Architecture-independent power bound for vibration energy harvesters

E Halvorsen[1], C P Le[1], P D Mitcheson[2] and E M Yeatman[2]

[1] Department of Micro and Nano Systems Technology, Vestfold University College, Raveien 197, N-3184 Borre, Norway

[2] Department of Electrical and Electronic Engineering, Imperial College, London SW7 2AZ, United Kingdom

E-mail: Einar.Halvorsen@hive.no

Abstract. The maximum output power of energy harvesters driven by harmonic vibrations is well known for a range of specific harvester architectures. An architecture-independent bound based on the mechanical input-power also exists and gives a strict limit on achievable power with one mechanical degree of freedom, but is a least upper bound only for lossless devices. We report a new theoretical bound on the output power of vibration energy harvesters that includes parasitic, linear mechanical damping while still being architecture independent. This bound greatly improves the previous bound at moderate force amplitudes and is compared to the performance of established harvester architectures which are shown to agree with it in limiting cases. The bound is a hard limit on achievable power with one mechanical degree of freedom and can not be circumvented by transducer or power-electronic-interface design.

1. Introduction

In designing energy harvesters, an important question is how to adjust the parameters of a chosen harvester architecture in order to optimize the performance? This question has been resolved for a number of harvester architectures under various operating conditons [1, 2, 3, 4] and also resulted in guidelines on which among the known architectures to choose for given operating conditions [2]. Another important question is to what extent it is possible to significantly improve ouput power beyond that of the known architectures by inventing new device concepts, either related to the mechanics, the electromechanical conversion or the power electronic interface? In order to answer this question, it is necessary to know the ultimate limits on power without having to specify the details of the transducer and electronic interface, i.e. an architecture-independent power bound is needed. Such a bound based on input-power exists [5, 6], but does not account for parasitic losses which can amount to as much as half the input power even under optimal conditions [1]. To tighten the bound, losses should be accounted for. In this contribution we present a new theoretical bound that takes linear mechanical damping into account without making any a priori assumptions on the the details of the transducer or the electronic subsystem beyond the transducer having only one mechanical port. The optimal output powers of the established harvester architectures are compared to this bound.

2. Power bound

We consider energy harvesting systems characterized by a single mass m and a single mechanical displacement x. The mass is acted upon by an external force F which would usually be the inertial force $F = -m\ddot{y}$ due to the displacement y of the device frame attached to a vibrating body. The proof-mass motion drives an electromechanical transducer which may or may not include an elastic suspension. Unwanted parasitic loss is modelled by the linear damping force $b\dot{x}$ characterized by a constant b. An equivalent circuit for the system is shown in figure 1.

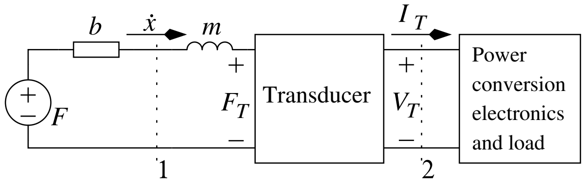

Figure 1. Energy harvesting system.

Newton's second law for the proof mass is $m\ddot{x} = -F_{\mathrm{T}} - b\dot{x} + F$ where F_{T} is the transducer force. As opposed to optimizing F_{T} for a given architecture, we here seek a least upper bound on the output power when we are not limited to already known architectures. This problem requires an approach that doesn't hinge on a priori knowledge of F_{T}. It is made possible by the power balance which follows from observing that the time average of the mechanical input power minus the time average of power lost in parasitic damping equals the average power delivered to the rest of the system. We therefore base the analysis on this energy difference expressed as

$$E[x; t_{\mathrm{b}}, t_{\mathrm{a}}] = \int_{t_{\mathrm{a}}}^{t_{\mathrm{b}}} \left[F(t)\dot{x}(t) - b(\dot{x}(t))^2\right] \mathrm{d}t = \int_{t_{\mathrm{a}}}^{t_{\mathrm{b}}} (F(t))^2 \mathrm{d}t/4b - b\int_{t_{\mathrm{a}}}^{t_{\mathrm{b}}} \left[\dot{x}(t) - F(t)/2b\right]^2 \mathrm{d}t. \quad (1)$$

for a time interval $[t_{\mathrm{a}}, t_{\mathrm{b}}]$ and seek the displacement waveform x that maximizes this quantity. This approach is equivalent to optimizing the average of the power $(F - b\dot{x})\dot{x}$ transferred at "1" in figure 1 and for long times it gives an upper bound on the power flow at later stages, such as the opportunity power at the output of the electromechanical transducer ("2" in the figure) or the final output power to a load or to a storage unit.

2.1. Unrestricted proof mass motion

The energy is always less than or equal to the first term on the r.h.s. of (1), so the average power is bounded by $\bar{P}(x; t_{\mathrm{b}}, t_{\mathrm{a}}) = E[x; t_{\mathrm{b}}, t_{\mathrm{a}}]/(t_{\mathrm{b}} - t_{\mathrm{a}}) \leq F_{\mathrm{rms}}^2/4b$ where F_{rms} is the rms force. The equality of the bound is reached if one can realize dynamics that give $\dot{x}(t) = F(t)/2b$. With a harmonic force $F = F_0 \cos\omega t$ of amplitude F_0 and angular frequency ω, $\bar{P} = F_0^2/8b$ when $t_{\mathrm{b}} - t_{\mathrm{a}} \to \infty$ and coincides with the optimal power point of the velocity damped generator (VDRG) [1, 2]. Therefore, no transducer or power electronic interface design can result in a better performance than the optimal VDRG when the forcing is harmonic and the displacement is unconstrained.

For a possibly irregular F, we insert $\dot{x}(t) = F(t)/2b$ into Newton's second law and solve for F_{T} to determine the transducer force under optimum operation. The result is $F_{\mathrm{T}} = F/2 - m\dot{F}/2b = b\dot{x} - m\ddot{x}$ which in the frequency domain amounts to complex conjugate load matching and cannot be achieved exactly over a band of frequencies with a passive load. The problem is evident for white noise which has infinite F_{rms}. It is conceivable to closely approximate the optimal force over a limited band of frequencies with an active control strategy as proposed in [7] for another target F_{T}. Such a system has bounded input power even with white noise input [8, 9].

2.2. *Strict limits on proof mass motion and periodic forcing*

We now seek an optimal proof mass trajectory $x = x(t)$ subject to the constraint that $-Z_\mathrm{l} \leq x(t) \leq Z_\mathrm{l}$ for a limit of displacement Z_l. We start with maximizing the energy (1) over the period T of the force $F(t)$. General solution methods exist for this type of inequality-constrained nonlinear programming problem [10, 11]. Based on the key ideas of these general methods, we present a specific approach for our problem.

The displacement constraint is turned into an equality constraint $x^2+s^2 = Z_\mathrm{l}^2$ by introducing an auxiliary real function $s = s(t)$ and is enforced by introducing a Lagrange multiplier $\lambda = \lambda(t)$. We first look for x, s and λ that make $S = \int_0^T \left[F\dot{x} - b\dot{x}^2 + \lambda(x^2 + s^2 - Z_\mathrm{l}^2)\right] \mathrm{d}t$ stationary with $\dot{x}$ piecewise differentiable and everywhere continuous. The first order variation of S w.r.t x, s and λ is then zero if

$$2b\ddot{x} + 2\lambda x = \dot{F}, \tag{2}$$
$$\lambda s = 0, \tag{3}$$
$$x^2 + s^2 = Z_\mathrm{l}^2. \tag{4}$$

Suppose now that x, s, λ solve (2)-(4), and consider another candidate periodic displacement waveform $\tilde{x}$ which we write $\tilde{x} = x + \Delta x$ and which also fulfils $|\tilde{x}| \leq Z_\mathrm{l}$. We permit $\dot{\tilde{x}}$ and $\Delta\dot{x}$ that have a finite number of step discontinuities, so that cases with impulse forces on the proof mass are within the set of admissible $\tilde{x}$. The converted energy (1) can then be written $E[\tilde{x}; T, 0] = E[x; T, 0] - 2\int_0^T \lambda x \Delta x \mathrm{d}t - b\int_0^T \Delta\dot{x}^2 \mathrm{d}t$ by help of an integration by parts, the periodic boundary condition, and (2). The rightmost integral is manifestly non-negative. When $\lambda \neq 0$, then $s = 0$ by (3), we have $|x| = Z_\mathrm{l}$ and therefore $x\Delta x \leq 0$. This leads to the conclusion that if $\lambda(t) \leq 0$ for all t, then also $\int_0^T \lambda x \Delta x \mathrm{d}t \geq 0$ and we have $E[\tilde{x}; T, 0] \leq E[x; T, 0]$ for all admissible $\tilde{x}$. Hence, a solution of (2-4) with $\lambda(t) \leq 0$, if it exists, maximizes the energy per cycle.

We reiterate the above arguments for a possibly aperiodic $\tilde{x}$ and find that the average power $\bar{P}$ over an arbitrary time interval $[t_\mathrm{a}, t_\mathrm{b}]$ can be bounded as $\bar{P}[\tilde{x}; t_\mathrm{b}, t_\mathrm{a}] \leq E[x; t_\mathrm{b}, t_\mathrm{a}]/(t_\mathrm{b} - t_\mathrm{a}) + (F - 2b\dot{x})\Delta x|_{t=t_\mathrm{a}}^{t_\mathrm{b}} / (t_\mathrm{b} - t_\mathrm{a})$. It follows that

$$P[\tilde{x}] \equiv \lim_{t_\mathrm{b} - t_\mathrm{a} \to \infty} \bar{P}[\tilde{x}; t_\mathrm{b}, t_\mathrm{a}] \leq P[x] = E[x; T, 0]/T. \tag{5}$$

Therefore the specified periodic solution also maximizes the long-time average power. It remains to show that such a solution actually exists. We will do so for a time-harmonic force.

2.3. *Optimum operation for a time harmonic force*

For a small enough force, the solution for unrestricted motion still applies, and with $F = F_0 \cos\omega t$ it is $x = (F_0/2b\omega)\sin\omega t$ up to a constant. Displacement-constrained operation will be reached at a critical force amplitude $F_\mathrm{c} = 2b\omega Z_\mathrm{l}$ with a critical power of $P_c = F_\mathrm{c}^2/8b = b\omega^2 Z_\mathrm{l}^2/2$.

For $F_0 \geq F_\mathrm{c}$, we note that while $\lambda \neq 0$ so that $s = 0$ by (3), the proof mass is at rest at one of the displacement limits according to (4). We can then use (2) to write the requirement of nonpositive λ as $\lambda = \dot{F}/2x = -F_0\omega\sin(\omega t)/2x \leq 0$ and conclude that the proof mass may only rest at $x = Z_\mathrm{l}$ if $\sin\omega t \geq 0$ and at $x = -Z_\mathrm{l}$ if $\sin\omega t \leq 0$. When $\lambda = 0$, we integrate (2) twice to get

$$\dot{x} = F_0(\cos\omega t - \cos\theta_1)/2b \quad \text{and} \quad x = F_0(\sin\omega t - \omega t\cos\theta_1)/2b\omega \tag{6}$$

where the integration constants were determined by requiring zero velocity at the limits. The limit $x(t_1) = Z_\mathrm{l}$ is reached at a time t_1 and a phase angle $\theta_1 = \omega t_1 \in [0, \pi/2]$ given by

$$\sin\theta_1 - \theta_1\cos\theta_1 = F_\mathrm{c}/F_0. \tag{7}$$

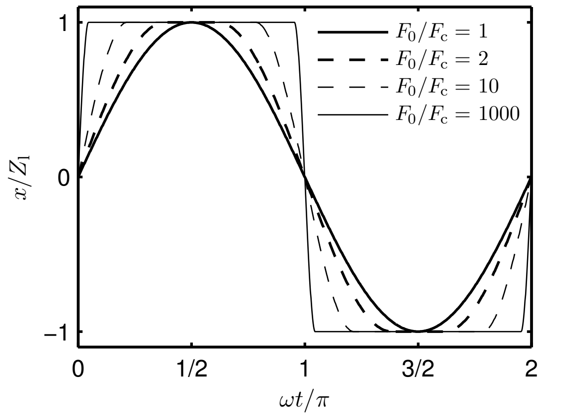

Figure 2. Displacement waveforms that maximize power delivered into the transducer.

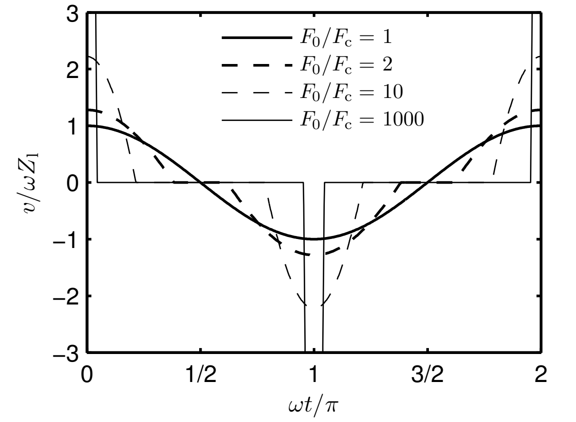

Figure 3. Velocity waveforms that maximize power delivered into the transducer.

Because $F(t-\pi/\omega) = -F(t)$, we have that $-x(t-\pi/\omega)$ is a solution of (2) if $x(t)$ is. We can therefore generate solutions for any $\omega t \in [n\pi-\theta_1, n\pi+\theta_1]$ with integer n by repeated application of this symmetry operation to (6). For the intervening intervals, the proof mass is at rest at either of the limits which is also a feature of the Coulomb-force parametric generators (CFPGs) [2]. The resulting waveforms are shown in figures 2 and 3.

To determine the output power bound for $F_0 \geq F_c$, we insert the optimal solution into (1) and divide by T. One way to write the result is

$$P/P_c \quad = \quad \frac{1}{\pi}\left(\frac{F_0}{F_c}\right)^2 (\sin 2\theta_1 - 2\theta_1 \cos 2\theta_1)\,. \tag{8}$$

Figure 4 shows this power bound over a range of force amplitudes. It is compared to the exact result for the optimal VDRG [2] which coincides with the bound for $F_0 \leq F_c$ as discussed above. For larger F_0, the initial asymptotics agrees to first order in $F_0/F_c - 1$, but as $F_0/F_c \to \infty$ the deviation increases and the power bound approaches asymptotically the mechanical input-power bound $P_{\text{in,bound}}/P_c = 8F_0/\pi F_c$ [5, 6] which is $4/\pi$ larger than for the VDRG. The CFPG has output power $P_{\text{CFPG}} = \beta P_{\text{in,bound}}$ with the parameter $\beta \to 1$ as the vibration amplitude goes to infinity. Hence, its performance coincides with the bound in this limit.

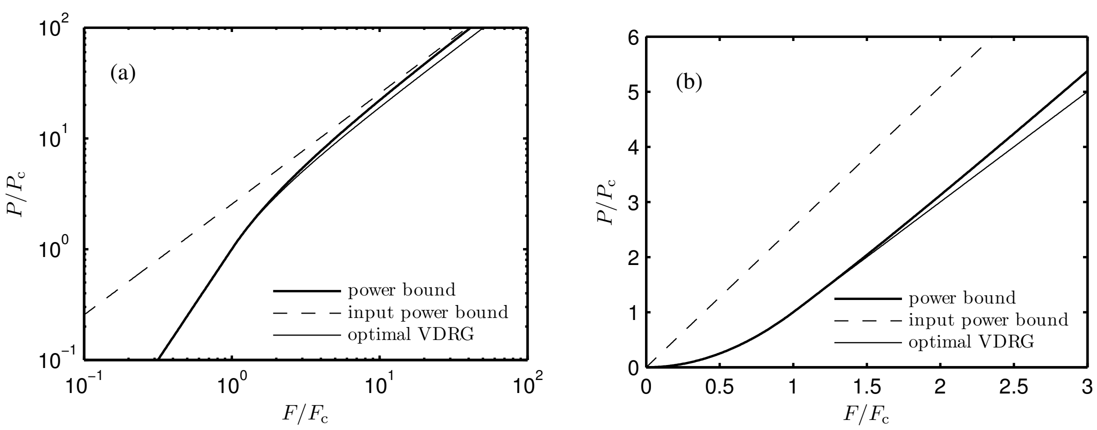

Figure 4. Output-power bound compared to optimal VDRG and input-power bound. (a) Wide force-amplitude range. (b) Moderate force amplitudes.

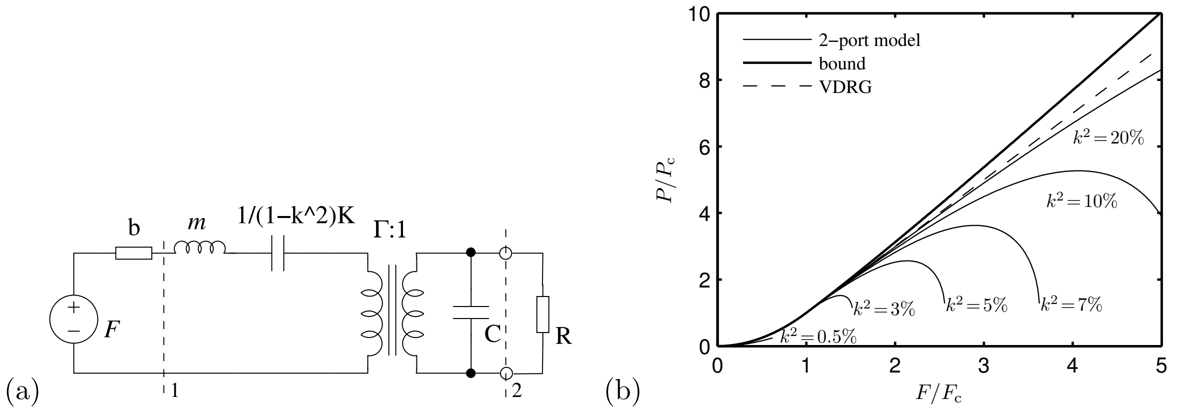

Figure 5. Linear two-port model. (a) Equivalent circuit. (a) Output power with $Q_{\mathrm{m}} = 100$.

3. Linear two-port devices

The linear-two port model has the equivalent circuit in figure 5a and performs equally with the VDRG without constraints and for $k^2 Q_{\mathrm{m}} > 1 + \sqrt{1 - k^2}$ with $k^2 = \Gamma^2/KC$, $Q_{\mathrm{m}} = m\omega_0/b$ and $\omega_0 = \sqrt{K/m}$ [3]. Its output power is shown in figure 5b where the load resistance has been adjusted to avoid displacement beyond the limit. The drive frequency is held fixed at the value near the resonant frequency that is optimal for unconstrained motion [3]. The two-port model does not perform better than the VDRG and even has a power drop before sufficient damping becomes impossible to achieve. However, the higher the coupling is, the further into the high-force regime is it possible to operate and the closer the model follows the VDRG. Therefore higher coupling than the critical one is advantageous for displacement-limited operation.

4. Concluding remarks

We have derived a new architecture-independent power bound which at small to intermediate force-amplitudes greatly improves on using input-power bound as a bound for output power. VDRG performance is equal or close to the new bound for small force amplitudes while the CFPG performance is close to it when the force approaches infinity. The bound is a fundamental limit which can not be circumvented by transducer or power-electronic-interface design. Multiple degrees of freedom and explicit treatment of other parasitic damping than linear are interesting extensions to pursue in future work.

Acknowledgments

This work was supported by the Research Council of Norway under grant no. 191282.

References

[1] Williams C B and Yates R B 1996 *Sens. Actuators A* **52** 8–11
[2] Mitcheson P D, Green T C, Yeatman E M and Holmes A S 2004 *J. Microelectromech. Syst.* **13** 429–440
[3] Renno J M, Daqaq M F and Inman D J 2009 *J. Sound Vibr.* **320** 386 – 405
[4] Halvorsen E 2008 *J. Microelectromech. Syst.* **17** 1061–1071
[5] Mitcheson P D, Yeatman E M, Rao G K, Holmes A S and Green T C 2008 *Proc. IEEE* **96** 1457–1486
[6] Ramlan R, Brennan M, Mace B and Kovacic I 2010 *Nonlinear Dynamics* **59**(4) 545–558
[7] Okamoto H, Onuki T, Nagasawa S and Kuwano H 2009 *J. Microelectromech. Syst.* **18** 1420 –1431
[8] Scruggs J 2009 *J. Sound Vibr.* **320** 707 – 725
[9] Halvorsen E 2013 *Phys. Rev. E* **87**(4) 042129
[10] Léonard D and Long N V 1992 *Optimal control theory and static optimization in economics* (Cambridge University Press)
[11] Zalmai G J 1985 *Journal of mathematical analysis and applications* **109** 426–452