

Higher symmetries of cotangent coverings for Lax-integrable multi-dimensional partial differential equations and Lagrangian deformations

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Abstract. We present examples of Lax-integrable multi-dimensional systems of partial differential equations with higher local symmetries. We also consider Lagrangian deformations of these equations and construct variational bivectors on them.

1. Introduction

Different approaches to integrability of partial differential equations (PDEs), [11, 22], are based on their diverse but related properties such as existence infinite hierarchies of (local or nonlocal) symmetries and/or conservation laws, zero-curvature representations (ZCR), bi-Hamiltonian structures, recursion operators, etc.

Much progress was achieved in the study of PDEs with two independent variables. In particular, in a big number of examples it was shown that a PDE integrable in the sense of presence of a ZCR with a non-removable parameter (we call such equations *Lax-integrable*) have infinite hierarchies of higher local symmetries, see, e.g., [11]. In the multidimensional case the situation looks different. As far as we know, no nontrivial examples of multi-dimensional equations with local higher symmetries were found, cf. [20, §6].

In this paper, we present five examples of multi-dimensional systems with higher local symmetries. All these systems are defined as cotangent coverings, [8, 9], for Lax-integrable PDEs with three or four independent variables, namely, for the r -th dispersionsless Dym equation, [2, 10, 12, 15, 17],

$$u_{ty} = u_x u_{xy} - u_y u_{xx}, \quad v_{ty} = u_x v_{xy} - u_y v_{xx} + 2(u_{xx} v_y - u_{xy} v_x), \quad (1)$$

the r -th dispersionsless KP equation, [2, 5, 10, 16, 17, 18],

$$u_{yy} = u_{tx} + u_y u_{xx} - u_x u_{xy}, \quad v_{yy} = v_{tx} + u_y v_{xx} - u_x v_{xy} + 2(u_{xy} v_x - u_{xx} v_y), \quad (2)$$

the Veronese web equation, [3, 7, 21],

$$u_{xy} = \frac{\alpha u_x u_{ty} + (1 - \alpha) u_y u_{tx}}{u_t},$$



$$\begin{aligned}
 v_{xy} = & \frac{\alpha u_x v_{ty} + (1 - \alpha) u_y v_{tx}}{u_t} \\
 & + \frac{1}{u_t^2} ((\alpha u_x u_{ty} + (1 - \alpha) u_y u_{tx}) v_t \\
 & + ((1 - 2\alpha) u_t u_{ty} - (1 - \alpha) u_y u_{tt}) v_x - (\alpha u_x u_{tt} + (1 - 2\alpha) u_t u_{tx}) v_y) \\
 & - \frac{v}{u_t^3} ((1 - \alpha) u_y u_{tt} u_{tx} - u_t u_{tx} u_{ty} + \alpha u_x u_{tt} u_{ty})
 \end{aligned} \tag{3}$$

with $\alpha \neq 0, 1$, the universal hierarchy equation, [10, 17],

$$u_{yy} = u_y u_{tx} - u_x u_{ty}, \quad v_{yy} = u_y v_{tx} - u_x v_{ty} + 2(u_{ty} v_x - u_{tx} v_y), \tag{4}$$

and for the 4-dimensional analogue of (1), [14],

$$u_{ty} = u_z u_{xy} - u_y u_{xz}, \quad v_{ty} = u_z v_{xy} - u_y v_{xz} + 2(u_{xz} v_y - u_{xy} v_z), \tag{5}$$

introduced in [10]. We show that these systems have local symmetries of the third order. Equations (1), (2) and (3) belong to the families of PDEs

$$u_{ty} = u_x u_{xy} + \varkappa u_y u_{xx}, \tag{6}$$

$$u_{yy} = u_{tx} + \left(\frac{1}{2}(\varkappa + 1)u_x^2 + u_y\right)u_{xx} + \varkappa u_x u_{xy}, \tag{7}$$

$$u_{xy} = \frac{\alpha u_x u_{ty} + \beta u_y u_{tx}}{u_t} \tag{8}$$

and are featured as the only representatives of their families whose ZCRs have a non-removable parameter. We make an observation that all the symmetries of the cotangent coverings of Equations (6) with $\varkappa \neq -1$, (7) with $\varkappa \neq -1$, and (8) with $\beta \neq 1 - \alpha$ up to order 5 are the point symmetries. This leads to a conjecture that the cotangent coverings of Equations (6), (7), (8) have local higher symmetries whenever these equations are Lax-integrable. If this claim turns out to be correct, it will be a valuable refinement of the observations of [20, §6].

In the last section, we consider *Lagrangian deformations* of cotangent coverings to some of the equations considered above and construct variational bivectors, see [8], on these deformations.

2. Basics

We briefly recall here the basic definitions from the geometry of PDEs, including the definition of the cotangent covering, see [8, 9], and its Lagrangian deformations. Let $\pi: \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n$ be the trivial bundle with the coordinates (x^1, \dots, x^n) in \mathbb{R}^n and (u^1, \dots, u^m) in \mathbb{R}^m and $J^\infty(\pi)$ be the space of its infinite jets. Local coordinates on $J^\infty(\pi)$ are $(x^i, u^\alpha, u_I^\alpha)$, where $I = (i_1, \dots, i_k)$ is a multi-index, and for every local section f of π and the corresponding infinite jet $j_\infty(f): \mathbb{R}^n \rightarrow J^\infty(\pi)$ the coordinate u_I^α is defined by $u_I^\alpha|_{j_\infty(f)} = \frac{\partial^{\#I} f^\alpha}{\partial x^I} = \frac{\partial^{i_1+\dots+i_n} f^\alpha}{(\partial x^1)^{i_1} \dots (\partial x^n)^{i_n}}$. We put $u^\alpha = u_{(0,\dots,0)}^\alpha$.

The vector fields

$$D_{x^k} = \frac{\partial}{\partial x^k} + \sum_{\#I \geq 0} \sum_{\alpha=1}^m u_{I+1_k}^\alpha \frac{\partial}{\partial u_I^\alpha}, \quad k = 1, \dots, n,$$

$(i_1, \dots, i_k, \dots, i_n) + 1_k = (i_1, \dots, i_k + 1, \dots, i_n)$, are called *total derivatives*. They commute everywhere on $J^\infty(\pi)$.

The *evolutionary derivation* associated to an arbitrary smooth function $\varphi: J^\infty(\pi) \rightarrow \mathbb{R}^m$ is the vector field

$$\mathbf{E}_\varphi = \sum_{\#I \geq 0} \sum_{\alpha=1}^m D_I(\varphi^\alpha) \frac{\partial}{\partial u_I^\alpha} \quad (9)$$

with $D_I = D_{(i_1, \dots, i_n)} = D_{x^{i_1}}^{i_1} \circ \dots \circ D_{x^{i_n}}^{i_n}$.

A system of PDEs $F_a(x^i, u_I^\alpha) = 0$, $\#I \leq s$, $a = 1, \dots, r$, of order $s \geq 1$ with $r \geq 1$ defines the submanifold $\mathcal{E} = \{(x^i, u_I^\alpha) \in J^\infty(\pi) \mid D_K(F_a(x^i, u_I^\alpha)) = 0, \#K \geq 0\}$ in $J^\infty(\pi)$.

A function $\varphi: \mathcal{E} \rightarrow \mathbb{R}^m$ is called a (*generator of an infinitesimal symmetry*) of \mathcal{E} when $\mathbf{E}_\varphi(F) = 0$ on \mathcal{E} . The symmetry φ is a solution to the *defining system*

$$\ell_\mathcal{E}(\varphi) = 0, \quad (10)$$

where $\ell_\mathcal{E} = \ell_F|_\mathcal{E}$ with the matrix differential operator

$$\ell_F = \left(\sum_{\#I \geq 0} \frac{\partial F_a}{\partial u_I^\alpha} D_I \right).$$

Solutions to (10) constitute the Lie algebra with respect to the *Jacobi bracket* $\{\varphi, \psi\} = \mathbf{E}_\varphi(\psi) - \mathbf{E}_\psi(\varphi)$ denoted by $\mathbf{sym}(\mathcal{E})$. The subalgebra of *contact symmetries* of \mathcal{E} is $\mathbf{sym}_0(\mathcal{E}) = \mathbf{sym}(\mathcal{E}) \cap C^\infty(J^1(\pi), \mathbb{R}^m)$. Symmetries from $\mathbf{sym}(\mathcal{E}) \setminus C^\infty(J^1(\pi), \mathbb{R}^m)$ are said to be *higher* ones.

Dually, a *cosymmetry* is a function $\psi: \mathcal{E} \rightarrow \mathbb{R}^r$ that satisfies the equation

$$\ell_\mathcal{E}^*(\psi) = 0,$$

where $\ell_\mathcal{E}^*$ denotes the formally adjoint operator

$$\ell_F^* = \left(\sum_{\#I \geq 0} (-1)^{\#I} D_I \circ \frac{\partial F_a}{\partial u_I^\alpha} \right)^T$$

restricted to \mathcal{E} .

A differential operator $\Delta: C^\infty(\mathcal{E}, \mathbb{R}^r) \rightarrow C^\infty(\mathcal{E}, \mathbb{R}^m)$ is called a *variational bivector* if it takes symmetries of the equation \mathcal{E} to its cosymmetries and satisfies $(\ell_\mathcal{E} \circ \Delta)^* = \ell_\mathcal{E} \circ \Delta$. Any such an operator defines a skew-symmetric bracket $\{\omega_1, \omega_2\}_\Delta = L_{\Delta(\delta\omega_1)}(\omega_2)$ on the space of conservation laws, where $\delta = \delta^{0, n-1}$ is the differential in the term E_1 of the Vinogradov \mathcal{C} -spectral sequence (see [19]) and L denotes the Lie derivative. If the variational Schouten bracket $[[\Delta, \Delta]]$ vanishes then Δ is a Hamiltonian operator, see [8].

Denote $\mathcal{W} = \mathbb{R}^l$, $l \leq \infty$, with coordinates w^0, \dots, w^s, \dots . Locally, a *differential covering* over \mathcal{E} is a trivial bundle $\tau: \tilde{\mathcal{E}} = \mathcal{E} \times \mathcal{W} \rightarrow \mathcal{E}$ equipped with the *extended total derivatives*

$$\tilde{D}_{x^k} = D_{x^k} + \sum_{s=0}^l T_k^s(x^i, u_I^\alpha, w^j) \frac{\partial}{\partial w^s} \quad (11)$$

such that $[\tilde{D}_{x^i}, \tilde{D}_{x^j}] = 0$. This yields the system of the *covering equations*

$$w_{x^k}^s = T_k^s(x^i, u_I^\alpha, w^j). \quad (12)$$

This over-determined system of PDEs is compatible whenever $(x^i, u_I^\alpha) \in \mathcal{E}$.

The *cotangent covering* is defined as follows. Let, as before, $\mathcal{E} \subset J^\infty(\pi)$ be given by the relations $F_1 = \dots = F_r = 0$, where $F_a = F(x^i, u_I^j)$. Consider the bundle $\pi': \mathbb{R}^r \times \mathbb{R}^m \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ with coordinates v^α in \mathbb{R}^r . Then $\mathcal{T}^*\mathcal{E} \subset J^\infty(\pi')$ is the equation obtained from \mathcal{E} by adding the relations $\ell_\mathcal{E}^*(v^\alpha) = 0$. The natural projection $\tau^*: \mathcal{T}^*\mathcal{E} \rightarrow \mathcal{E}$ is an infinite-dimensional covering. Note that $\mathcal{T}^*\mathcal{E}$ is always an Euler-Lagrange equation with the Lagrangian density $\mathcal{L} = (F_1 v^1 + \dots + F_r v^r) dx^1 \wedge \dots \wedge dx^n$. Consequently, the spaces of symmetries $\mathbf{sym}(\mathcal{E})$ and cosymmetries $\mathbf{cosym}(\mathcal{E})$ coincide for $\mathcal{T}^*\mathcal{E}$.

3. Higher symmetries of cotangent coverings

In this section we present the results of computation of the third order symmetries for Systems (1)–(5). The computations were held in the JETS software, [1, 6]. We use the notation $u^1 = u$, $u^2 = v$, $x^1 = t$, $x^2 = x$, $x^3 = y$, and $x^4 = z$ in the formulas of the previous section.

Proposition 1. *The cotangent coverings of the Lax-integrable cases of the r -th dispersionsless Dym equation, r -th modified dispersionsless KP equation, the Veronese web equation, the universal hierarchy equation, and the 4-dimensional r -th dispersionsless Dym equation have the following symmetries up to the third order:*

1) System (1):

$$\begin{aligned} \varphi_0(A_0) &= (A_0 u_t + A_{0,t}(x u_x - u) + \frac{1}{2} A_{0,tt} x^2, A_0 v_t + A_{0,t}(x v_x + 2v)), \\ \varphi_1(A_1) &= (A_1 u_x + A_{1,t} x, A_1 v_x), \\ \varphi_2(A_2) &= (A_2, 0), \\ \varphi_3(A_3) &= (0, A_3(2u_t - 3u_x^2) + A_{3,t}(2x u_x + u) - \frac{1}{2} A_{3,tt} x^2), \\ \varphi_4(A_4) &= (0, 2A_4 u_x - A_{4,t} x), \\ \varphi_5(A_5) &= (0, A_5), \\ \varphi_6(B_0) &= (B_0 u_y, B_0 v_y), \\ \varphi_7(B_1) &= (0, B_1 u_y^{-2}), \\ \varphi_8 &= (x u_x - 2u, x v_x), \\ \varphi_9 &= (0, v), \\ \varphi_{10} &= (0, u_{ttt} - 3u_x(u_{ttx} - u_x u_{txx} - u_{tx} u_{xx}) - u_x^3 u_{xxx} - \frac{3}{2}(u_{tx}^2 + u_x^2 u_{xx}^2)), \\ \varphi_{11} &= (0, u_{ttt} - 2u_x u_{txx} + u_x^2 u_{xxx} - u_{xx}(u_{tx} - u_x u_{xx})), \\ \varphi_{12} &= (0, u_{txx} - u_x u_{xxx} - \frac{1}{2} u_{xx}^2), \\ \varphi_{13} &= (0, u_{xxx}), \\ \varphi_{14} &= (0, (2u_y u_{xxy} - u_{xy}^2) u_y^{-2}), \\ \varphi_{15} &= (0, (u_y u_{xyy} - u_{xy} u_{yy}) u_y^{-3}), \\ \varphi_{16} &= (0, (2u_y u_{yyy} - 3u_{yy}^2) u_y^{-4}), \end{aligned}$$

where $A_i = A_i(t)$, $B_i = B_i(y)$ are arbitrary functions of their arguments;

2) System (2):

$$\begin{aligned} \varphi_0(A_0) &= (A_0 u_t + (x u_x + y u_y - u) A_{0,t} + (\frac{1}{2} u_x y^2 - x y) A_{0,tt} - \frac{1}{6} y^3 A_{0,ttt}, \\ &\quad A_0 v_t + (x v_x + y v_y + 2v) A_{0,t} + \frac{1}{2} y^2 v_x A_{0,tt}), \\ \varphi_1(A_1) &= (A_1 u_y + A_{1,t}(y u_x - x) - \frac{1}{2} A_{1,tt} y^2, A_1 v_y + A_{1,t} y v_x), \\ \varphi_2(A_2) &= (A_2 u_x - A_{2,t} y, A_2 v_x), \\ \varphi_3(A_3) &= (A_3, 0), \end{aligned}$$

$$\begin{aligned}
 \varphi_4(A_4) &= (0, 2A_4(u_t + 2u_x^3 + 3u_x u_y) + A_{4,t}(3y u_x^2 + 2x u_x + 2y u_y + u) \\
 &\quad + A_{4,tt}y(y u_x + x) + \frac{1}{6}A_{4,ttt}y^3), \\
 \varphi_5(A_5) &= (0, A_5(2u_y + 3u_x^2) + A_{5,t}(2y u_x + x) + \frac{1}{2}A_{5,tt}y^2), \\
 \varphi_6(A_6) &= (0, 2A_6 u_x + A_{6,t}y), \\
 \varphi_7(A_7) &= (0, A_7), \\
 \varphi_8 &= (2x u_x + y u_y - 3u, 2x v_x + y v_y + 4v + 2t u_t + u_x(2x + 3y u_x + 4t u_x^2) \\
 &\quad + 2u_y(3t u_x + y) + u), \\
 \varphi_9 &= (y u_x + 2t u_y, y v_x + 2t v_y), \\
 \varphi_{10} &= (0, v), \\
 \varphi_{11} &= (0, u_{xxx}), \\
 \varphi_{12} &= (0, u_{xxy} + u_x u_{xxx} + \frac{1}{2}u_{xx}^2), \\
 \varphi_{13} &= (0, u_{txx} + u_{xxx}(u_x^2 + u_y) + u_x u_{xxy} + u_{xx}(u_{xy} + u_x u_{xx})), \\
 \varphi_{14} &= (0, 2u_{txy} + 4u_x u_{txx} + 2u_x u_{xxx}(u_x^2 + 2u_y) + 2u_{xxy}(u_x^2 + u_y) + u_{xx}^2(3u_x^2 + 2u_y) \\
 &\quad + u_{xy}^2 + 2u_{xx}(u_{tx} + 2u_x u_{xy})), \\
 \varphi_{15} &= (0, u_{ttt} + u_{txx}(3u_x^2 + 2u_y) + 2u_x u_{txy} + u_{xxx}(u_x^4 + 3u_y u_x^2 + u_y^2) + u_x u_{xy}^2 \\
 &\quad + u_x u_{xxy}(u_x^2 + 2u_y) + u_{tx}(u_{xy} + 3u_x u_{xx}) + u_x u_{xx}^2(2u_x^2 + 3u_y) \\
 &\quad + u_{xx}(u_{ty} + u_{xy}(3u_x^2 + 2u_y))), \\
 \varphi_{16} &= (0, 2u_{tty} + 6u_x u_{ttt} + 4u_x u_{txx}(2u_x^2 + 3u_y) + 2u_{txy}(2u_y + 3u_x^2) \\
 &\quad + 2u_x u_{xxx}(u_x^4 + 4u_x^2 u_y + 3u_y^2) + 2u_{xxy}(u_x^4 + 3u_x^2 u_y + u_y^2) + 3u_{tx}^2 \\
 &\quad + 6u_{tx}(u_x u_{xy} + u_{xx}(u_y + 2u_x^2)) + u_{xx}^2(5u_x^4 + 12u_x^2 u_y + 3u_y^2) \\
 &\quad + u_{xy}^2(3u_x^2 + 2u_y) + 2u_{ty}(u_{xy} + 3u_x u_{xx}) + 4u_x u_{xx} u_{xy}(2u_x^2 + 3u_y)), \\
 \varphi_{17} &= (0, u_{ttt} + 3(2u_x^2 + u_y)u_{ttt} + 3u_x u_{tty} + (5u_x^4 + 12u_x^2 u_y + 3u_y^2)u_{txx} \\
 &\quad + (5u_y u_x^4 + 6u_y^2 u_x^2 + u_x^6 + u_y^3)u_{xxx} + u_x(u_y + u_x^2)(3u_y + u_x^2)u_{xxy} \\
 &\quad + 12u_x u_y u_{tx} u_{xx} + 10u_x^3 u_{tx} u_{xx} + 6u_x u_{tx}^2 + 2u_x(2u_x^2 + 3u_y)u_{txy} \\
 &\quad + 5u_x^4 u_{xx} u_{xy} + 12u_x^2 u_y u_{xx} u_{xy} + 3u_y u_{tx} u_{xy} + 6u_x^2 u_{tx} u_{xy} + 3u_x u_y u_{xy}^2 \\
 &\quad + 3u_y^2 u_{xx} u_{xy} + 3u_x^5 u_{xx}^2 + 6u_x u_y^2 u_{xx}^2 + 10u_y u_x^3 u_{xx}^2 \\
 &\quad + 3(u_{tx} + 2u_{xx} u_x^2 + u_y u_{xx} + u_x u_{xy})u_{ty} + 2u_x^3 u_{xy}^2),
 \end{aligned}$$

where $A_i = A_i(t)$ are arbitrary functions of t ;

3) System (3):

$$\begin{aligned}
 \varphi_1(P_1) &= (P_1 u_t, P_1 v_t + P_{1,t} v), \\
 \varphi_2(Q_1) &= (Q_1 u_x, Q_1 v_x), \\
 \varphi_3(R_1) &= (R_1 u_y, R_1 v_y), \\
 \varphi_4(S_1) &= (S_1, -S_{1,u} v), \\
 \varphi_5 &= (u, 0), \\
 \varphi_6(P_2) &= (0, P_2 u_t^{-1}), \\
 \varphi_7(Q_2) &= (0, Q_2 u_t u_x^{-2}), \\
 \varphi_8(R_2) &= (0, R_2 u_t u_y^{-2}), \\
 \varphi_9(S_2) &= (0, S_2 u_t), \\
 \varphi_{10} &= (0, (2u_t u_x u_{ttt} - 2u_x u_{tt} u_{tx} - u_t u_{tx}^2) u_t^{-2} u_x^{-2}),
 \end{aligned}$$

$$\begin{aligned}
 \varphi_{11} &= (0, (2 u_t u_y u_{tty} - 2 u_y u_{tt} u_{ty} - u_t u_{ty}^2) u_t^{-2} u_y^{-2}), \\
 \varphi_{12} &= (0, (2 u_t u_x u_{txx} - u_x u_{tx}^2 - 2 u_t u_{tx} u_{xx}) u_t^{-1} u_x^{-3}), \\
 \varphi_{13} &= (0, (2 u_t u_y u_{tyy} - u_y u_{ty}^2 - 2 u_t u_{ty} u_{yy}) u_t^{-1} u_y^{-3}), \\
 \varphi_{14} &= (0, (2 u_t u_{ttt} - 3 u_{tt}^2) u_t^{-3}), \\
 \varphi_{15} &= (0, u_t (2 u_x u_{xxx} - 3 u_{xx}^2) u_x^{-4}), \\
 \varphi_{16} &= (0, u_t (2 u_y u_{yyy} - 3 u_{yy}^2) u_y^{-4}),
 \end{aligned}$$

where $P_i = P_i(t)$, $Q_i = Q_i(x)$, $R_i = R_i(y)$, $S_i = S_i(u)$ are arbitrary functions of their arguments;

4) System (4):

$$\begin{aligned}
 \varphi_0(A_0) &= (A_0 u_t - A_{0,t} u, A_0 v_t + 2 A_{0,t} v), \\
 \varphi_1(A_1) &= (A_1, 0), \\
 \varphi_2(A_2) &= (0, 2 A_2 u_t + A_{2,t} u), \\
 \varphi_3(A_3) &= (0, A_3), \\
 \varphi_4(B_0) &= (B_0 u_x + B_{0,x} y u_y, B_0 v_x + B_{0,x} y v_y), \\
 \varphi_5(B_1) &= (B_1 u_y, B_1 v_y), \\
 \varphi_6(B_2) &= (0, (2 B_2 u_x - B_{2,x} y u_y) u_y^{-3}), \\
 \varphi_7(B_3) &= (0, B_3 u_y^{-2}), \\
 \varphi_8 &= (y u_y + u, y v_y), \\
 \varphi_9 &= (0, v), \\
 \varphi_{10} &= (0, u_{ttt}), \\
 \varphi_{11} &= (0, (u_y^2 u_{ttt} - u_x u_y u_{tty} + u_x u_{ty}^2 - u_y u_{tx} u_{ty}) u_y^{-3}), \\
 \varphi_{12} &= (0, (2 u_y u_{tty} - u_{ty}^2) u_y^{-2}), \\
 \varphi_{13} &= (0, (2 u_x^2 u_y^2 (u_x u_{ttt} - 3 u_{txy}) - 2 u_x^4 u_y u_{tty} - 2 u_y^3 (u_{xxy} - 2 u_x u_{txx}) \\
 &\quad + 3 u_x u_y^2 (u_x u_{tx}^2 - 2 u_{ty} u_{xx} - 2 u_{tx} u_{xy}) - 4 u_x^2 u_y u_{ty} (2 u_x u_{tx} - 3 u_{xy}) \\
 &\quad + u_y^2 (3 u_{xy}^2 + 2 u_y u_{tx} u_{xx}) + 5 u_x^4 u_{ty}^2) u_y^{-6}), \\
 \varphi_{14} &= (0, (2 u_y^4 u_{xxx} + 2 u_x^4 u_y (u_y u_{ttt} - 5 u_{txy} u_{ty}) - 4 u_x^3 u_y^2 (2 u_{txy} - u_{tx}^2) \\
 &\quad + 6 u_x^2 u_y^2 (u_y u_{txx} - 2 u_{ty} u_{xx} - 2 u_{tx} u_{xy}) - 2 u_x^5 (u_y u_{tty} - 3 u_{ty}^2) \\
 &\quad - 6 u_y^3 (u_x u_{xxy} + u_{xx} u_{xy} - u_x u_{tx} u_{xx}) + 4 u_x u_y u_{xy} (3 u_y u_{xy} + 5 u_x^2 u_{ty})) u_y^{-7}), \\
 \varphi_{15} &= (0, (u_x^2 (2 u_y u_{tty} - 3 u_{ty}^2) + u_y^2 (2 u_{txy} - 2 u_x u_{ttt} - u_{tx}^2) \\
 &\quad - 2 u_y u_{ty} (u_{xy} - 2 u_x u_{tx})) u_y^{-4}), \\
 \varphi_{16} &= (0, (u_x^3 (u_y u_{tty} - 2 u_{ty}^2) + u_x u_y^2 (2 u_{txy} - u_{tx}^2) - u_y^2 (u_x^2 u_{ttt} - u_{ty} u_{xx} - u_{tx} u_{xy}) \\
 &\quad - u_y^3 u_{txx} + 3 u_x u_y (u_x u_{tx} u_{ty} - u_{ty} u_{xy})) u_y^{-5}),
 \end{aligned}$$

where $A_i = A_i(t)$, $B_i = B_i(x)$ are arbitrary functions of their arguments;

5) System (5):

$$\begin{aligned}
 \varphi_0(A_0) &= (A_0 u_x - A_{0,x} u + A_{0,t} z, A_0 v_x + 2 A_{0,x} v), \\
 \varphi_1(A_1) &= (A_1, 0), \\
 \varphi_2(A_2) &= (0, 2 A_2 u_x + A_{2,x} u - A_{2,t} z), \\
 \varphi_3(A_3) &= (0, A_3), \\
 \varphi_4(B_0) &= (B_0 u_y, B_0 v_y),
 \end{aligned}$$

$$\begin{aligned}
 \varphi_5(B_1) &= (0, B_1 u_y^{-2}), \\
 \varphi_6 &= (t u_t + u, t v_t), \\
 \varphi_7 &= (z u_z - u, z v_z), \\
 \varphi_8 &= (u_z, v_z), \\
 \varphi_9 &= (0, v), \\
 \varphi_{10} &= (0, u_{xxx}), \\
 \varphi_{11} &= (0, (2 u_y u_{xxy} - u_{xy}^2) u_y^{-2}), \\
 \varphi_{12} &= (0, (u_y u_{xyy} - u_{xy} u_{yy}) u_y^{-3}), \\
 \varphi_{13} &= (0, (2 t u_y^2 u_{xxx} - z (2 u_y u_{xxy} - u_{xy}^2)) u_y^{-2}), \\
 \varphi_{14} &= (0, (t u_y (2 u_y u_{xxy} - u_{xy}^2) - 2 z (u_y u_{xyy} - u_{xy} u_{yy})) u_y^{-3}), \\
 \varphi_{15} &= (0, (z^2 (u_y u_{xyy} - u_{xy} u_{yy}) - t z u_y (2 u_y u_{xxy} - u_{xy}^2) + t^2 u_y^3 u_{xxx}) u_y^{-3}), \\
 \varphi_{16} &= (0, (2 u_y u_{yyy} - 3 u_{yy}^2) u_y^{-4}), \\
 \varphi_{17} &= (0, (z (2 u_y u_{yyy} - 3 u_{yy}^2) - 2 t u_y (u_y u_{xyy} - u_{xy} u_{yy})) u_y^{-4}), \\
 \varphi_{18} &= (0, (z^2 (2 u_y u_{yyy} - 3 u_{yy}^2) - 4 t z u_y (u_y u_{xyy} - u_{xy} u_{yy}) \\
 &\quad + t^2 u_y^2 (2 u_y u_{xxy} - u_{xy}^2)) u_y^{-4}), \\
 \varphi_{19} &= (0, (z^3 (2 u_y u_{yyy} - 3 u_{yy}^2) - 6 t z^2 u_y (u_y u_{xyy} - u_{xy} u_{yy}) \\
 &\quad + 3 t^2 z u_y^2 (2 u_y u_{xxy} - u_{xy}^2) - 2 t^3 u_y^4 u_{xxx}) u_y^{-4}),
 \end{aligned}$$

where $A_i = A_i(t, x)$, $B_i = B_i(y, z)$ are arbitrary functions of their arguments.

All the functions above are assumed to be smooth and the second subscript (as in, say, $A_{3,t}$) denotes the corresponding derivative.

Thus, we have the following higher symmetries:

For Systems (1), (3), (4):	$\varphi_{10}, \dots, \varphi_{16},$
For System (2):	$\varphi_{11}, \dots, \varphi_{17},$
For System (5):	$\varphi_{10}, \dots, \varphi_{19}.$

4. Bivectors on the deformed cotangent covering

Let \mathcal{E} be an equation and $\tau^*: \mathcal{T}^*\mathcal{E} \rightarrow \mathcal{E}$ be its tangent covering. As it was mentioned in Section 2, $\mathcal{T}^*\mathcal{E}$ is an Euler-Lagrange equation. Let $\mathcal{T}^*\mathcal{E}$ be given by $F(x, u) = 0$, $\ell_{\mathcal{E}}^*(x, u, v) = 0$. Then we say that the system

$$\tilde{\mathcal{E}}: \quad \ell_{\mathcal{E}}^*(x, u, v) + G(x, u, v) = 0, \quad F(x, u) + H(x, u, v) = 0 \quad (13)$$

is a *Lagrangian deformation* (LD) if $\tilde{\mathcal{E}}$ is also an Euler-Lagrange equation. We know at least two meaningful examples of this construction: (i) The system

$$u_{ty} = u_{xy}u_x - u_{xx}u_y, \quad v_{ty} = 2(u_{xx}v_y - u_{xy}v_x) + u_xv_{xy} - u_yv_{xx} - 2(u_{xx}u_y + 2u_{xy}u_x),$$

see [15], is an LD of (1); (ii) The Dunajski equation, [4],

$$\Theta_{wx} + \Theta_{zy} + \Theta_{xx}\Theta_{yy} - \Theta_{xy}^2 = f, \quad f_{xw} + f_{yz} + \Theta_{yy}f_{xx} + \Theta_{xx}f_{yy} - 2\Theta_{xy}f_{xy} = 0$$

is an LD of the 2nd Heavenly Equation.

The identical map from $\mathbf{cosym}(\tilde{\mathcal{E}})$ to $\mathbf{sym}(\tilde{\mathcal{E}})$ is always a bivector on any LD $\tilde{\mathcal{E}}$ and we pose the following problem: *Given an equation \mathcal{E} , can $\mathcal{T}^*\mathcal{E}$ be deformed in such a way that the resulting LD $\tilde{\mathcal{E}}$ will admit nontrivial bivectors?*

We studied LDs for Equations (1), (4), and (5) and obtained the following result:

Proposition 2. *Equations (1), (4), and (5) possess two-parameter families of LDs with $H = 0$ and $G = \delta L_{\kappa_1, \kappa_2} / \delta u$ and three-dimensional spaces of bivectors $\Delta = \alpha_0 \text{id} + \alpha_1 \Delta^1 + \alpha_2 \Delta^2$, $\Delta^i = (\Delta_{jk}^i)$, $i, j, k = 1, 2$, where*

For Equation (1):

$$\begin{aligned} L_{\kappa_1, \kappa_2} &= (\kappa_1 u_y + \kappa_2) u_x^2, \\ \Delta_{11}^1 &= u_y D_x^{-1} \circ u_y^{-2} \circ D_y, \\ \Delta_{12}^1 &= 0, \\ \Delta_{21}^1 &= (v_y - \kappa_2) D_x^{-1} \circ u_y^{-2} \circ D_y + u_y^{-2} D_x^{-1} \circ (v_y - \kappa_2) \circ D_y, \\ \Delta_{22}^1 &= u_y^{-2} D_x^{-1} \circ u_y \circ D_y, \\ \Delta_{11}^2 &= \frac{1}{2} u_x + \frac{1}{2} D_x^{-1} \circ (D_t - 2u_x D_x), \\ \Delta_{12}^2 &= 0, \\ \Delta_{21}^2 &= v + \kappa_1 u D_x - \frac{1}{2} D_x^{-1} \circ (2\kappa_1 u D_x + v_{xx}) - \kappa_1 D_x^{-1} \circ (D_t - 2u_x D_x), \\ \Delta_{22}^2 &= \frac{1}{2} u D_x - u_x + \frac{1}{2} D_x^{-1} \circ (D_t - u D_x^2). \end{aligned}$$

For Equation (4):

$$\begin{aligned} L_{\kappa_1, \kappa_2} &= u_t (\kappa_1 u_y + \kappa_2 u_x), \\ \Delta_{11}^1 &= u D_t^{-1} \circ u_y^{-2} \circ D_y, \\ \Delta_{12}^1 &= 0, \\ \Delta_{21}^1 &= (v_y + \kappa_2) D_t^{-1} \circ u_y^{-2} \circ D_y + u_y^{-2} D_t^{-1} \circ (v_y + \kappa_2) \circ D_y, \\ \Delta_{22}^1 &= u_y^{-2} D_t^{-1} \circ u_y \circ D_y, \\ \Delta_{11}^2 &= u D_t - D_y^{-1} \circ (u_{yt} + u D_y D_t), \\ \Delta_{12}^2 &= 0, \\ \Delta_{21}^2 &= -2v D_t - D_y^{-1} \circ (v_{yt} + 2\kappa_2 D_t - 2v D_y D_t), \\ \Delta_{22}^2 &= u D_t + D_y^{-1} \circ (u D_y D_t - 2u_{yt}). \end{aligned}$$

For Equation (5):

$$\begin{aligned} L_{\kappa_1, \kappa_2} &= u_x (\kappa_1 u_z + \kappa_2 u_y), \\ \Delta_{11}^1 &= -u D_x + D_z^{-1} \circ (u_{xz} + D_t + u D_x D_z), \\ \Delta_{12}^1 &= 0, \\ \Delta_{21}^1 &= -D_z^{-1} \circ (v_{xz} + 2(v_z + \kappa_2) D_x), \\ \Delta_{22}^1 &= -u D_x + 2u_x + D_z^{-1} \circ (u_{xz} + D_t + u D_x D_z), \end{aligned}$$

$$\begin{aligned}\Delta_{11}^2 &= u_y D_x^{-1} \circ u_y^{-2} \circ D_y, \\ \Delta_{12}^2 &= 0, \\ \Delta_{21}^2 &= -u_y^{-2} D_x^{-1} \circ (v_y + \varkappa_1) \circ D_y - (v_y + \varkappa_1) D_x^{-1} \circ u_y^{-2} \circ D_y, \\ \Delta_{22}^2 &= u_y^{-2} D_x^{-1} \circ u_y D_y.\end{aligned}$$

5. Concluding remarks

Remark 1. The identical bivector in Equation (1), after transforming the system to the evolutionary form, corresponds to the first Hamiltonian operator from [15], while the bivector Δ^1 corresponds to the second one.

Remark 2. It can be shown that the LDs for Equations (4) and (5) presented above are in a sense trivial, because the deformed equations are transformed to the initial cotangent space by a point transformation. As for Equation (1), there exists a covering $\tau: \tilde{\tilde{\mathcal{E}}} \rightarrow \tilde{\mathcal{E}}$ (that corresponds to a differential substitution) such that $\tilde{\mathcal{E}}$ is equivalent to $\mathcal{T}^*\mathcal{E}$.

Remark 3. As it was mentioned above, all computations were done using the JETS (MAPLE 15), [1, 6], software. We used the IBM Blade cluster consisting of 3 working machines, each of them quad-core Intel Xeon: one E5460, 2.66 GHz, 56 GB RAM two E5430, 2.66 GHz, 16 GB RAM. The classification tree contained 3446 branches, 15 of them were abandoned due to lack of memory. The results presented are due to successful processing of 11 branches only. Other branches are to be processed in future. We also plan to consider LDs of a more general nature.

Remark 4. Actually, presenting results of Proposition 2 in the operator form is not complete and is due to the common tradition only. It is more adequate to deal with them as with Bäcklund transformations between the tangent (see [8]) and cotangent spaces of the equation at hand (cf. [14]), but the operator form seems to be “more visual”.

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