

Quantum Lie algebra solitons

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Abstract. We construct a special type of quantum soliton solutions for quantized affine Toda models. The elements of the principal Heisenberg subalgebra in the affinised quantum Lie algebra are found. Their eigenoperators inside the quantized universal enveloping algebra for an affine Lie algebra are constructed to generate quantum soliton solutions.

1. Conformal and affine abelian Toda systems: classical region

1.1. Lie-algebraic setting

Let M be a manifold, \mathbb{R}^2 or \mathbb{C}^1 , with the standard coordinates z_{\pm} . For \mathbb{C}^1 we suppose that $z_- = z_+^*$. Let G be a complex simple Lie group [25] of rank r with the Lie algebra \mathcal{G} endowed with the principal grading. In the decomposition $\mathcal{G} = \oplus_{m \in \mathbb{Z}} \mathcal{G}_m$, the subspace \mathcal{G}_0 is abelian. Denote by G_0 and G_{\pm} the subspaces corresponding to \mathcal{G}_0 and $\oplus_{m > 1} \mathcal{G}_{\pm m}$ respectively. Denote by h_i and $x_{\pm i}$ Cartan and Chevalley generators of \mathcal{G} . In the principal grading, the elements of \mathcal{G}_0 and $\mathcal{G}_{\pm 1}$, respectively, satisfy the defining relations

$$[h_i, h_j] = 0, [h_i, x_{\pm j}] = \pm k_{ij} x_{\pm j}, [x_{+i}, x_{-j}] = \delta_{ij} h_i, \quad 1 \leq i, j \leq r, \quad (1)$$

where k is Cartan matrix of \mathcal{G} .

1.2. Conformal Toda systems

Conformal abelian Toda fields $\phi = \sum_{i=1}^r h_i \phi_i$ satisfy the equations [26, 18]

$$\partial_+ \partial_- \phi + \frac{4\eta^2}{\beta} \sum_{i=1}^r m_i \frac{\alpha_i}{\alpha_i^2} e^{\beta \alpha_i \cdot \phi} = 0, \quad (2)$$

with some coupling constant β and a length scale factor η . The formal general solution [18, 17] to (2), found in holomorphically factorisable form, is given by

$$e^{-\beta \lambda_j \cdot \phi} = \langle \Lambda_j | \gamma_+^{-1} \mu_+^{-1} \mu_- \gamma_- | \Lambda_j \rangle, \quad (3)$$

where $\gamma_{\pm}(z_{\pm})$ and $\mu_{\pm}(z_{\pm})$ are holomorphic and antiholomorphic mappings $M \rightarrow G_0$, $M \rightarrow G_{\pm}$ to the Gauss decomposition subgroups G_0 , G_{\pm} , respectively; $|\Lambda_i\rangle$ is the highest vector of the i -th fundamental representation of \mathcal{G} . Mappings $\mu_{\pm}(z_{\pm})$ satisfy the initial value problem

$$\partial_{\pm} \mu_{\pm} = \mu_{\pm} \kappa_{\pm}, \quad \kappa_{\pm}(z_{\pm}) = \sum_{i=1}^r \phi_i^0 x_{\pm i}, \quad (4)$$



where κ_{\pm} are mappings $M \rightarrow \mathcal{G}_{\pm 1}$ that can be also represented as

$$\kappa_{\pm}(z_{\pm}) = \gamma_{\pm}^{-1} E_{\pm 1} \gamma_{\pm}, \quad E_{\pm 1} = \sum_{i=1}^r m_i x_{\pm i}, \quad (5)$$

with some nonzero constants m_i . Note that to obtain parametric solutions from (3), in particular instantons [17], it is pretty enough to take the screening functions $\phi_i^0(z_{\pm}) = c_{\pm i} z_{\pm}^{n_i}$ parametrized by constants $c_{\pm i}$.

1.3. Affine Toda systems

Affine Toda fields satisfy the equations

$$\partial_+ \partial_- \phi + \frac{4\eta^2}{\beta} \left(\sum_{i=1}^r m_i \frac{\alpha_i}{\alpha_i^2} e^{\beta \alpha_i \cdot \phi} - \frac{\psi}{\psi^2} e^{-\beta \psi \cdot \phi} \right) = 0. \quad (6)$$

Here ∂_{\pm} stand for the partial derivatives with respect to z^{\pm} , η conventionally denotes a real inverse length scale, and β is an imaginary coupling constant. The coefficients are arranged in such a way that $\phi = 0$ is a constant solution. The formal general solution to (6) was found in [21]

$$e^{-\beta \lambda_j \cdot \phi} = e^{-\beta \lambda_j \cdot \phi^0} \frac{\langle \Lambda_j | \gamma_+^{-1} \mu_+^{-1} \mu_- \gamma_- | \Lambda_j \rangle}{\langle \Lambda_0 | \gamma_+^{-1} \mu_+^{-1} \mu_- \gamma_- | \Lambda_0 \rangle^{m_j}}, \quad 1 \leq i \leq r, \quad (7)$$

with some quite clear different meaning of the ingredients here. Namely, $\widehat{\mathcal{G}}$ is an affine Kac–Moody Lie algebra [14] of rank $r+1$, k (in the relations of type (1)) is an affine (degenerated with the single zero eigenvalue) matrix. For more details on affine Lie algebras see [14]. Moreover, it is convenient to enlarge Cartan subalgebra of $\widehat{\mathcal{G}}$ by the derivative element d , such that $[d, h_i] = 0$, $[d, x_{\pm i}] = \pm x_{\pm i}$, and then completed Cartan subalgebra has dimension $r+2$. Positive integers m_i in (7) are defined as the lowest for which $\sum_i k_{ji} m_i = 0$. In terms of these integers the dual Coxeter numbers of $\widehat{\mathcal{G}}$ are written as $\tilde{h} = \sum_{i=0}^r m_i$, while the centre of $\widehat{\mathcal{G}}$ is $c = \sum_{i=0}^r m_i h_i$. Finally, $|\Lambda_i\rangle$ is the highest vector of the i -th fundamental representation of affine algebra $\widehat{\mathcal{G}}$. Note that, contrarily to those for finite dimensional systems, the general solution (7) has a rather complicated structure. In particular, it can be represented as infinite series, though absolutely convergent ones.

1.4. Solitonic specialisation on classical level

In [22] a remarkable specialisation of the general solution (7) which led to soliton solutions was suggested. Before we explain it, let us introduce some useful information and notations. The subspaces $\widehat{\mathcal{G}}_{\pm 1}$ of an affine Lie algebra $\widehat{\mathcal{G}}$ endowed with the principal grading contain elements $\widehat{E}_{\pm 1} = \sum_{i=0}^r \sqrt{m_i} x_{\pm i}$ of the principal Heisenberg subalgebra of $\widehat{\mathcal{G}}$. These elements, thanks to the defining relations, satisfy a very important condition

$$[\widehat{E}_{+1}, \widehat{E}_{-1}] = c.$$

The whole principal Heisenberg subalgebra of $\widehat{\mathcal{G}}$ commuting with \widehat{E}_{+1} , modulo the central term c , is spanned by $\{\widehat{E}_m\}$ with m being exponents of $\widehat{\mathcal{G}}$. One can normalise the invariant bilinear form on $\widehat{\mathcal{G}}$ in such a way that

$$[\widehat{E}_m, \widehat{E}_{-n}] = \delta_{mn} m c.$$

The subspace of $\widehat{\mathcal{G}}$, $\widehat{\mathcal{G}}_m$, $m \neq 0$ contains, in addition to \widehat{E}_m , the elements \widehat{F}_m^i , $1 \leq i \leq r$, satisfying the relations

$$[\widehat{E}_m, \widehat{F}_n^i] = \chi_i \widehat{F}_{m+n}^i,$$

with some constants χ_i that belong to the root space of $\widehat{\mathcal{G}}$. Then the generating elements for \widehat{F}_n^i , $\widehat{F}^i := \widehat{F}^i(t) = \sum_{n \in \mathbb{Z}} t^{-n} \widehat{F}_n^i$ with a complex parameter t , is an eigenvector of \widehat{E}_m ,

$$[\widehat{E}_m, \widehat{F}^i(t)] = \chi_i^{(m)} t^m \widehat{F}^i(t). \quad (8)$$

The simplest solitonic specialisation found in [22] consists in the following. Let \widehat{G} be an infinite-dimensional group with the Lie algebra $\widehat{\mathcal{G}}$. One chooses mappings γ_{\pm} in (5) to be unit elements of the subgroup \widehat{G}_0 of \widehat{G} , and writes solutions to (4) as

$$\mu_{\pm} = \mu_{\pm}^0 e^{\eta^{\pm} z_{\pm} \widehat{E}_{\pm 1}}.$$

with some constants η^{\pm} , and constant mapping $\mu^0 \equiv (\mu_+^0)^{-1} \mu_-^0 : M \rightarrow \mathcal{G}$ independent of the coordinates z_{\pm} . The elements $\widehat{E}_{\pm 1}$ can be removed from (7) if one chooses μ^0 in the form

$$\mu^0 = \prod_{i=1}^N Q_i e^{\widehat{F}^i},$$

where $Q_i \in \mathbb{R}$ are some constants, and uses relations (8), namely that $[\widehat{E}_{\pm 1}, \widehat{F}^i] = \chi_i^{(\pm 1)} t^m \widehat{F}^i(t)$. Then (7) delivers an N -soliton solutions [5] to the affine Toda model characterized by the parameters Q_i and χ_i^{\pm} , $i = 1, \dots, N$.

2. Quantum Lie algebras

2.1. The quantised universal enveloping algebra $U_q(sl_2)$

In the spirit of [4, 13], the quantised enveloping algebra $U_q(sl_2)$ is an associative algebra generated by X^+ , X^- , H with q -deformed commutation relations

$$X^+ X^- - X^- X^+ = (q^H - q^{-H}) (q - q^{-1})^{-1}, \quad H X^{\pm} - X^{\pm} H = \pm 2 X^{\pm}.$$

It possesses a Hopf algebra structure with the deformed adjoint action

$$(ad_{X^{\pm}})_q a = X^{\pm} a q^{H/2} - q^{\mp 1} q^{H/2} a X^{\pm}, \quad (ad_H)_q a = H a - a H,$$

for all $a \in U_q(sl_2)$.

2.2. The quantum algebra $(sl_2)_q$

In [3, 8] a new deformed basis for the generators in the quantised universal enveloping algebra $U_q(sl_2)$ was introduced

$$X_h^{\pm} = \sqrt{2(q + q^{-1})^{-1}} q^{-H/2} X^{\pm}, \quad H_h = 2(q + q^{-1})^{-1} (q X^+ X^- - q^{-1} X^- X^+), \quad (9)$$

which form a three dimensional subspace $(sl_2)_q$ generated by $\{X_h^+, X_h^-, H_h\}$ in $U_q(sl_2)$ closed under the quantum Lie bracket

$$[a, b]_h := (ad_a)_q b, \quad a, b \in (sl_2)_q. \quad (10)$$

The generators (9) possess the following commutation relations:

$$\begin{aligned} [H_h, X_h^{\pm}]_h &= \pm 2 q^{\pm 1} X_h^{\pm}, & [X_h^{\pm}, H_h]_h &= \mp 2 q^{\mp 1} X_h^{\pm}, & [X_h^+, X_h^-]_h &= H_h, \\ [X_h^-, X_h^+]_h &= -H_h, & [H_h, H_h]_h &= 2(q - q^{-1}) H_h, & [X_h^{\pm}, X_h^{\pm}]_h &= 0. \end{aligned}$$

The algebra $(sl_2)_q$ is not a Lie algebra in the standard sense. The generators (9) do not satisfy Jacobi identity and deformed Lie bracket (10) is not skew-symmetric. Although the q -analogue of Jacobi identity for $(sl_2)_q$ is missing, nevertheless (10) is q -skew-symmetric in accordance with [8]. By q -skew-symmetry we mean a symmetry under q -conjugation (which we will denote with a tilde), the automorphism of $(sl_2)_q$ defined by $q \mapsto 1/q$. Then for an element in $(sl_2)_q$ $(a X_h^+ + b X_h^- + c H_h) \sim \tilde{a} X_h^+ + \tilde{b} X_h^- + \tilde{c} H_h$, $a, b, c \in \mathbb{C}$, the q -deformed Lie bracket satisfies $[a, b]_h = [\tilde{b}, \tilde{a}]_h$.

2.3. The affinisation of $(sl_2)_q$

In this subsection we introduce the affinisation of the algebra $(sl_2)_q$. Denote $\mathcal{G} = (sl_2)_q$. Let $\mathcal{L} = \mathbb{C}[t, t^{-1}]$ be the algebra of Laurent polynomials in t and $\mathcal{L}(\mathcal{G}) = \mathcal{L} \otimes_{\mathbb{C}} \mathcal{G}$. Introduce the complex vector space $(sl_2)_q^t$: $\tilde{\mathcal{L}}(\mathcal{G}) = \mathcal{L}(\mathcal{G}) \oplus \mathbb{C}c \oplus \mathbb{C}d$. This is a loop algebra $\mathcal{L}(\mathcal{G})$ completed with the derivation d (acting as $t \frac{d}{dt}$ in \mathcal{L} and trivially on c) extended by one dimensional center c corresponding to \mathbb{C} -valued q -deformed 2-cocycle $\Psi_q(a, b) = (x|y)_h \Phi(P, Q)$, $\Phi(P, Q) = \text{Res}_t \frac{dP}{dt} Q$ on $\mathcal{L}(\mathcal{G})$. Here $(x|y)_h$ is a non-degenerate bilinear form on $(sl_2)_q$ and P, Q are polynomials in t . We define the q -deformed Lie bracket in this algebra as

$$\begin{aligned} & [t^m \otimes x \oplus \omega c \oplus \nu d, \quad t^n \otimes y \oplus \omega_1 c \oplus \nu_1 d]_h \\ & = (t^{m+n} \otimes [x, y]_h + \nu n t^n \otimes y - n \nu_1 m t^m \otimes x) \oplus m \delta_{m+n,0} (x|y)_h c, \end{aligned} \quad (11)$$

where $x, y \in \mathcal{G}$; $\nu, \omega, \nu_1, \omega_1 \in \mathbb{C}$. Now we introduce generators that form the affinisation of the quantum algebra $(sl_2)_q$

$$H_1 = 1 \otimes H_h, \quad H_0 = 1 \otimes (c - H_h), \quad e_1 = 1 \otimes X_h^+, \quad e_0 = t \otimes X_h^-, \quad f_1 = 1 \otimes X_h^-, \quad f_0 = t^{-1} \otimes X_h^+.$$

Then we derive the adjoint action:

$$\begin{aligned} & [H_0, e_0]_h = 2e_0 q^{-1}, \quad [H_0, f_0]_h = -2f_0 q^{-1}, \quad [H_0, e_1]_h = -2e_1 q, \quad [H_0, f_1]_h = 2f_1 q^{-1}, \\ & [H_1, e_1]_h = 2e_1 q, \quad [H_1, f_1]_h = -2f_1 q^{-1}, \quad [H_1, e_0]_h = 2e_0 q^{-1}, \quad [H_1, f_0]_h = 2q f_0, \\ & [H_0, H_1]_h = -2(q - q^{-1})H_1, \quad [H_0, H_0]_h = 2(q - q^{-1})H_1, \\ & [H_1, H_1]_h = 2(q - q^{-1})H_1, \quad [H_1, H_0]_h = -2(q - q^{-1})H_1, \\ & [e_0, f_0]_h = H_0, \quad [e_1, f_1]_h = H_1, \quad [f_0, e_0]_h = -H_0, \quad [f_1, e_1]_h = -H_1, \\ & [e_0, H_0]_h = -2q e_0, \quad [e_0, H_1]_h = 2q e_0, \quad [e_1, H_0]_h = 2e_1 q, \quad [e_1, H_1]_h = -2e_1 q, \\ & [f_0, H_0]_h = 2q^{-1} f_0, \quad [f_0, H_1]_h = -2q^{-1} f_0, \quad [f_1, H_1]_h = 2q f_1, \quad [f_1, H_0]_h = -2q f_1. \end{aligned}$$

2.4. The Heisenberg subalgebra and eigenvectors of the q -deformed adjoint action

Recall that the basic point in the construction of solitonic solutions [22] to the affine Toda equations (6) is the existence of eigenvectors with respect to elements of the Heisenberg subalgebra of underlying affine algebra. Here we define the elements of the principal Heisenberg subalgebra

$$\hat{E}_h^+ = 1 \otimes X_h^+ + t \otimes X_h^-, \quad \hat{E}_h^- = 1 \otimes X_h^- + t^{-1} \otimes X_h^+,$$

so that the following generating series for $\zeta \in \mathbb{C}$,

$$\begin{aligned} \mathcal{F}_q &= \sum_{k=-\infty}^{+\infty} \zeta^k A_k, \quad A_{2m} = (q + q^{-1})^m t^m \otimes (-H_h), \\ A_{2m+1} &= (q + q^{-1})^m t^m \otimes q^{-1} X_h^+ - t^{m+1} \otimes q X_h^- + (X_h^- | X_h^+)_h, \quad m \in \mathbb{Z}, \end{aligned} \quad (12)$$

is an eigenvector of \hat{E}_h^\pm with respect to the bracket (11) with eigenvalues $\zeta^{\pm 1}$. Using \mathcal{F}_q one can find a quantum analogue of solitonic solutions corresponding to the affinisation of the quantum algebra $(sl_2)_q$.

3. Quantum group solutions

3.1. Formal quantum group solutions

There are a few ways to quantize the conformal affine Toda models [6, 9, 10, 11, 12, 15, 16, 20]. In [20] the light cone quantization was performed, appropriate quantum equations and Lax

pairs found, and formal quantum solutions [20, 6, 15, 16] constructed. Previously, in [15, 16] it was shown that the formal quantum solution in terms of Heisenberg field operators and in the Yang–Feldman perturbative formalism for the quantized conformal affine Toda equations can be written in the form (7), (3) but with ingredients replaced by their quantum analogues, namely the group-like elements ${}_q\gamma_{\pm}$, ${}_q\mu_{\pm}$ of the quantized universal enveloping algebras $U(\mathcal{G})$ in the Gauss decomposition [2], and vectors $|\Lambda\rangle_q$ in highest weight fundamental representations for corresponding quantum group [1].

The quantised affine Toda Heisenberg field operators (see [20] for the conformal Toda model) associated to an affine Lie algebra $\widehat{\mathcal{G}}$ satisfy the equations

$$\partial_+ \partial_- \phi^{(q)} + \frac{4\eta^2}{\beta} \left(\sum_{i=1}^r m_i \frac{\alpha_i}{\alpha_i^2} : e^{\beta \alpha_i \cdot \phi^{(q)}} : - \frac{\psi}{\psi^2} : e^{-\beta \psi \cdot \phi^{(q)}} : \right) = 0.$$

We can get quantum group deformed solutions to the affine Toda equations corresponding to $U_q(\widehat{\mathcal{G}})$. Starting from the general solution (7) to the affine Toda equations for $\widehat{\mathcal{G}}$ we replace the vectors of fundamental highest weight $\widehat{\mathcal{G}}$ -representations of by the fundamental highest weight representation vectors of $U_q(\widehat{\mathcal{G}})$. One shows [20] that the quantum Toda system possess the Heisenberg field operator solution of the form (dots denoting normal ordering) for $1 \leq i \leq r$,

$$: e^{-\beta \lambda_j^{(q)} \cdot \phi^{(q)}} :=: e^{-\beta \lambda_j^{(q)} \cdot \phi_0^{(q)}} : {}_q\langle \Lambda_j | {}_q\gamma_+^{-1} {}_q\mu_+^{-1} {}_q\mu_- {}_q\gamma_- | \Lambda_j \rangle_q [{}_q\langle \Lambda_0 | {}_q\gamma_+^{-1} {}_q\mu_+^{-1} {}_q\mu_- {}_q\gamma_- | \Lambda_0 \rangle_q]^{-m_j}, \quad (13)$$

where $\phi_0^{(q)}$ are free field operators, and $\partial_{\pm} {}_q\mu_{\pm} = {}_q\mu_{\pm} {}_q\kappa_{\pm}$, ${}_q\gamma_{\pm}$, ${}_q\mu_{\pm}$, and ${}_q\kappa_{\pm}$ are mappings $M \rightarrow U(\widehat{\mathcal{G}})$.

3.2. Quantum group soliton solutions

Quantum solutions [9, 10, 11] generated by quantum soliton operators [24] and corresponding to classical soliton solutions [22] can be obtained from the quantum formal solution (13). For that purpose we put ${}_q\gamma_{\pm} = Id$, i.e., the unit mapping, and let ${}_q\kappa_{\pm}(z_{\pm}) = {}_q\gamma_{\pm}^{-1} {}_q\widehat{E}_{\pm 1} {}_q\gamma_{\pm}$, where ${}_q\widehat{E}_{\pm 1}$ are supposed to be generators of a Heisenberg subalgebra in $U_q(\widehat{\mathcal{G}})$. Then ${}_q\mu_{\pm}$ are of the form

$${}_q\mu_{\pm} = {}_q\mu_{\pm}^0 e^{\eta^{\pm} z_{\pm} {}_q\widehat{E}_{\pm 1}}.$$

By letting ${}_q\mu^0 = ({}_q\mu_+^0)^{-1} {}_q\mu_-^0 = e^{Q\mathcal{F}}$ for $Q \in \mathbb{R}$, we therefore obtain from (13)

$$: e^{-\beta \Lambda_i \cdot \phi^{(q)}} :=: e^{-\beta \lambda_j \cdot \phi_0^{(q)}} : {}_q\langle \Lambda_j | e^{-q\widehat{E}^+ z_+} e^{Qq\mathcal{F}} e^{q\widehat{E}^- z_-} | \Lambda_j \rangle_q [{}_q\langle \Lambda_0 | e^{-q\widehat{E}^+ z_+} e^{Qq\mathcal{F}} e^{q\widehat{E}^- z_-} | \Lambda_0 \rangle_q]^{-m_j}. \quad (14)$$

Here $|\Lambda_i\rangle_q$ denotes the highest vector in the i -th fundamental representation of $U_q(\widehat{\mathcal{G}})$. Though the main fundamental problem remains unsolved, i.e., a suitable analogue for the principal Heisenberg subalgebra of $U(\widehat{\mathcal{G}})$ is unknown, we are still able to deduce certain soliton solutions from (14) by means of the algebraic considerations in subsection 2. To illustrate that we construct an example (for the case of $(\widehat{sl}_2)_q^t$) in the next subsection.

3.3. Example: soliton solutions from quantum Lie algebra $(sl_2)_q^t$

The case $\widehat{\mathcal{G}} = \widehat{sl}_2$ of the affine Toda system corresponds to the sine–Gordon equation [17]. We use the quantum Lie algebra constructions given in section 2 to generate quantum group soliton solutions based on $(sl_2)_q^t$. Starting from (13), we put ${}_q\mu_{\pm} = {}_q\mu_{\pm}^0 e^{\eta^{\pm} z_{\pm} \widehat{E}_{\pm 1}^t}$, ${}_q\mu^0 = ({}_q\mu_+^0)^{-1} {}_q\mu_-^0 = e^{Q\mathcal{F}_q}$ for the group-like mappings associated with $(sl_2)_q^t$, and include the fundamental highest

weight vectors $|\Lambda_j\rangle_q^t$, $j = 0, \dots, r$ (some examples of finite-dimensional representations for $(sl_2)_q^t$ are given in Appendix 4). Then we obtain for $j = 0, 1$

$$\begin{aligned} : e^{-\beta\lambda_j\cdot\phi^{(q)}} : &:= e^{-\beta\lambda_j\cdot\phi_0^{(q)}} : {}^t_q\langle\Lambda_j|e^{-\widehat{E}_h^+z+e^{Q\mathcal{F}_q}\widehat{E}_h^-z-}|\Lambda_j\rangle_q^t \left[{}^t_q\langle\Lambda_0|e^{-\widehat{E}_h^+z+e^{Q\mathcal{F}_q}\widehat{E}_h^-z-}|\Lambda_0\rangle_q^t \right]^{-m_j} \\ &=: e^{-\beta\lambda_j\cdot\phi_0^{(q)}} : {}^t_q\langle\Lambda_j|F|\Lambda_j\rangle_q^t \left[\langle\Lambda_0|F|\Lambda_0\rangle_q^t \right]^{-m_j}, \end{aligned}$$

where we have denoted $F = \exp\left(Qe^{-2z+\zeta-2\frac{1}{t}z-\zeta^{-1}}\mathcal{F}_q\right)$. Thus, for $j = 0, 1$ we have

$$\begin{aligned} : e^{-\beta\lambda_0\cdot\phi^{(q)}} : &:= e^{-\beta\lambda_0\cdot\phi_0^{(q)}} :, \\ : e^{-\beta\lambda_1\cdot\phi^{(q)}} : &:= e^{-\beta\lambda_1\cdot\phi_0^{(q)}} : \exp\left(-Qe^{-2z+\zeta-2\frac{1}{t}z-\zeta^{-1}}\Xi(\zeta, t, q)\right), \\ \Xi(\zeta, t, q) &\equiv \sum_{m=-\infty}^{+\infty} \zeta^{2m} t^m (q + q^{-1})^m. \end{aligned}$$

Finally, we would like to make several remarks. The construction of a quantum Lie algebra $(\mathcal{G})_h$ find its extensions to all semisimple Lie algebras \mathcal{G} in [8]. For cases (other than sl_2) of quantum Lie algebras associated to \mathcal{G} , one can also derive formulae for generators of their Heisenberg subalgebras. In these notes we only discussed the case of the principal grading [17] of \mathcal{G} . Similar constructions of Toda systems associated to the homogeneous and non-abelian [23] gradings of corresponding Lie algebras can be found. Conformal affine Toda systems have natural generalizations [7] which involve in the construction generators of higher grading subspaces of Lie algebras. Multi-solitonic solutions for the generalized Toda systems were found in [7]. Generalizations of the constructions mentioned in this paragraph for quantum Lie algebras will be given in a forthcoming publication.

4. Appendix: representations for $(sl_2)_h$

4.1. A two-dimensional representation

In [3, 8] a two-dimensional q -representation of $(sl_2)_h$ was given in the form

$$\pi(X_h^+) = \sqrt{(q+q^{-1})/2} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \pi(H_h) = \begin{pmatrix} q & 0 \\ 0 & -q^{-1} \end{pmatrix}, \quad \pi(X_h^-) = \pi(X_h^+)^t,$$

satisfying $\pi([a, b]_h) = \pi(a)\pi(b) - (\tilde{\pi}(b)\tilde{\pi}(a))$, $\pi(\tilde{a}) = \tilde{\pi}(a)$, with $\tilde{h} = -h$, $h \in sl_2$, $\tilde{q} = q^{-1}$. In [8] it is mentioned that there exist similar construction of $(sl_2)_h$ - q -representations of any dimension.

4.2. Fundamental highest weight representations of $(sl_2)_q$

The highest weight vector v_0 of the fundamental representation of the quantum Lie algebra $(sl_2)_q$ satisfies the conditions

$$H_h v_0 = v_0, \quad X_h^+ v_0 = 0, \quad X_h^- v_0 = v_1.$$

Thanks to definition (9) of generators $(sl_2)_h$ we have

$$[H, X_h^+]_h = [H, X_h^+] = 2X_h^+, \quad [H, X_h^-]_h = [H, X_h^-] = -2X_h^-, \quad [H, H_h]_h = [H, H_h] = 0.$$

The element H lies in Cartan subalgebra of $(sl_2)_q$. Therefore

$$H_h(Hv_0) = (Hv_0), \quad Hv_0 = \lambda v_0.$$

Using the definition (9) we find that

$$H_h = 2(q + q^{-1})^{-1} (qX^+X^- - q^{-1}X^-X^+) = q^H(X_h^+X_h^- - X_h^-X_h^+),$$

and the action of H_h on v_0 in such a form gives

$$q^H(X_h^+X_h^- - X_h^-X_h^+)v_0 = v_0.$$

Finally $X_h^\pm v_1 = q^{-\xi}v_0$, where ξ can be written in the following fashion

$$2(q + q^{-1})^{-1} (qX^+X^- - q^{-1}X^-X^+) v_0 = 2(q + q^{-1})^{-1} [H]v_0 = v_0,$$

$$2q(q^\xi - q^{-\xi})(q + q^{-1})^{-1}(q - q^{-1})^{-1} = 1.$$

This procedure can be recurrently continued for all v_n of the basis of the representation.

4.3. Fundamental highest weight representation for the affinised $(sl_2)_q$

Above we introduced a quantum affine algebra $(sl_2)_q^t$ as an affinisation of $(sl_2)_q$. The highest weight vector of the i -th ($i = 1, 2$) fundamental representation of $(sl_2)_q^t$ possesses the properties similar to the properties of the highest weight vector of fundamental representation of $(sl_2)_q$. The actions of $h_{1,2}$, $e_{1,2}$ and $f_{1,2}$ generators on highest weight vectors $v_0^{(1)}$ and $v_0^{(0)}$ are given by

$$\begin{aligned} h_1 v_0^{(1)} &= v_0^{(1)}, h_1 v_0^{(0)} = 0, h_0 v_0^{(1)} = 0, h_0 v_0^{(0)} = v_0^{(0)}, e_1 v_0^{(0)} = e_1 v_0^{(1)} = e_0 v_0^{(0)} = e_0 v_0^{(1)} = 0, \\ f_0 v_0^{(1)} &= f_1 v_0^{(0)} = 0, f_0 v_0^{(0)} = v_1^{(0)}, f_1 v_0^{(0)} = v_1^{(0)}, \end{aligned}$$

where superscripts correspond to representation and subscripts label the vectors of the basis. In the same way as for $(sl_2)_q$, we have

$$\begin{aligned} e_1 v_1^{(1)} &= q^{-H} v_0^{(1)}, e_1 v_1^{(0)} = q^{-\lambda^{(1)}} v_0^{(1)}, h v_0^{(1)} = \lambda^{(1)} v_0^{(1)}, e_1 v_1^{(0)} = 0, h = 1 \otimes H, \\ e_0 v_1^{(0)} &= q^{-H} v_0^{(0)}, e_0 v_1^{(0)} = q^{-\lambda^{(0)}} v_0^{(0)}, h v_0^{(0)} = \lambda^{(0)} v_0^{(0)}, e_0 v_1^{(1)} = 0, \\ H_1 v_1^{(1)} &= (1 - 2qq^{-(\lambda^{(1)}-2)}) v_1^{(1)}, H_1 v_1^{(0)} = 2q^{-1} q^{(2+\lambda_0^{(0)})} v_1^{(0)}. \end{aligned}$$

References

- [1] Chari V and Pressley A 1994 *A guide for quantum groups* (Cambridge University Press)
- [2] Damaskinski E V, Kulish P P, Lyakhovsky V V and Sokolov M A 1996 *Symmetry methods in physics*, 1 (Dubna, Joint Inst. Nuclear Res., Dubna)
- [3] Delius G W and Hueffmann A 1996 J. Phys. A29
- [4] Drinfeld V G 1985 Sov. Math. Dokl. 32
- [5] Faddeev L D and Takhtajan L A 1987 *Hamiltonian methods in the theory of solitons* (Springer-Verlag)
- [6] Fedoseev I A and Leznov A N 1984 Phys. Lett. 141B, 1-2
- [7] Ferreira L A, Gervais J-L, Guillen, J S and Saveliev M V 1996 Nuclear Phys. B 470 1-2
- [8] Gould M D and Delius G W 1997 Commun. Math. Phys. 185
- [9] Hollowood T 1991 PUPT-1286, hep-th/9110010
- [10] Hollowood T 1993 Int. J. Mod. Phys. A8
- [11] Hollowood T 1993 Phys. Lett. B300
- [12] Hollowood T and Mansfield P 1990 Nucl. Phys. B 330 2-3
- [13] Jimbo M 1985 Lett. Math. Phys. 10, 63
- [14] Kac V G 1990 *Infinite dimensional Lie algebras*. Third edition (Cambridge university Press, Cambridge)
- [15] Leznov A N and Fedoseev I A 1982 Theor. Math. Phys. 53 3
- [16] Leznov A N and Mukhtarov M A 1987 Theor. Math. Phys. 71 1
- [17] Leznov A N and Saveliev M V 1992 *Group-Theoretical Methods for Integration of Non-Linear Dynamical Systems*. Progress in Physics Series, v. 15, (Birkhauser-Verlag, Basel)

- [18] Leznov A N and Saveliev M V 1979 Lett. Math. Phys. 3
- [19] Lyubashenko V and Sudbery 1995 arXiv:q-alg/9510004
- [20] Mansfield P 1983 Nucl. Phys. B 222
- [21] Olive D I Turok N and Underwood J W R 1993 Nucl. Phys. B401
- [22] Olive D I, Turok N and Underwood J W R 1993 Nucl. Phys. B 409
- [23] Razumov A V, Saveliev M V and Zuevsky A 1999 *In the Memorial volume dedicated to M. V. Saveliev, Dubna*
- [24] Saveliev M V and Zuevsky A B 2000 Internat. J. Modern Phys. A 15 24
- [25] Serr J-P 1992 *Lie Algebras and Lie Groups, 2nd ed* (Lecture Notes in Mathematics), Vol. 1500
- [26] Toda M 1975 Phys. Rep. 18C 1