

Derivation of kinetic equations from non-Wiener stochastic differential equations

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Abstract. Kinetic differential-difference equations containing terms with fractional derivatives and describing α -stable Levy processes with $0 < \alpha < 1$ have been derived in a unified manner in terms of one-dimensional stochastic differential equations controlled merely by the Poisson processes.

1. Introduction

Stochastic differential equations (SDE) provide a straightforward and efficient technique of deriving kinetic equations to describe the dynamics of an open system both for classic cases [1] and quantum ones [2,3]. Nevertheless, before works [4-7] only Wiener-type SDEs were used to derive kinetic equations for both classic and quantum cases. In [4-7] density matrix equations for several models of quantum open systems were derived which allowed predicting and describing such effects as stabilization of excited quantum states of atomic ensembles and photon systems on the basis of quantum non-Wiener-type SDEs.

In classic physics, anomalous diffusion [8,9], single molecule radiation in disordered media [10] and other processes with exceptional (non-Gaussian) statistics and kinetics related to the Levy stochastic processes are in the front line of research. However, kinetic equations for such systems were derived not from SDEs with a noise source but by means of other indirect methods. For example, in work [11] the Chandrasekhar method [12] was used. In the meantime, it is common knowledge that both Wiener - and Levy processes can be defined on the basis of Poisson processes [13]. The Poisson-controlled SDEs appear to be non-Wiener type SDEs. The Poisson process with the intensity λ follows simple and intuitively obvious incremental algebra [14]:

$$dN(t)dN(t) = dN(t), \quad dN(t)dt = dt dt = 0, \quad \langle dN(t) \rangle = \lambda dt \quad (1)$$

which should seemingly simplify the derivation of not only kinetic equations but also simplify general theoretical analysis. Though until recently such an approach had not been available. In work [15] it was shown as a simple example that kinetic equations for the Levy processes can also be directly derived from SDEs.

2. The simplest kinetic equation derivation from SDE

The simplest classic stochastic process controlled by the Poisson process is an electric current in an electron tube. The appropriate SDE takes the following form [1]:



$$dX(t) = -\gamma X(t)dt + \beta dN(t), \quad -\gamma q_0 + \beta = 0, \quad (2)$$

where the value q_0 is the elementary charge. Hence it follows that a kinetic equation can be easily derived with the help of the method analogous to derivation [1] of the Fokker-Planck equation from the Wiener SDE when considering the increment $df(X(t)) = f(X(t) + dX(t)) - f(X(t))$ for any continuously differentiable function $f(x)$:

$$df(X(t)) = \frac{1}{1!} f'(X(t))dX(t) + \frac{1}{2!} f''(X(t))[dX(t)]^2 + \dots = f'(X(t))[-\gamma X(t)dt + \beta dN(t)] + \frac{1}{2!} f''(X(t))[dX(t)]^2 + \dots = -f'(X(t))\gamma X(t)dt + \{f(X(t) + \beta) - f(X(t))\}dN(t).$$

Upon averaging $\langle f(N(t)) \rangle = \int f(x)p(x,t)dx$ over probability density function $p(x,t)$ we have

$$\int f(x) \frac{\partial}{\partial t} p(x,t)dx = - \int f'(x)\gamma xp(x,t)dx + \lambda \int [f(x + \beta) - f(x)]p(x,t)dx,$$

Whence the differential-difference kinetic equation follows

$$\frac{\partial}{\partial t} p(x,t) = \frac{\partial}{\partial x} [\gamma xp(x,t)] + \lambda [p(x - \beta, t) - p(x, t)]. \quad (3)$$

3. Transition from microscopic to macroscopic description in terms of SDE

Equations (2) and (3) describe phenomena at microscopic level. In order to have a description at macroscopic level, it is necessary to renormalize initial equations [16,17]. For this purpose let us separate out the compensated Poisson process $N_c(t) = N(t) - \lambda t$, $\langle N_c(t) \rangle = 0$ in equation (2) and write it over in the form

$$dX(t) = (-\gamma X(t) + \lambda\beta)dt + \beta dN_c(t).$$

Then renormalization involves substitutions of $X(t)$ and $N_c(t)$ for $\tilde{X}(t) = X(st)/\phi(s)$ and $\tilde{N}_c(t) = N_c(st)/\phi(s)$ where the scaling function $\phi(s) \rightarrow \infty$ at $s \rightarrow \infty$:

$$d\tilde{X}(t) = (-\gamma s \tilde{X}(t) + \frac{\lambda s \beta}{\phi(s)})dt + \beta d\tilde{N}_c(t).$$

As a result, the increment

$$df(\tilde{X}(t)) = f'(\tilde{X}(t))[-\gamma s \tilde{X}(t) + \frac{\lambda s \beta}{\phi(s)}]dt + \beta d\tilde{N}_c(t) + \frac{1}{2!} \beta^2 f''(\tilde{X}(t))[d\tilde{N}_c(t)]^2 + \dots$$

and its average contains terms ($n \geq 2$) except for the time increment

$$d\tilde{N}_c(t), \quad \langle d\tilde{N}_c(t) \rangle = 0, \quad [d\tilde{N}_c(t)]^n = [\phi(s)]^{-n} dN(st), \quad \langle [d\tilde{N}_c(t)]^n \rangle = s[\phi(s)]^{-n} \lambda dt.$$

If we set $s = \phi(s) = \lambda \gg \gamma$ (for dimensionless quantities), then all terms with $n > 2$ within the average increment equation $\langle df(\tilde{X}(t)) \rangle$ can be neglected, thus we derive a conventional Fokker-Planck equation

$$\frac{\partial}{\partial t} p(x,t) = \frac{\partial}{\partial x} [\tilde{\gamma}(x - I)p(x,t)] + \frac{1}{2} \beta^2 \frac{\partial^2}{\partial x^2} p(x,t), \quad (4)$$

where a macroscopic quantity – average current magnitude $I = \beta/\gamma$ – was introduced, along with renormalized rate $\tilde{\gamma} = \lambda\gamma$.

A scaled compensated Poisson process $\tilde{N}_c(t)$ at $s = \lambda \rightarrow \infty$ appears as a conventional Wiener process $W(t)$ with increment algebra [1]

$$dW(t)dW(t) = dt, \quad dW(t)dt = dt dt = 0, \quad \langle dW(t) \rangle = 0. \quad (5)$$

Note that the relation of the scaled compensated Poisson process to the Wiener process is generally established by means of the characteristic function. In a similar way, one can consider rescaling of compensated Poisson process only.

Terms of SDE containing dt , $dW(t)$ and $dN(t)$ are conventionally named as drift, diffuse and jump.

4. The derivation of general kinetic equation from SDE

Let us consider the simplest general SDE with drift, diffuse and jump terms:

$$dX(t) = a(X, t)dt + b(X, t)dW(t) + \beta dJ(t), \quad \beta > 0, \quad (6)$$

where a jump term is represented as the increment of composite Poisson process $J(t) = \sum_{\delta} \delta N(\delta, t)$.

Poisson processes $N(\delta, t)$ with various δ are independent and different by intensity $\lambda(\delta)$ for different values of jump δ . Analogously, all Poisson processes included into $J(t)$ are statistically independent of the Wiener process, $dW(t)dN(\delta, t) = 0$.

Pursuing the described conventional scheme we have the increment

$$\begin{aligned} df(X) &= \sum_{k=1}^{\infty} \frac{1}{k!} \frac{\partial^k f(X)}{\partial X^k} (a(X, t)dt + b(X, t)dW(t) + \beta dJ(t))^k = \\ &= \left(\frac{\partial f(X)}{\partial X} a(X, t) + \frac{1}{2} \frac{\partial^2 f(X)}{\partial X^2} b^2(X, t) \right) dt + \frac{\partial f(X)}{\partial X} b(X, t) dW(t) + \sum_{\delta} \{f(X + \beta\delta) - f(X)\} dN(\delta, t). \end{aligned}$$

Upon averaging with regard to the increment algebra drift and diffuse terms lead to a conventional Fokker-Planck equation. Transformation of a jump term depends on the function $\lambda(\delta)$.

If we take the Levy measure [13,14] in place of $\lambda(\delta)$

$$\lambda(\delta) = \begin{cases} a_L \delta^{-(\alpha+1)}, & \delta > 0; \\ b_L |\delta|^{-(\alpha+1)}, & \delta < 0, \end{cases} \quad a_L > 0, \quad b_L > 0, \quad (7)$$

then the composite Poisson process will describe α -stable Levy process. In addition,

$$\langle \sum_{\delta > 0} \{f(X + \beta\delta) - f(X)\} dN(\delta, t) \rangle = \int p(x, t) \int_0^{\infty} a_L \frac{f(x + \beta\delta) - f(x)}{\delta^{\alpha+1}} d\delta.$$

Introducing a designation for the Marchaud fractional derivative

$$\mathbf{D}_{\pm}^{\alpha} f(x) := \frac{\alpha}{\Gamma(1-\alpha)} \int_0^{\infty} d\xi \frac{f(x) - f(x \mp \xi)}{\xi^{\alpha+1}}, \quad 0 < \alpha < 1$$

and making transformations analogous to those ones in [15], we obtain a Fokker-Planck generalized equation

$$\begin{aligned} \frac{\partial p(x, t)}{\partial t} &= -\frac{\partial}{\partial x} [a(x, t)p(x, t)] + \frac{1}{2} \frac{\partial^2}{\partial x^2} [b^2(x, t)p(x, t)] - \\ &- a_L \beta^{\alpha} \frac{\Gamma(1-\alpha)}{\alpha} \mathbf{D}_{+}^{\alpha} p(x, t) - b_L \beta^{\alpha} \frac{\Gamma(1-\alpha)}{\alpha} \mathbf{D}_{-}^{\alpha} p(x, t), \end{aligned} \quad (8)$$

which contains terms with fractional derivatives together with conventional terms. In case of good functions fractional derivatives on a line and Marchaud derivatives coincide $\mathbf{D}_{\pm}^{\alpha} f(x) = D_{\pm}^{\alpha} f(x)$ and according to equation (8) at $a_L = b_L$ terms with a fractional derivative coincide with analogous terms in work [11].

The represented derivation has shown strength and universality of the approach in terms of non-Wiener stochastic differential equations and extended an ordinary derivation of Fokker-Planck equation in case of the Gaussian process [1] to the Levy non-Gaussian processes with parameter $0 < \alpha < 1$. It is worth noting that the Levy measure can also be obtained from the rescaling procedure of noise term analogous to the one presented in [16,17].

Physical effects and regularities described by terms with fractional derivatives have not been discussed in this paper because they are analogous to the investigated ones in [11]. The main results of the paper consist in the demonstration of the unified derivation of kinetic equation for different cases from the non-Wiener SDE only, including the justification of terms with fractional derivatives.

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5. References

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