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Exactly solvable central potentials related to Romanovski polynomials

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Abstract. In this paper, we apply a simple transformation method to construct exactly solvable potentials of Schrödinger equation in any arbitrary dimensional Euclidean space. The normalized wave functions of the constructed potentials are obtained in terms of Romanovski polynomials.

1. Introduction

Exact solution of Schrödinger equation with a physical potential is of utmost importance in non-relativistic quantum mechanics. Successful solution of Schrödinger equation provides analytical form of the normalized wave function and quantized energy eigenvalues. However, a very few quantum systems yield exact solutions for potentials of physical interest. Along the years, many authors have tried different methods to obtain the exact solution of the Schrödinger equation [1–6]. To get the bound state wave functions in terms of Romanovski polynomials, we apply a simple transformation method [6] which comprises of a co-ordinate transformation followed by a functional transformation. By applying this method, we transform the second order ordinary differential equation satisfied by Romanovski polynomials to standard radial Schrödinger equation in D-dimensional Euclidean space and thus try to construct exactly solvable potentials. The motivation for doing this work comes from the fact that exact solutions of Schrödinger equation in terms of Romanovski polynomials is not so widespread [7,8] like other orthogonal polynomials [1–5, 9, 10].

The paper is organized as follows. In section 2, the formalism of the theory is given. In section 3, exact solution of Schrödinger equation in terms of Romanovski polynomials is discussed. The conclusions are discussed in section 4.

2. Formalism

We start with the linear second order differential equation satisfied by a special function $Q(r)$

$$Q''(r) + M(r)Q'(r) + J(r)Q(r) = 0 \tag{1}$$

where a prime denotes differentiation with respect to its argument. $Q(r)$ will later be identified as one of the orthogonal polynomials.

The transformation method comprises of the following two steps

$$r \to g(r) \quad \text{and} \quad \psi(r) = f^{-1}(r)Q(g(r)) \tag{2}$$

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We implement the above prescription to equation (1) and obtain

$$\psi''(r) + \left[\frac{d}{dr}\ln\frac{f^2(r)\exp(\int M(g)dg)}{g'(r)}\right]\psi'(r) + \left(\frac{f''(r)}{f(r)} - \frac{g''(r)}{g'(r)}\frac{f'(r)}{f(r)} + g'(r)M(g)\frac{f'(r)}{f(r)} + g'^2 J(g)\right)\psi(r) = 0 \tag{3}$$

The radial Schrödinger equation in D-dimensional Euclidean space is($\hbar = 1 = 2m$)

$$\psi''(r) + \frac{(D-1)}{r}\psi'(r) + \left(E_n - V(r) - \frac{\ell(\ell+D-2)}{r^2}\right)\psi(r) = 0 \tag{4}$$

Consistency of equations (3) and (4) demand that

$$\frac{d}{dr}\ln\frac{f^2(r)\exp(\int M(g)dg)}{g'(r)} = \frac{(D-1)}{r} \tag{5}$$

which fixes the form of $f(r)$ as

$$f(r) = Nr^{\frac{(D-1)}{2}}g'^{\frac{1}{2}}\left[\exp\left(-\int M(g)dg\right)\right]^{\frac{1}{2}} \tag{6}$$

where N is the integration constant and plays the role of the normalization constant of the wave functions.
Using (5) and (6) in equation (3) yields

$$\frac{\psi''(r)}{\psi(r)} + \frac{(D-1)}{r}\frac{\psi'(r)}{\psi(r)} = -\frac{1}{2}\{g,r\} + \frac{g'^2(r)}{4}\left[M^2(g) + 2M'(g) - 4J(g)\right] - \frac{(D-1)(D-3)}{4r^2} \tag{7}$$

where the Schwartzian derivative symbol [11], $\{g,r\}$ is defined as

$$\{g,r\} = \frac{g'''(r)}{g'(r)} - \frac{3}{2}\frac{g''^2(r)}{g'^2(r)}$$

From equations (2) and (6), the expression for normalisable wave function is

$$\psi(r) = Nr^{-\frac{(D-1)}{2}}g'^{-\frac{1}{2}}\left[\exp\left(\int M(g)dg\right)\right]^{\frac{1}{2}}Q(g(r)) \tag{8}$$

The radial wave function $\psi(r) = \frac{u(r)}{r}$ has to satisfy the boundary condition $u(r) = 0$, in order to rule out singular solutions [12].

Expression (7) can be cast in the standard Schrödinger equation form (equation (4)) if we can write

$$-(E_n - V(r)) = -\frac{1}{2}\{g,r\} + \frac{g'^2(r)}{4}\left[M^2(g) + 2M'(g) - 4J(g)\right] - \frac{(D-1)(D-3)}{4r^2} \tag{9}$$

Once we choose a particular orthogonal polynomial, $Q(g)$, to construct an exact solution of the Schrödinger equation, the characteristic functions of the polynomial $M(g)$, $J(g)$ get specified. We have to choose one or more than one terms containing the function $g(r)$ in expression (9) and put it equal to a constant to get the energy eigenvalues E_n. The procedure is worked out in detail for Romanovski polynomials in the next section.
It is interesting to note that when the generated potential is purely non-power law, the potential given by expression (9) has a term $(D-1)(D-3)/(4r^2)$ which behaves as constant background attractive inverse square potential in any arbitrary dimension except for dimensions 1 and 3. For power law cases, this background potential and the potential coming from Schwartzian derivative unite to give the correct centrifugal barrier potential in arbitrary dimensions.

3. Construction of exactly solvable potentials from Romanovski polynomials

Identifying

$$Q(g(r)) = R_n^{(\alpha,\beta)}(g) \tag{10}$$

as the Romanovski polynomial and its characteristic functions $M(g)$ and $J(g)$ are

$$M(g) = \frac{2(1-\alpha)g+\beta}{1+g^2} \quad ; \quad J(g) = -\frac{n(n-1)+2n(1-\alpha)}{1+g^2} \tag{11}$$

Using equations (10) and (11) in equation (7) yields

$$\begin{aligned}\frac{\psi''(r)}{\psi(r)} + \frac{(D-1)}{r}\frac{\psi'(r)}{\psi(r)} &= \frac{1}{4}(4(1-\alpha)+\beta^2)\frac{g'^2}{(1+g^2)^2} + (n(n-1)+2n(1-\alpha))\frac{g'^2}{1+g^2} \\ &- \alpha\beta\frac{gg'^2}{(1+g^2)^2} - \alpha(1-\alpha)\frac{g^2g'^2}{(1+g^2)^2} - \frac{1}{2}\frac{g'''}{g'} + \frac{3}{4}\frac{g''^2}{g'^2} - \frac{(D-1)(D-3)}{4r^2}\end{aligned} \tag{12}$$

and equation (8) yields

$$\psi(r) = Nr^{-\frac{(D-1)}{2}}g'^{-\frac{1}{2}}(1+g^2)^{\frac{1-\alpha}{2}}\exp\left(\frac{\beta}{2}\tan^{-1}g\right)R_n^{(\alpha,\beta)}(g) \tag{13}$$

To convert equation (12) into a standard stationary state Schrödinger equation, we make one or more terms of the right hand side of equation (12) a constant quantity. This enables us to get the energy eigenvalues E_n, the functional form of $g(r)$ and subsequently potential $V(r)$ and wave function $\psi(r)$.

(i) As a first case, let us choose

$$\frac{g'^2}{(1+g^2)^2} = p_1^2 \tag{14}$$

where p_1^2 is a real positive constant independent of r. Equation (14) gives the functional form of $g(r)$ as $g(r) = \tan p_1 r$. Using this value in equation (12) yields

$$E_n = -\left[n(n-1) - \alpha(2n-1) - 2\alpha - 1 + \frac{\beta^2}{4}\right]p_1^2 \tag{15}$$

$$V(r) = Ap_1^2\tan^2 p_1 r + Bp_1^2\tan p_1 r - \frac{(D-1)(D-3)}{4r^2} \tag{16}$$

$$\psi(r) = Nr^{-\frac{(D-1)}{2}}(\sec p_1 r)^{-(\alpha+1)}\exp\left(\frac{\beta}{2}p_1 r\right)R_n^{\alpha+1,\beta}(\tan p_1 r) \tag{17}$$

in equation (16), A and B are constants.

$$\alpha = \frac{n(n-1)-A}{2n-1} \quad ; \quad \beta = \frac{B(2n-1)}{1+A-n(n+1)} \tag{18}$$

(ii) Proceeding similarly let,

$$\frac{g'^2}{1+g^2} = p_2^2 \tag{19}$$

where p_2^2 is a real positive constant independent of r. Equation (19) gives the functional form of $g(r)$ as $g(r) = \sinh p_2 r$. Using this value in equation (12) yields

$$E_n = -\left[n(n-2\alpha+1) + \alpha(\alpha-1) + \frac{1}{4}\right] p_2^2 \tag{20}$$

$$V(r) = Ap_2^2 \operatorname{sech}^2(p_2 r) + Bp_2^2 \operatorname{sech}(p_2 r)\tanh(p_2 r) - \frac{(D-1)(D-3)}{4r^2} \tag{21}$$

$$\psi(r) = Nr^{-\frac{(D-1)}{2}} (\cosh p_2 r)^{-(\alpha-\frac{1}{2})} \exp\left[\frac{\beta}{2}\tan^{-1}(\sinh p_2 r)\right] R_n^{\alpha,\beta}(\sinh p_2 r) \tag{22}$$

where A, B are constants.

$$A = -\left[\alpha^2 - \frac{1}{4}(1+\beta^2)\right]; \qquad B = -\alpha\beta$$

The third choice $gg'^2/(1+g^2)^2$ in equation (12) is integrable to have $r = r(g)$, but not invertible. The choice g'''/g' equal to constant gives a similar system as in case (ii). The remaining terms $g^2 g'^2/(1+g^2)^2$ and g''^2/g'^2 do not yield potentials of physical interest.

4. Conclusions

We have obtained exactly solvable potentials of radial Schrödinger equation in D-dimensional Euclidean space whose bound state solutions are given in terms of Romanovski polynomials. We have presented a simple transformation method which consists of co-ordinate transformation followed by a functional transformation. The transformation method is performed on the linear second order differential equation satisfied by a particular special function to retrieve radial Schrödinger equation. The constructed potentials are non-power law type with a background inverse square potential $(D-1)(D-3)r^{-2}$ which vanishes for $D = 1$ and $D = 3$. The method can also be applied to construct exactly solvable potentials of Schrödinger equation from other orthogonal polynomials.

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