written in general as [7, 8]

$$\vartheta(p_j, p_k) = 2 \sum_{\substack{r=2 \\ s \geq r+1 \\ r+s \text{ odd}}}^{\infty} c_{r,s}(\lambda) \left(\frac{\lambda}{16\,\pi^2} \right)^{\frac{r+s-1}{2}} \left[q_r(p_j)\, q_s(p_k) - q_s(p_j)\, q_r(p_k) \right] . \tag{1}$$

where $q_n(p)$ is the elementary magnon n-th charge. The strong coupling expansion of the coefficient functions $c_{r,s}(\lambda)$ is

$$c_{r,s}(\lambda) = c_{r,s}^{(0)} + \frac{1}{\sqrt{\lambda}}\, c_{r,s}^{(1)} + \dots, \tag{2}$$

where the leading term $c_{r,s}^{(0)} = \delta_{r+1,s}$ is fixed by the matching with the classical finite-gap equations, and it is the same AFS contribution [9] as in the $AdS_{5,4}$ cases.

Our strategy to compute the $c_{r,s}^{(1)}$ relies on the study of a simple solution, the $SU(2)$ circular spinning string as a guiding example. While the 1-loop correction to the dressing phase and the $c_{r,s}^{(1)}$ coefficients can be computed, in principle in a more general way, without refering to any particular solution, in the Algebraic Curve (AC) framework, following the approach of [10], we will need anyway an independent calculation of the 1-loop energy, done for the particular solution of the rigid circular string using the World-Sheet (WS) method, which is independent of the integrability of the model: we need this particular example both to understand regularization issues and to check the independence of the result from the missing modes in the AC, realtive to the T^4 component of the background.

The paper is organised as follows: in section 2 we briefly review the AC setup for the $AdS_3 \times S^3 \times T^4$ model, and present the calculation of the coefficients applying the AC method naively. Section 3 is devoted to the same calculation in the WS approach. In section 4 we discuss then the origin of the mismatch between the two results for the particular case of the circular string and show that the difference is due to regularization issues in the sum over the modes. The Section 5 briefly describes the second example of the folded string solution, and finally we discuss open problems.

2. Algebraic Curve determination of the $c_{r,s}^{(1)}$
Introducing the function of the spectral parameter $\widehat{\alpha}(x)$ and the resolvents G, H defined in terms of the Bethe roots $x_{a,k}$ as

$$\widehat{\alpha}(x) = \frac{4\pi}{\sqrt{\lambda}} \frac{x^2}{x^2 - 1} \, ,$$

$$G_a(x) = \sum_{k=1}^{K_a} \frac{\widehat{\alpha}(x_{a,k})}{x - x_{a,k}}, \qquad H_a(x) = \sum_{k=1}^{K_a} \frac{\widehat{\alpha}(x)}{x - x_{a,k}}, \qquad \overline{G}(x) = G(1/x), \quad \overline{H}(x) = H(1/x). \tag{3}$$

the finite-gap equations can be written as:

$$2\,\pi\,n_1 = -H_2 - \overline{H}_{\overline{2}} + \frac{G_2(0) + x\,G_2'(0)}{x^2 - 1} - \frac{G_{\overline{2}}(0) + x\,G_{\overline{2}}'(0)}{x^2 - 1}, \tag{4}$$

$$2\,\pi\,n_2 + \frac{4\,\pi\,\mathcal{J}\,x}{x^2 - 1} = 2\,H_2 - H_1 - H_3 + 2\,\overline{H}_{\overline{2}} - \overline{H}_{\overline{1}} - \overline{H}_{\overline{3}} + 2\,\frac{G_{\overline{2}}(0) - G_2(0)}{x^2 - 1}, \tag{5}$$

$$2\,\pi\,n_3 = -H_2 - \overline{H}_{\overline{2}} + \frac{G_2(0) + x\,G_2'(0)}{x^2 - 1} - \frac{G_{\overline{2}}(0) + x\,G_{\overline{2}}'(0)}{x^2 - 1}, \tag{6}$$

$$2\,\pi\,n_{\overline{1}} = -H_{\overline{2}} - \overline{H}_2 - \frac{G_2(0) + x\,G_2'(0)}{x^2 - 1} + \frac{G_{\overline{2}}(0) + x\,G_{\overline{2}}'(0)}{x^2 - 1}, \tag{7}$$

$$2\,\pi\,n_{\overline{2}} - \frac{4\,\pi\,\mathcal{J}\,x}{x^2 - 1} = 2\,H_{\overline{2}} - H_{\overline{1}} - H_{\overline{3}} + 2\,\overline{H}_2 - \overline{H}_1 - \overline{H}_3 - 2\,\frac{G_{\overline{2}}(0) - G_2(0)}{x^2 - 1}, \tag{8}$$

$$2\,\pi\,n_{\overline{3}} = -H_{\overline{2}} - \overline{H}_2 - \frac{G_2(0) + x\,G_2'(0)}{x^2 - 1} + \frac{G_{\overline{2}}(0) + x\,G_{\overline{2}}'(0)}{x^2 - 1}. \tag{9}$$

Following on the work of [11], we define the quasimomenta in terms of the resolvents as ($\mathcal{J}$ is the angular momentum of the string):

$$p_1 = -p_4 \;\; = \;\; -\frac{1}{2}\,H_1 - \frac{1}{2}\,\overline{H}_{\overline{1}} - \frac{1}{2}\,H_3 - \frac{1}{2}\,\overline{H}_{\overline{3}} - \frac{2\,\pi\,\mathcal{J}\,x}{x^2 - 1} + \frac{x}{x^2 - 1}[G_2'(0) - G_{\overline{2}}'(0)], \tag{10}$$

$$p_2 = -p_3 \;\; = \;\; H_2 + \overline{H}_{\overline{2}} - \frac{1}{2}\,H_1 - \frac{1}{2}\,\overline{H}_{\overline{1}} - \frac{1}{2}\,H_3 - \frac{1}{2}\,\overline{H}_{\overline{3}} - \frac{2\,\pi\,\mathcal{J}\,x}{x^2 - 1}, \tag{11}$$

$$p_{\overline{1}} = -p_{\overline{4}} \;\; = \;\; -\frac{1}{2}\,H_{\overline{1}} - \frac{1}{2}\,\overline{H}_1 - \frac{1}{2}\,H_{\overline{3}} - \frac{1}{2}\,\overline{H}_3 + \frac{2\,\pi\,\mathcal{J}\,x}{x^2 - 1} + \frac{x}{x^2 - 1}[G_{\overline{2}}'(0) - G_2'(0)], \tag{12}$$

$$p_{\overline{2}} = -p_{\overline{3}} \;\; = \;\; H_{\overline{2}} + \overline{H}_2 - \frac{1}{2}\,H_{\overline{1}} - \frac{1}{2}\,\overline{H}_1 - \frac{1}{2}\,H_{\overline{3}} - \frac{1}{2}\,\overline{H}_3 + \frac{2\,\pi\,\mathcal{J}\,x}{x^2 - 1}. \tag{13}$$

Up to winding contributions, we have

$$p_{1,2,3,4}(x) = p_{\overline{1},\overline{2},\overline{3},\overline{4}}(1/x) \; . \tag{14}$$

The above finite-gap equations are obtained with $p_i - p_j = 2\,\pi\,n_{ij}$ where

$$(i, j) = (1, 2), (2, 3), (3, 4), \qquad (\overline{1}, \overline{2}), (\overline{2}, \overline{3}), (\overline{3}, \overline{4}). \tag{15}$$

The eight functions $p_i(x)$ and their branch cuts define the classical algebraic curve for the $AdS_3 \times S^3 \times T^4$ background: Note that here the algebraic curve is a connected sum of two pieces interchanged by the $x \to 1/x$ transformation, while in the $AdS_5 \times S^5$ case the curve is a single connected invariant piece. In the recent paper [4] a second, different, set of BE has been derived, using the bootstrap method: It is worth to note that these two versions of the BE, while they are inequivalent at the quantum level, share the same finite-gap limit, and so they lead to equivalent classical algebraic curves.

2.1. Semiclassical one-loop dressing factor

In the semiclassical quantisation of the AC, the first correction to the phase is encoded in a set of potentials V_I correcting the classical quasimomenta that characterize the curve: these potentials compute the effect of the quantum fluctuations around the classical solution; for each quasi-momentum p_I the correction V_I is obtained summing over all the fluctuations connecting the sheets of the AC. The total phase corrections to the Bethe equations are obtained by evaluating

$\mathcal{V} = V_I - V_J$. For the middle node 2 equation (the other BA equations are not corrected) $\mathcal{V}(x) \equiv V_2(x) - V_3(x)$

$$\mathcal{V}(x) = \int_{-1}^{1} \frac{dy}{2\pi} \left[\left(G_2(y) + \overline{G}_{\bar{2}}(y) \right)' \frac{\widehat{\alpha}(x)}{x - y} + \left(\overline{G}_2(y) + G_{\bar{2}}(y) \right)' \frac{\widehat{\alpha}(1/x)}{1/x - y} \right]. \tag{16}$$

If we excite only the node 2, it reduces to

$$\mathcal{V}_2(x) = \int_{-1}^{1} \frac{dy}{2\pi} \left[G_2'(y) \frac{\widehat{\alpha}(x)}{x - y} + \overline{G}_2'(y) \frac{\widehat{\alpha}(1/x)}{1/x - y} \right], \tag{17}$$

where the notation is $\int_{-1}^{1} = \frac{1}{2} \int_{C^+} + \frac{1}{2} \int_{C^-}$ and the half circumferences $C^{\pm}$ (and their orientation) are defined in the caption of figure 4 of [10].

At this point we can evaluate $\mathcal{V}_2(x)$ for large x: Using the relation between the resolvent G_2 and the conserved charges Q_n,

$$G_2(y) = -\sum_{n=0}^{\infty} Q_{n+1} y^n, \tag{18}$$

the resulting function of y is not singular anywhere on the circle $|y| = 1$ and the integration is trivial. The result is

$$\mathcal{V}_2(x) = \frac{\widehat{\alpha}(x)}{2\pi} \sum_{r=2}^{\infty} \sum_{s=1}^{\infty} \widehat{c}_{r,s} \frac{Q_r}{x^s}, \tag{19}$$

$$\widehat{c}_{r,s} = -4 \left(1 - \frac{1}{2} \delta_{s,1} \right) \frac{1 - (-1)^{r+s}}{2} \frac{r - 1}{r + s - 2}. \tag{20}$$

Here $\widehat{c}_{r,s}$ are the (naive) prediction of the algebraic curve method for the values of the $c_{r,s}^{(1)}$ coefficients which parametrize the phase. A serious problem of this result is that the coefficients $\widehat{c}_{r,s}$ are not antisymmetric, which is an important consistency requirement.

If we now apply the AC method to the particular case of the $SU(2)$ circular string, we get for the 1-loop contribution of the dressing phase to the energy as:

$$\delta E_1^{AC} = \frac{1}{\sqrt{m^2 + \mathcal{J}^2}} \left(m^2 + \mathcal{J}^2 \log \frac{\mathcal{J}^2}{m^2 + \mathcal{J}^2} \right) - \frac{m^2 \left(2\mathcal{J} \left(\mathcal{J} - \sqrt{\mathcal{J}^2 + m^2} \right) + m^2 \right)}{2 \left(\mathcal{J}^2 + m^2 \right)^{3/2}}. \tag{21}$$

In the next sections we show that the part that breaks the antisymmetry in the coefficients also induces a mismatch with the WS calculation for the non-analytic term in the one-loop energy. This disagreement turns out to be due to a regularization ambiguity in the sum over fluctuation frequencies. Once this regularization problem is fixed, the algebraic curve approach result agrees with the string theory result, the the antisymmetry of $c_{r,s}$ is recovered, and the result can be consistently interpreted as a phase.

3. One-loop correction to the energy for the $SU(2)$ string: WS approach

The 1-loop correction to the energy in the WS approach is obtained as usual as a sum over the fluctuation frequences around the classical solution: at the classical level the solution is identical than that of the AdS_5 case, and the frequences in the AdS_3 background can be easily obtained by truncating a certain subset of the fluctuations fields and adjusting the number of independent fluctuations, i.e. removing two bosonic frequencies that correspond to fluctuations

in the transverse directions of S^5, and halve the AdS and fermionic contributions. There are also four additional bosonic and four fermionic massless modes coming from the T^4 part of the background, but their contributions cancel among themselves. The result for the one-loop correction to the energy is given by:

$$E_1 = \sum e(n) , \tag{22}$$

$$e(n) = \sqrt{1 + \frac{(n + \sqrt{n^2 - 4m^2})^2}{4(\mathcal{J}^2 + m^2)}} + \sqrt{1 + \frac{n^2}{\mathcal{J}^2 + m^2}} - 2\sqrt{1 + \frac{n^2 - m^2}{\mathcal{J}^2 + m^2}} . \tag{23}$$

It is straightforward to check that this sum is UV finite. The dressing contribution to E_1 is obtained isolating the non-analytic part in the sum, following the same method as in [7]: expanding $e(n)$ at large $\mathcal{J}$ and separating out the convergent (i.e. regular) and divergent (i.e. singular) parts, we can write $e(n) = e_{\text{reg}}^{\text{sum}}(n) + e_{\text{sing}}^{\text{sum}}(n)$. To deal with the singular part we define $e^{\text{int}}(x) = e(\mathcal{J}x)$ and expand it for large $\mathcal{J}$ at fixed x, getting $e^{\text{int}}(x) = e_{\text{reg}}^{\text{int}}(x) + e_{\text{sing}}^{\text{int}}(x)$ where $e_{\text{sing}}^{\text{int}}$ is the part whose integral is divergent at $x = 0$. The regular part in one regime is in fact equal to the singular part in the other regime.

$$e_{\text{sing}}^{\text{int}}(x) = e_{\text{reg}}^{\text{sum}}(\mathcal{J}x), \qquad e_{\text{sing}}^{\text{sum}}(n) = e_{\text{reg}}^{\text{int}}(n/\mathcal{J}), \tag{24}$$

so that finally we get:

$$E_1 = E_1^{\text{analytic}} + E_1^{\text{non-analytic}} , \tag{25}$$

$$E_1^{\text{analytic}} = \sum e_{\text{reg}}^{\text{sum}}(n), \qquad E_1^{\text{non-analytic}} \equiv \delta E_1 = \int\limits_{-\infty}^{\infty} \mathcal{J} dx \, e_{\text{reg}}^{\text{int}}(x) . \tag{26}$$

leading to:

$$\delta E_1^{AdS_3} = \frac{1}{\sqrt{\mathcal{J}^2 + m^2}} \left(m^2 + \mathcal{J}^2 \log \frac{\mathcal{J}^2}{m^2 + \mathcal{J}^2} \right) , \tag{27}$$

Clearly there is a mismatch with the previous AC results, the difference being the term:

$$\Delta E_1 = -\frac{m^2 \left(2\mathcal{J} \left(\mathcal{J} - \sqrt{\mathcal{J}^2 + m^2} \right) + m^2 \right)}{2 \left(\mathcal{J}^2 + m^2 \right)^{3/2}} \tag{28}$$

4. Origin of the mismatch

Both problems, the non-antisymmetry of the coefficients and the mismatch ΔE_1, can be traced back to a regularization issue in the sum over the frequencies of the quantum fluctuations: in the WS approach the natural cut-off is a common mode number N, while in the AC approach the cut-off is a common radius for the contour integral in the spectral plane defining the potentials; this difference is translated in a reordering of the terms in the sum over the frequences to compute the 1-lopp energy $E_1 \sim \sum\limits_{n \in Z} (\omega_n^B - \omega_n^F)$, and is the origin of the ΔE_1 term. We can repeat the calculation of $\mathcal{V}$ enforcing the regularization of the AC curve to reproduce the WS result for the circular string, solving both problems at once. At the level of the potential $\mathcal{V}(x)$ the choice of the new regularization corresponds to an integration by parts, i.e. we start with the potential correcting the Bethe equation

$$\mathcal{V}(x) = \int_{-1}^{1} \frac{dy}{2\pi} \left[\left(G_2(y) + \overline{G}_{\overline{2}}(y) \right)' \frac{\widehat{\alpha}(x)}{x - y} + \left(\overline{G}_{\overline{2}}(y) + G_{\overline{2}}(y) \right)' \frac{\widehat{\alpha}(1/x)}{1/x - y} \right]. \tag{29}$$

and integrating by parts we define the potential $\widehat{\mathcal{V}}(x)$, correspondin to the new regularization:

$$\widehat{\mathcal{V}}(x) = \int_{-1}^{1} \frac{dy}{2\,\pi} \Big[\big(G_2(y) + \overline{G}_{\overline{2}}(y) \big)' \frac{\widehat{\alpha}(x)}{x-y} - \big(\overline{G}_2(y) + G_{\overline{2}}(y) \big) \Big(\frac{\widehat{\alpha}(1/x)}{1/x - y} \Big)' \Big]. \tag{30}$$

The large x expansion of $\widehat{\mathcal{V}}(x)$ is

$$\widehat{\mathcal{V}}(x) = \frac{\widehat{\alpha}(x)}{2\,\pi} \sum_{r=1}^{\infty} \sum_{s=1}^{\infty} \frac{c_{r,s}^{(1)}\, Q_r - \overline{c}_{r,s}^{(1)}\, \overline{Q}_r}{x^s}, \tag{31}$$

where the expansion coefficients are now *antisymmetric*:

$$c_{r,s}^{(1)} = 2\, \frac{1-(-1)^{r+s}}{2}\, \frac{s-r}{r+s-2}\,, \qquad\qquad \overline{c}_{r,s}^{(1)} = -2\, \frac{1-(-1)^{r+s}}{2}\, \frac{r+s-2}{s-r}. \tag{32}$$

These coefficients (only $c_{r,s}^{(1)}$ is actually contributing) now lead precisely to the string theory expression $\delta E_1^{AdS_3}$ for the non-analytic term in the circular $SU(2)$ string case.

Using the new regularization, we can now compute the scattering phases between magnons, following [10]: We identify the dressing phase contribution in the BE as:

$$e^{i\,\widehat{\mathcal{V}}(x_{2,i})} = \prod_{k\neq j}^{K_2} \sigma^2(x_{2,j}, x_{2,k}) \prod_{k}^{K_{\overline{2}}} \sigma^{-2}(x_{2,j}, x_{\overline{2},k}) = \prod_{k\neq j}^{K_2} e^{i\,\vartheta(x_{2,j}, x_{2,k})} \prod_{k}^{K_{\overline{2}}} e^{-i\,\widetilde{\vartheta}(x_{2,j}, x_{\overline{2},k})}. \tag{33}$$

Using the discrete definition of the function G and integrating over y we obtain

$$\vartheta(x,y) = -\frac{\widehat{\alpha}(x)\,\widehat{\alpha}(y)}{2\,\pi\,(x-y)^2} \Big[2 \log\Big(\frac{x+1}{x-1}\frac{y-1}{y+1} \Big) + 2\, \frac{(x-y)\,(x^2+y^2-2)}{(x^2-1)(y^2-1)} \Big], \tag{34}$$

$$\widetilde{\vartheta}(x,y) = -\frac{\widehat{\alpha}(x)\,\widehat{\alpha}(y)}{2\,\pi\,(1-x\,y)^2} \Big[2 \log\Big(\frac{x+1}{x-1}\frac{y-1}{y+1} \Big) - 2\, \frac{(x-y)\,(x^2 y^2 -1)}{(x^2-1)\,(y^2-1)} \Big]. \tag{35}$$

5. Folded string and open problems
The choice of regularization used in the previous section allows to separate the original potential $\mathcal{V} = \mathcal{V}_{phase} + \delta\mathcal{V}$, where only the first part can be consistently interpreted as a phase, while the second is understood as a regularization effect. Nevertheless the prescription is based on the agreement between the AC and WS approaches in the particular case of the $SU(2)$ curcular string. If we consider as a second example the $SL(2)$ folded string solution, we can repeat the same steps: computing the 1-loop dressing contribution to the energy for the folded string, the AC and WS give different result, and the mismatch is again due to the different regularizations. While we can remove the discrepancy and get a consistent result on the string theory side, if we compare the string prediction with the BE, using the phase derived in the previous section we still have a discrepancy.

$$\text{WS} \equiv \text{AC-reg. mismatch}: \qquad E_1^{\text{dressing}} = \frac{\coth^{-1}\big(\sqrt{\mathcal{J}^2+1} \big)}{2\mathcal{J}^3\sqrt{\mathcal{J}^2+1}}\, \mathcal{S}^2 + \mathcal{O}(\mathcal{S}^3),$$

$$\text{BE with } c_{r,s} \text{ coeff.}: \qquad E_1^{\text{dressing}} = \Big[\frac{\coth^{-1}\big(\sqrt{\mathcal{J}^2+1} \big)}{2\mathcal{J}^3\sqrt{\mathcal{J}^2+1}} + \frac{1}{2\,\mathcal{J}^4\,\sqrt{\mathcal{J}^2+1}} \Big]\, \mathcal{S}^2 + \mathcal{O}(\mathcal{S}^3).$$

where the first line is the string theory result for a folded string with semiclassical spin $\mathcal{S}$ and angular momentum $\mathcal{J}$, the second line is the result obtained from the BE equations, assuming the coefficients $c_{r,s}$ for the dressing phase.

This disagreement calls for a deeper understanding of the role of $\delta\mathcal{V}$; while the regularization/antisymmetrization prescription works perfectly in the $SU(2)$ sector, the comparison between string theory and BE predictions for more general cases is still problematic.

Another open issue is related to the fact that the magnon phases in eq.(35) don't satisfy the crossing relations found in [4]: to get crossing respecting phases the coefficients $c_{r,s}$ with $r = 1$ should be modified with an additional factor $1/2$ [12, 13]. The relative phases for the magnon scattering can be found starting from the original potential in eq.(16), and making the resulting phase antisymmetric by hand $\theta^{AC,asym}(x,y) = 1/2[\theta^{AC}(x,y) - \theta^{AC}(y,x)]$. But this phase, while crossing symmetric, gives a contribution to the 1-loop energy not in agreement with the string theory WS prediction, even in the case of the circular string solution.

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