Gravitational redshift in Kerr field

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Abstract. Generally gravitational redshift has been calculated without consideration of rotation of a body. Neglecting the rotation, the geometry of space time can be described by using the well-known spherically symmetric Schwarzschild geometry. Rotation has great effect on general relativity, which gives new challenges on gravitational redshift. When rotation is taken into consideration spherical symmetry lost and off diagonal terms appears in the metric and the geometry of space time can be described by using the Kerr solution, which is the exact solution of the Einstein's field equations known at present. In this paper we will derive the expression for gravitational redshift for rotating source in Kerr field, and also apply the derived expression to calculate the gravitational redshift in case of Sun under Newtonian approximation of angular momentum.

1. Introduction

A line spectrum emitted by some atoms located, for example, on the sun, looks the same there as the spectrum emitted by the same atoms located on the earth would appear on it. If, however, we observe on the earth the spectrum emitted by the atoms located on the sun, then, its lines appear to be shifted with respect to the lines of the same spectrum emitted on the earth. Each line with frequency ω will be shifted through the interval $\Delta\omega < 0$, i.e. the shift occurs in the direction of lower frequency. The phenomenon we have described is called the "redshift". The first claimed confirmation of the predicted gravitational redshift came from the measurement of the apparent radial velocity of Sirius B, the white dwarf companion to Sirius (Adams 1925 [1]), identified as the difference from the motion expected in its 50 year orbit. The first Experimental verification of Gravitational redshift is gravitational redshift in nuclear resonance (Pound and Rebka 1959 [2]). An improved version of the experiment of Pound and Rebka has been performed to measure the effect of gravity (Pound and Snider 1965 [3]) making use of the Mossbauer Effect; the result found was (0.9990 ± 0.0076) times the value 4.905×10^{-15} of $2gh/c^2$ predicted from principle of equivalence. The redshift of the solar potassium absorption line at 7699 angstrom has been measured (Snider 1972 [4]) by means of an atomic - beam resonance - scattering technique. Gravitational redshift of sun has been also measured by (Krisher et al. 1993 [5]). A expression for gravitational redshift factor has been also calculated (P D Nunez and M Nowakowski, 2010 [6]) by applying small perturbation to the Schwarzschild geometry and two main highlighted results are the derivation of a maximum angular velocity depending only on the mass of the object and a possible estimates of the radius.

Covariant form of metric tensor in Boyer – Lindquist coordinates with signature (+,-,-,-) is expressed as

$$ds^2 = g_{tt}c^2dt^2 + g_{rr}dr^2 + g_{\theta\theta}d\theta^2 + g_{\phi\phi}d\phi^2 + 2g_{t\phi}cdtd\phi \tag{1}$$

Non-zero components $g_{\mu\nu}$ of Kerr field are given as follows

$$g_{tt} = \frac{\Delta - a^2\sin^2\theta}{\rho^2} \tag{2}$$

$$g_{rr} = -\frac{\rho^2}{\Delta} \tag{3}$$

$$g_{\theta\theta} = -\rho^2 \tag{4}$$

$$g_{\phi\phi} = -\frac{\left[(r^2+a^2)^2 - a^2\Delta\sin^2\theta\right]\sin^2\theta}{\rho^2} \tag{5}$$

$$g_{t\phi} = \frac{2mra\sin^2\theta}{\rho^2} \tag{6}$$

With

$$\rho^2 = r^2 + a^2\cos^2\theta \tag{7}$$

$$\Delta = r^2 - 2mr + a^2 = r^2 - r_g r + a^2 \tag{8}$$

Where the parameters, $m = \frac{GM}{c^2}$, e and $a = \frac{J}{Mc}$ are respectively associated with mass, charge and angular momentum per unit mass (rotation parameter) of the source. Where Schwarzschild radius, $r_g = 2m$.

For equatorial plane, $\theta = \frac{\pi}{2}$ the metric elements becomes,

$$g_{tt} = 1 - \frac{2m}{r} = 1 - \frac{r_g}{r} \tag{9}$$

$$g_{\phi\phi} = -(r^2 + a^2 + \frac{2ma^2}{r}) = -(r^2 + a^2 + \frac{r_g a^2}{r}) \tag{10}$$

$$g_{t\phi} = \frac{2ma}{r} = \frac{r_g a}{r} \tag{11}$$

2. Gravitational redshift from rotating body

The lagrangian of the test particle can be expressed as,

$$L = \frac{g_{ij}}{2}\dot{x}^i\dot{x}^j \tag{12}$$

(Dot over a symbol denotes ordinary differentiation with respect to an affine parameter λ)
From lagrangian of the test particle, we can obtain the momentum of the test particle as,

$$P_i = \frac{\partial L}{\partial \dot{x}^i} = g_{ii}\dot{x}^j \tag{13}$$

Thus corresponding momentum in the coordinates of t, r, θ and ϕ are expressed as

$$P_t \equiv -E = g_{tt}\dot{t} + g_{\phi\phi}\dot{\phi} \tag{14}$$

$$P_r = g_{rr}\dot{r} \tag{15}$$
$$P_\theta = g_{\theta\theta}\dot{\theta} \tag{16}$$
$$P_\phi \equiv L = g_{t\phi}\dot{t} + g_{\phi\phi}\dot{\phi} \tag{17}$$

Since the space time of the Kerr family is stationary and axially symmetric, and g_{ij} is a function of r and θ only, the momenta P_t and P_ϕ are conserved along the geodesics. So we obtain two constants of motion: one is corresponding to the conservation of energy (E) and the other is the angular momentum (L) about the symmetry axis.

Considering Stationary and axis symmetric space time, so a faraway observer can measure rotating body angular velocity (Ω) given by (Straumann N 1984 [7]),

$$\Omega = \frac{d\phi}{dt} = \left(\frac{d\phi}{d\tau}\right)\left(\frac{d\tau}{dt}\right) = \frac{u^\phi}{u^t} \tag{18}$$

Four velocity of a stationary point on the surface can be written as,

$$u^\mu = \left(u^t, 0, 0, u^\phi\right) \tag{19}$$

Using equation (18) in above equation (19)

$$u^\mu = \left(u^t, 0, 0, \Omega u^t\right) \tag{20}$$

Through the normalization condition of the four velocity given by (Landau LD 1980 [8])

$$u_\mu u^\mu = 1 = g_{\mu\nu} u^\mu u^\nu \tag{21}$$

$$g_{tt}u^t u^t + 2g_{t\phi}u^t u^\phi + g_{\phi\phi}u^\phi u^\phi = 1 \tag{22}$$

Using equation (20) in above equation (22) , we obtain the time-like component of the four velocities in terms of the metric components of Kerr field and angular velocity of rotation (Ω).

$$u^t = (g_{tt} + 2g_{t\phi}\Omega + g_{\phi\phi}\Omega^2)^{1/2} \tag{23}$$

Any observer measures the frequency $(\omega)$of a photon following null geodesic $x^\mu(\lambda)$can be calculated by the expression given by (Carroll SM 2004 [9]),

$$\omega = u^\mu \frac{dx^\mu}{d\lambda} = u^\mu g_{\mu\nu} \frac{dx^\nu}{d\lambda} \tag{24}$$

If photon is emitted at $r = \theta =$ constant, $dr = d\theta = 0$, then the frequency (ω) can be expressed as

$$\omega = u^t(g_{tt}\dot{t} + g_{t\phi}\dot{\phi}) + u^\phi(g_{t\phi}\dot{t} + g_{\phi\phi}\dot{\phi}) \tag{25}$$

Using equation (14 and 17) in above equation (25), we can write

$$\omega = u^t(-E) + u^\phi(L) = u^t(-E + \Omega L) \tag{26}$$

Using equation (23) in above equation (26), we can write the expression of frequency observed as

$$\omega = \frac{(-E + \Omega L)}{(g_{tt} + 2g_{t\phi}\Omega + g_{\phi\phi}\Omega^2)^{1/2}} \tag{27}$$

In general relativity gravitational redshift (Z) and redshift factor (g) is defined as

$$\frac{1}{Z+1} = g = \frac{\omega_{observed}}{\omega_{emitted}} \tag{28}$$

From equation (27) and equation (28), we can write the expression of gravitational redshift (Z) and gravitational factor (g) as,

$$\frac{1}{Z+1} = g = (g_{tt} + 2g_{t\phi}\Omega + g_{\phi\phi}\Omega^2)^{1/2} \tag{29}$$

We can conclude from equation (29) that the gravitational redshift will be affected by rotation of source because the expression contains the angular velocity of rotation of source in Kerr field. If we consider angular velocity (Ω) of source is zero then we can obtain the corresponding gravitational redshift from a static body of same mass.

3. Applications

Using equation (29), with the values of metric coefficients of Kerr field, we can calculate the numerical values of gravitational redshift from a rotating body.
Let us consider Sun as a rotating body, in case of Sun using the following values of Mass $(M) = 2.00\times10^{30}$, Radius $(r) = 0.70\times10^{6}$ km, Schwarzschild radius (r_g)= 3 km, Period (P)= 28 days.
Angular velocity of rotation,

$$\Omega = \frac{2\pi}{P} = 2.5982\times10^{-6}\,\text{rad/s} \tag{30}$$

Considering the Newtonian approximation of angular momentum,

$$J = \frac{2Mr^2\Omega}{5} \tag{31}$$

Rotation parameter,

$$a = \frac{J}{Mc} = \frac{2r^2\Omega}{5c} = 1.69749\text{ km} \tag{32}$$

The calculated value of gravitational redshift (Z) for Sun at equator $\theta = \frac{\pi}{2}$, is 2.11748247×10^{-6} and the corresponding value of redshift of same body (Sun) neglecting the rotation is 2.11746703×10^{-6}. The above calculation indicates that due to rotation effect there is change in gravitational redshift.

4. Acknowledgement

A Dubey is thankful to Sarani Chakraborty and Tanay Ghosh, Department of Physics, Assam University Silchar, for their useful discussions and suggestions.

References

[1] Adams W S, 1925, Observatory, **48**, 337.
[2] Pound R V and Rebka G V Jr., 1959, *Phys. Rev. Lett.,* Volume **3**, Number 9.
[3] Pound R V and Snider J L, 1965, *Physical review*, Volume **140**, Number 3B.
[4] Snider J L, *Phys. Rev. Lett.,* 1972, Volume **28**, Number 13.
[5] Krisher T P, Morabito D and Anderson J D , 1993, *Phys. Rev. Lett.,* **70**, 2213.
[6] Nunez P D and Nowakowski M, 2010, *J. Astrophys. Astr.*, **31**, 105–119.
[7] Straumann N, 1984, *General Relativity with applications to Astrophysics*, Springer Verlag.
[8] Landau L D and Lifshitz E M, 1980, *The Classical Theory of Fields,* Volume **2**, 4th edition Butterworth-Heinemann.
[9] Carroll SM, 2004, *Space time and Geometry,* Pearson Addison Wesley.