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Moments of two noncommutative random variables in terms of their joint quantum operators

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Abstract. The computation of the generating function of the joint moments of two non–commutative gaussiano–poissonian random variables, from the commutators of their quantum operators, is presented first. An example of such random variables is also included in the paper.

1. Introduction

In this paper we apply the algorithm for recovering the joint moments from the commutators of the quantum operators, and the first order moments, outlined in [9], to two noncommutative random variables X and Y. These are not classical random variables, but symmetric operators densely defined on a Hilbert space. We must also mention the fact that, the algorithm mentioned above was extended to q–commutators in [4], and the authors of that paper worked out the joint moments of both commutative and noncommutative random variables.
The multi–dimensional case is much harder than the one–dimensional one, and for this reason we restrict our attention to the dimension $d = 2$ in this paper.

The paper is structured as follows. In section 2 we present a little bit of the background concerning Noncommutative Probability and the joint quantum operators, generated by a finite family of noncommutative random variables. In section 3 we pose a two–dimensional commutator problem and apply the algorithm, from [9], to find a recursive formula for the joint moments of the two random variables involved in that commutator problem. In section 4, we use the recursive formula, obtained in the previous section, to compute the joint Laplace transform of the two random variables from section 3. Finally, in section 5, we show the existence of the operators involved in the commutator problem from section 3.

2. Background

We present in this section a basic background of Noncommutative Probability and joint quantum operators generated by a family of not necessary commutative random variables. The definition of a noncommutative probability space is taken from [13], and the facts about the joint quantum operators from [10].

Let $(H, \langle\cdot, \cdot\rangle)$ be a Hilbert space over the field of real numbers $\mathbb{R}$, and let $X_1, X_2, \ldots, X_d$ be d symmetric densely defined linear operators on H. We denote by $\mathcal{A}$ the unital algebra generated by $X_1, X_2, \ldots, X_d$. It is clear that $\mathcal{A}$ is in fact, the space of all operators of the form $f(X_1, X_2, \ldots, X_d)$, where f is a polynomial of d noncommutative variables $x_1, x_2, \ldots, x_d$. We make the assumption that there exists a nonzero element ϕ of H, such that ϕ belongs to the domain of g, for any $g \in \mathcal{A}$. We can normalize ϕ and assume that:

$$\parallel \phi \parallel \quad = \quad 1, \tag{1}$$

where $\parallel \cdot \parallel$ denotes the norm of the Hilbert space H. We fix ϕ and call it the *vacuum vector*.

Definition 2.1 *We call the pair $(\mathcal{A}, \phi)$ a* probability space supported by H*. We call every element g of $\mathcal{A}$, a* random variable*, and define its* expectation *as:*

$$E[g] \quad := \quad \langle g\phi, \phi\rangle. \tag{2}$$

We also define the following equivalence relation (see [10]):

Definition 2.2 *Let $(\mathcal{A}, \phi)$ and $(\mathcal{A}', \phi')$ be two probability spaces supported by two Hilbert spaces H and H', and let E and E' denote their expectations. Let $X_1, X_2, \ldots, X_d$ be operators from $\mathcal{A}$, and $X_1', X_2', \ldots, X_d'$ operators from $\mathcal{A}'$. We say that the random vectors $(X_1, X_2, \ldots, X_d)$ and $(X_1', X_2', \ldots, X_d')$ are* moment equal *and denote this fact by $(X_1, X_2, \ldots, X_d) \equiv (X_1', X_2', \ldots, X_d')$, if for any polynomial $p(x_1, x_2, \ldots, x_d)$ of d noncommutative variables, we have:*

$$E\left[p\left(X_1, X_2, \ldots, X_d\right)\right] = E'\left[p\left(X_1', X_2', \ldots, X_d'\right)\right]. \tag{3}$$

We define the subspace F, of H, as:

$$F \quad := \quad \{g\phi \mid g \in \mathcal{A}\}. \tag{4}$$

For any non–negative integer n, we define the space F_n as the set of all vectors, of H, of the form $f(X_1, X_2, \ldots, X_d)\phi$, where f is a polynomial of total degree less than or equal to n. We observe that, since F_n is a finite–dimensional subspace of H, F_n is a closed subspace of H, for all $n \geq 0$. We have:

$$F_0 \subset F_1 \subset F_2 \subset \cdots \subset F \subset H \tag{5}$$

and

$$F \quad = \quad \cup_{n=0}^{\infty} F_n. \tag{6}$$

We define $G_0 := F_0$, and for all $n \geq 1$, $G_n := F_n \ominus F_{n-1}$, that means, G_n is the orthogonal complement of F_{n-1} into F_n. For any $n \geq 0$, we call G_n the *homogenous chaos space of order* n generated by $X_1, X_2, \ldots, X_d$. We also define the space:

$$\mathcal{H} \quad := \quad \oplus_{n=0}^{\infty} G_n,$$

and call $\mathcal{H}$ the *chaos space* generated by $X_1, X_2, \ldots, X_d$. It is not hard to see that $\mathcal{H}$ is the closure of the space $\mathcal{A}\phi := \{g\phi \mid g \in \mathcal{A}\}$ in H.

We have the following lemma:

Lemma 2.3 *For any $i \in \{1, 2, \ldots, d\}$ and any non–negative integer n:*

$$X_i G_n \quad \perp \quad G_k, \tag{7}$$

for all $k \neq n-1$, n, and $n+1$, where $\perp$ means "orthogonal to".

We present a proof of the above lemma, that follows exactly the proof of the same lemma, in the commutative case, from [1].

Proof. Let $i \in \{1, 2, \ldots, d\}$ be fixed.
If $k \notin \{n-1, n, n+1\}$, then either $k \geq n+2$ or $k \leq n-2$.

Case 1: If $k \geq n+2$, then let $u \in G_n$ and $v \in G_k$. Since $u \in G_n$, there exists f a polynomial of degree at most n, such that:

$$u = f(X_1, X_2, \ldots, X_d)\phi. \tag{8}$$

Since the polynomial $x_i f(x_1, x_2, \ldots, x_d)$ has degree at most $n+1$, we have:

$$\begin{aligned} X_i u &= X_i f(X_1, X_2, \ldots, X_d)\phi \\ &\in F_{n+1}. \end{aligned}$$

Because $n+1 < k$, we have $G_k \perp F_{n+1}$, and so:

$$v \perp X_i u.$$

Case 2: If $k \leq n-2$, then let $u \in G_n$ and $v \in G_k$. We can see as before that $X_i v \in F_{k+1}$. Using first the fact that X_i is a symmetric operator, and second that $G_n \perp F_{k+1}$ since $n > k+1$, we obtain:

$$\begin{aligned} \langle X_i u, v \rangle &= \langle u, X_i v \rangle \\ &= 0. \end{aligned}$$

□

To ease our notation, we write: $X = (X_1, X_2, \ldots, X_d)$ and $f(X) = f(X_1, X_2, \ldots, X_d)$ for every polynomial f of d variables.
Let n be a fixed nonnegative integer, and $f(X)\phi \in G_n$. Then, according to the previous lemma, for any $i \in \{1, 2, \ldots, d\}$:

$$X_i f(X)\phi \in G_{n-1} \oplus G_n \oplus G_{n+1}. \tag{9}$$

Thus, there exist and are unique $f_{i,n-1}(X)\phi \in G_{n-1}$, $f_{i,n}(X)\phi \in G_n$, and $f_{i,n+1}(X)\phi \in G_{n+1}$, such that:

$$X_i f(X)\phi = f_{i,n-1}(X)\phi + f_{i,n}(X)\phi + f_{i,n+1}(X)\phi. \tag{10}$$

We define the following three families of operators:

$$D_n^-(i) : G_n \to G_{n-1}, \tag{11}$$

defined as:

$$D_n^-(i) f(X)\phi := f_{i,n-1}(X)\phi. \tag{12}$$

Since f has degree n, while $f_{i,n-1}$ has degree $n-1$, we see that $D_n^-(i)$ decreases the degree of a homogenous polynomial by 1 unit, and so we call $D_n^-(i)$ an *annihilation* operator.

$$D_n^0(i) : G_n \to G_n, \tag{13}$$

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defined as:

$$D_n^0(i)f(X)\phi \quad := \quad f_{i,n}(X)\phi. \tag{14}$$

Since f has degree n, and $f_{i,n}$ has degree n, too, we see that $D_n^0(i)$ preserves the degree of a homogenous polynomial, and so we call $D_n^0(i)$ a *preservation* operator.

$$D_n^+(i) : G_n \to G_{n+1}, \tag{15}$$

defined as:

$$D_n^+(i)f(X)\phi \quad := \quad f_{i,n+1}(X)\phi. \tag{16}$$

Since f has degree n, while $f_{i,n+1}$ has degree $n+1$, we see that $D_n^+(i)$ increases the degree of a homogenous polynomial by 1 unit, and so we call $D_n^+(i)$ a *creation* operator.
If we restrict the domains of the operators X_1, X_2, $\ldots$, X_d to the space G_n, then we can see from (10) that, for all $i \in \{1, 2, \ldots, d\}$, we have:

$$X_i|_{G_n} \quad = \quad D_n^-(i) + D_n^0(i) + D_n^+(i). \tag{17}$$

We have defined, so far, the annihilation, preservation, and creation operators only separately on each homogenous chaos space G_n, for $n \geq 0$. We can extend this partial definition, in a linear way, to the whole space $F = \mathcal{A}\phi$. More precisely, if f is a polynomial of degree k, where k is a nonnegative integer, then $f(X)\phi \in F_k$, and we know that:

$$F_k \quad = \quad G_0 \oplus G_1 \oplus \cdots \oplus G_k. \tag{18}$$

Thus, there exist and are unique $f_0(X)\phi \in G_0$, $f_1(X)\phi \in G_1$, $\ldots$, $f_k(X)\phi \in G_k$, such that:

$$f(X)\phi \quad = \quad f_0(X)\phi + f_1(X)\phi + \cdots + f_k(X)\phi. \tag{19}$$

We define:

$$a^-(i)f(X)\phi \quad := \quad D_0^-(i)f_0(X)\phi + \cdots + D_k^-(i)f_k(X)\phi, \tag{20}$$

$$a^0(i)f(X)\phi \quad := \quad D_0^0(i)f_0(X)\phi + \cdots + D_k^0(i)f_k(X)\phi, \tag{21}$$

and

$$a^+(i)f(X)\phi \quad := \quad D_0^+(i)f_0(X)\phi + \cdots + D_k^+(i)f_k(X)\phi. \tag{22}$$

We call $a^-(i)$ an *annihilation* operator, $a^0(i)$ a *preservation* operator, and $a^+(i)$ a *creation* operator.
We also call the operators $\{a^-(i)\}_{1 \leq i \leq d}$, $\{a^0(i)\}_{1 \leq i \leq d}$, and $\{a^+(i)\}_{1 \leq i \leq d}$, the *minimal joint quantum operators* generated by the family $\{X_i\}_{1 \leq i \leq d}$. The domain of the minimal joint quantum operators is understood to be the space F.
Restricting the domain of each X_i, for $i \in \{1, 2, \ldots, d\}$, to the space F, we have now:

$$X_i \quad = \quad a^-(i) + a^0(i) + a^+(i), \tag{23}$$

for all $i \in \{1, 2, \ldots, d\}$.
For each $1 \leq i \leq d$, we call the equality (23), the *minimal quantum decomposition* of X_i.

One should not forget that, for any $i \in \{1, 2, \ldots, d\}$ and $n \geq 0$, $a^-(i) : G_n \to G_{n-1}$, $a^0(i) : G_n \to G_n$, and $a^+(i) : G_n \to G_{n+1}$, where $G_{-1} := \{0\}$ is the null space.
If for each $n \geq 0$, we denote by P_n the orthogonal projection of $\mathcal{H}$ onto its closed subspace F_n, and restrict the domain of P_n to F, then it is not hard to see that:

$$a^-(i) = \sum_{n=1}^{\infty} P_{n-1}X_iP_n, \tag{24}$$

$$a^0(i) = \sum_{n=0}^{\infty} P_nX_iP_n, \tag{25}$$

$$a^+(i) = \sum_{n=0}^{\infty} P_{n+1}X_iP_n, \tag{26}$$

for all $i \in \{1, 2, \ldots, d\}$. We can see from these three formulas that, since X_i is symmetric, for all u and v in F, and all $i \in \{1, 2, \ldots, d\}$, we have:

$$\langle a^+(i)u, v\rangle = \langle u, a^-(i)v\rangle, \tag{27}$$

$$\langle a^0(i)u, v\rangle = \langle u, a^0(i)v\rangle. \tag{28}$$

If U and V are two operators, then we define their commutator $[U, V]$ as:

$$[U, V] := UV - VU. \tag{29}$$

It was proven in [1], [2], and [10], that $X_1, X_2, \ldots, X_d$ commute among themselves if and only if the following conditions hold, for any $i, j \in \{1, 2, \ldots, d\}$:

$$\left[a^-(i), a^-(j)\right] = 0, \tag{30}$$

$$\left[a^-(i), a^0(j)\right] = \left[a^-(j), a^0(i)\right], \tag{31}$$

$$\left[a^-(i), a^+(j)\right] - \left[a^-(j), a^+(i)\right] = \left[a^0(j), a^0(i)\right]. \tag{32}$$

Example 2.4 *Let $X_1, X_2, \ldots, X_d$ be classic random variables defined on the same probability space $(\Omega, \mathcal{F}, P)$, having finite moments of any order. Let $H := L^2(\Omega, \mathcal{F}, P)$ and $\phi = 1$, i.e., the constant random variable equal to 1. We can view $X_1, X_2, \ldots, X_d$ as multiplication operators on the space $\mathcal{A}1 \subset H$, where $\mathcal{A}$ is the algebra of the random variables of the form $f(X_1, X_2, \ldots, X_d)$, where f is a polynomial of d variables. It is obvious that $X_iX_j = X_jX_i$, for all $1 \leq i < j \leq d$.*

The following definition is taken from [10].

Definition 2.5 *Let H be a Hilbert space, $(\mathcal{A}, \phi)$ a probability space supported by H, and $\{X_i\}_{1\leq i\leq d}$ elements of $\mathcal{A}$, that are symmetric operators. Let $\{a_{x_i}^-\}_{1\leq i\leq d}$, $\{a_{x_i}^0\}_{1\leq i\leq d}$, and $\{a_{x_i}^+\}_{1\leq i\leq d}$ be three families of linear operators, defined on subspaces of H, such that, ϕ belongs to the domain of $a_{x_{i_1}}^{\epsilon_1} a_{x_{i_2}}^{\epsilon_2} \cdots a_{x_{i_n}}^{\epsilon_n}$, for all $n \geq 1$, $(i_1, i_2, \ldots, i_n) \in \{1, 2, \ldots, d\}^n$, and $(\epsilon_1, \epsilon_2, \ldots, \epsilon_n) \in \{-, 0, +\}^n$. We say that these families of operators form* a joint quantum decomposition of $\{X_i\}_{1\leq i\leq d}$ relative to $\mathcal{A}$, *if the following conditions hold:*

$$X_i = a_{x_i}^- + a_{x_i}^0 + a_{x_i}^+, \tag{33}$$

$$\left(a_{x_i}^+\right)^* |\mathcal{A}\phi = a_{x_i}^- |\mathcal{A}\phi, \tag{34}$$

$$a_{x_i}^- H_n \subset H_{n-1}, \tag{35}$$

$$a_{x_i}^0 H_n \subset H_n, \tag{36}$$

for all $1 \leq i \leq d$ and $n \geq 0$, where $H_{-1} := \{0\}$, $H_0 := \mathbb{R}\phi$, and H_k is the vector space spanned by all vectors of the form $a^+_{x_{i_1}} a^+_{x_{i_2}} \cdots a^+_{x_{i_k}} \phi$, where $i_1, i_2, \ldots, i_k \in \{1, 2, \ldots, d\}$, for all $k \geq 1$. We call $a^-_{x_i}$ an annihilation *operator, $a^0_{x_i}$ a* preservation *operator, and $a^+_{x_i}$ a* creation *operator, for all $1 \leq i \leq d$.*

For all $1 \leq i \leq d$, the fact that $\left(a^+_{x_i}\right)^* |\mathcal{A}\phi = a^-_{x_i}|\mathcal{A}\phi$ is understood as:

$$\langle a^+_{x_i} u, v\rangle = \langle u, a^-_{x_i} v\rangle,$$

for all u and v in $\mathcal{A}\phi$, where $\langle \cdot, \cdot\rangle$ denotes the inner product of H. Because, for all $i \in \{1, 2, \ldots, d\}$, X_i is a symmetric operator, it follows from (34), that:

$$\left(a^0_{x_i}\right)^* |\mathcal{A}\phi = a^0_{x_i}|\mathcal{A}\phi. \tag{37}$$

The following lemma from [10] refers to the uniqueness of the joint quantum decomposition of a finite family of symmetric random variables.

Lemma 2.6 *Let $\{X_i\}_{1\leq i\leq d}$ be a family of symmetric random variables in a noncommutative probability space $(\mathcal{A}, \phi)$, and $\{a^-_{x_i}\}_{1\leq i\leq d}$, $\{a^0_{x_i}\}_{1\leq i\leq d}$, and $\{a^+_{x_i}\}_{1\leq i\leq d}$ a joint quantum decomposition of $\{X_i\}_{1\leq i\leq d}$ relative to $\mathcal{A}$. Let $\mathcal{A}'$ be the algebra generated by $\{X_i\}_{1\leq i\leq d}$, and $\{a^-(i)\}_{1\leq i\leq d}$, $\{a^0(i)\}_{1\leq i\leq d}$, and $\{a^+(i)\}_{1\leq i\leq d}$ be the minimal joint quantum decomposition of $\{X_i\}_{1\leq i\leq d}$. Then for any $i \in \{1, 2, \ldots, d\}$ and any $\epsilon \in \{-, 0, +\}$, we have:*

$$a^\epsilon_{x_i}|\mathcal{A}'\phi = a^\epsilon(i)|\mathcal{A}'\phi. \tag{38}$$

Moreover, if $\mathcal{A}''$ denotes the algebra generated by $\cup_{i=1}^d \{a^-_{x_i}, a^0_{x_i}, a^+_{x_i}\}$, then

$$\mathcal{A}''\phi = \mathcal{A}'\phi, \tag{39}$$

and $\{a^-_{x_i}\}_{1\leq i\leq d}$, $\{a^0_{x_i}\}_{1\leq i\leq d}$, and $\{a^+_{x_i}\}_{1\leq i\leq d}$ is a joint quantum decomposition of $\{X_i\}_{1\leq i\leq n}$ relative to $\mathcal{A}''$.

3. A commutator problem

Let us consider two symmetric "random variables" X and Y, whose joint quantum operators satisfy the following commutation relationships:

$$\left[a^-_x, a^-_y\right] = 0, \tag{1}$$
$$\left[a^-_x, a^+_x\right] = I, \tag{2}$$
$$\left[a^-_x, a^+_y\right] = bI, \tag{3}$$
$$\left[a^-_y, a^+_y\right] = I, \tag{4}$$
$$\left[a^-_x, a^0_x\right] = pa^-_x + p'a^-_y, \tag{5}$$
$$\left[a^-_x, a^0_y\right] = 0, \tag{6}$$
$$\left[a^-_y, a^0_x\right] = 0, \tag{7}$$
$$\left[a^-_y, a^0_y\right] = q'a^-_x + qa^-_y. \tag{8}$$

We assume that $E[X] = E[Y] = 0$.

Taking the adjoint in both sides of each of the relations (1), (3), (5), and (8), we obtain the following commutation relationships:

$$\begin{aligned}\left[a_y^+, a_x^+\right] &= 0, &(9)\\ \left[a_y^-, a_x^+\right] &= bI, &(10)\\ \left[a_x^0, a_x^+\right] &= pa_x^+ + p'a_y^+, &(11)\\ \left[a_y^0, a_y^+\right] &= q'a_x^+ + qa_y^+. &(12)\end{aligned}$$

From the Jacobi commutation identity:

$$\left[a_y^-, \left[a_x^0, a_x^+\right]\right] + \left[a_x^0, \left[a_x^+, a_y^-\right]\right] + \left[a_x^+, \left[a_y^-, a_x^0\right]\right] = 0,$$

we conclude that $p' = -pb$. Similarly, we can see that $q' = -qb$. Thus the commutation relationships (5) and (8) can be written as:

$$\begin{aligned}\left[a_x^-, a_x^0\right] &= p\left(a_x^- - ba_y^-\right), &(13)\\ \left[a_y^-, a_y^0\right] &= q\left(a_y^- - ba_x^-\right). &(14)\end{aligned}$$

We assume that $|b| \neq 1$, $b \neq 0$, $p \neq 0$, and $q \neq 0$. It follows that $XY \neq YX$, since if $XY = YX$, then we would have:

$$\begin{aligned}0 &= [X, Y]\\ &= \left[a_x^- + a_x^0 + a_x^+, a_y^- + a_y^0 + a_y^+\right]\\ &= \sum_{(\epsilon_1,\epsilon_2)\in\{-,0,+\}^2} \left[a_x^{\epsilon_1}, a_y^{\epsilon_2}\right].\end{aligned}$$

In the above sum of commutators, we have $[a_x^{\epsilon_1},\, a_y^{\epsilon_2}] + [a_x^{\epsilon_2},\, a_y^{\epsilon_1}] = 0$, for all $\epsilon_1 \neq \epsilon_2$. Moreover, $[a_x^-,\, a_y^-] = [a_x^+,\, a_y^+] = 0$. Thus we obtain:

$$\begin{aligned}\left[a_x^0, a_y^0\right] &= [X, Y]\\ &= 0.\end{aligned}$$

Using now the Jacobi identity:

$$\left[a_x^-, \left[a_x^0, a_y^0\right]\right] + \left[a_x^0, \left[a_y^0, a_x^-\right]\right] + \left[a_y^0, \left[a_x^-, a_x^0\right]\right] = 0,$$

and the fact that $bpq \neq 0$, we conclude that $a_y^- = ba_x^-$. Taking the adjoint in both sides of this equality we get $a_y^+ = ba_x^+$. It follows now from (2) and (4) that $b^2 = 1$, which is not possible since $|b| \neq 1$. Therefore we are dealing with non–commutative random variables.

We will show at the end of the paper that, for each $b \in (-1, 1)$, $p \neq 0$, and $q \neq 0$, there is a model in which these commutation relationships are realized. The necessary condition $|b| \leq 1$ follows from the Schwarz inequality, as it will be explained below. We exclude the case $b = \pm 1$ from our discussion.

Since both X and Y are symmetric, using Schwarz inequality, we obtain:

$$\begin{aligned}|E[XY]| &= |\langle XY\phi, \phi\rangle|\\ &= |\langle Y\phi, X\phi\rangle|\\ &\leq (\langle Y\phi, Y\phi\rangle)^{1/2} (\langle X\phi, X\phi\rangle)^{1/2}\\ &= \left(\langle X^2\phi, \phi\rangle\right)^{1/2} \left(\langle Y^2, \phi\rangle\right)^{1/2}\\ &= \left(E\left[X^2\right]\right)^{1/2} \left(E\left[Y^2\right]\right)^{1/2}. &(15)\end{aligned}$$

Because $E[X] = E[Y] = 0$, we have $a_x^0\phi = a_y^0\phi = 0$. Since $a_x^-\phi = a_y^-\phi = 0$, we have:

$$\begin{aligned} E[XY] &= \langle (a_x^- + a_x^0 + a_x^+)(a_y^- + a_y^0 + a_y^+)\phi, \phi\rangle \\ &= \langle (a_x^- + a_x^0 + a_x^+)a_y^+\phi, \phi\rangle \\ &= \langle a_y^+\phi, (a_x^+ + a_x^0 + a_x^-)\phi\rangle \\ &= \langle a_y^+\phi, a_x^+\phi\rangle \\ &= \langle a_x^- a_y^+\phi, \phi\rangle \\ &= \langle (a_x^- a_y^+ - a_y^+ a_x^-)\phi, \phi\rangle \\ &= \langle [a_x^-, a_y^+]\phi, \phi\rangle \\ &= \langle bI\phi, \phi\rangle \\ &= b. \end{aligned} \tag{16}$$

Similarly, we have:

$$E[X^2] = 1 \tag{17}$$

and

$$E[Y^2] = 1. \tag{18}$$

It follows now from (15), (16), (17), and (18) that:

$$|b| \leq 1. \tag{19}$$

Inside the algebra $\mathcal{A}$, we can consider the bi–module $M(X,\, Y)$ (left module over the ring $\mathcal{P}(X)$ of all polynomials of X, and right module over the ring $\mathcal{P}(Y)$ of all polynomials of Y) generated by the identity operator I. That means, $M(X,\, Y)$ is the vector space spanned by the monomials of the form $\{X^m Y^n\}_{m\geq 0, n\geq 0}$, in which all factors of X are to the left of all factors of Y. We show below a method for computing the expectation of all monomials from $M(X,\, Y)$.

Lemma 3.1 *If X and Y satisfy the commutation relationships (1)–(8), then for any m and n non–negative integers, with $m \geq 1$, we have:*

$$\begin{aligned} E[X^m Y^n] &= (m-1)E[X^{m-2}Y^n] + nbE[X^{m-1}Y^{n-1}] \\ &\quad + (1-b^2)\binom{m-1}{2} pE[X^{m-3}Y^n] \\ &\quad + p^2 \sum_{0\leq j<i\leq m-2} \langle X^{m-3-j}(a_x^- - ba_y^-)X^j Y^n\phi, \phi\rangle \\ &\quad - (m-1)bpqR_2, \end{aligned}$$

where

$$R_2 := \sum_{0\leq k\leq n-1} \langle X^{m-2}Y^{n-1-k}(a_y^- - ba_x^-)Y^k\phi, \phi\rangle.$$

Proof. Let m and n be two fixed non–negative integers, such that $m \geq 1$. We apply the commutator method from [9]. According to that method:

$$\begin{aligned} E[X^m Y^n] &= \langle X^m Y^n\phi, \phi\rangle \\ &= \langle (a_x^+ + a_x^0 + a_x^-)X^{m-1}Y^n\phi, \phi\rangle \\ &= \langle X^{m-1}Y^n\phi, a_x^-\phi\rangle + \langle X^{m-1}Y^n\phi, a_x^0\phi\rangle + \langle a_x^- X^{m-1}Y^n\phi, \phi\rangle \\ &= 0 + \langle X^{m-1}Y^n\phi, E[X]\phi\rangle + \langle a_x^- X^{m-1}Y^n\phi, \phi\rangle \\ &= \langle a_x^- X^{m-1}Y^n\phi, \phi\rangle, \end{aligned}$$

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since $E[X]=0$. We swap now a_x^- and $X^{m-1}Y^n$, using the product rule for commutators:

$$[A, B_1 \cdots B_k] = \sum_{i=1}^{k} B_1 \cdots B_{i-1}\,[A, B_i]\,B_{i+1} \cdots B_k,$$

for all operators A, B_1, …, B_k, and the fact that $a_x^-\phi = 0$, and obtain:

$$\begin{aligned}
E[X^mY^n] &= \langle a_x^- X^{m-1}Y^n\phi, \phi\rangle \\
&= \langle X^{m-1}Y^n a_x^-\phi, \phi\rangle + \langle [a_x^-, X^{m-1}Y^n]\phi, \phi\rangle \\
&= \sum_{0\le i\le m-2} \langle X^{m-2-i}[a_x^-, X]X^iY^n\phi, \phi\rangle \\
&\quad + \sum_{0\le j\le n-1} \langle X^{m-1}Y^{n-1-j}[a_x^-, Y]Y^j\phi, \phi\rangle.
\end{aligned}$$

Let us observe now that:

$$\begin{aligned}
[a_x^-, X] &= [a_x^-, a_x^+ + a_x^0 + a_x^-] \\
&= [a_x^-, a_x^+] + [a_x^-, a_x^0] \\
&= I + p(a_x^- - ba_y^-)
\end{aligned}$$

and

$$\begin{aligned}
[a_x^-, Y] &= [a_x^-, a_y^+] + [a_x^-, a_y^0] + [a_x^-, a_y^-] \\
&= bI + 0 + 0 \\
&= bI.
\end{aligned}$$

Therefore, we have:

$$\begin{aligned}
E[X^mY^n] &= \sum_{0\le i\le m-2} \langle X^{m-2-i}\{I + p(a_x^- - ba_y^-)\}X^iY^n\phi, \phi\rangle \\
&\quad + \sum_{0\le j\le n-1} \langle X^{m-1}Y^{n-1-j}bIY^j\phi, \phi\rangle \\
&= (m-1)E[X^{m-2}Y^n] + nbE[X^{m-1}Y^{n-1}] \\
&\quad + p\sum_{0\le i\le m-2} \langle X^{m-2-i}(a_x^- - ba_y^-)X^iY^n\phi, \phi\rangle.
\end{aligned}$$

Let us swap now the operators $a_x^- - ba_y^-$ and X^iY^n, using again the product rule for commutators, and the fact that $(a_x^- - ba_y^-)\phi = 0$. We obtain:

$$\begin{aligned}
E[X^mY^n] &= (m-1)E[X^{m-2}Y^n] + nbE[X^{m-1}Y^{n-1}] \\
&\quad + p\sum_{0\le i\le m-2} \langle X^{m-2-i}(a_x^- - ba_y^-)X^iY^n\phi, \phi\rangle \\
&= (m-1)E[X^{m-2}Y^n] + nbE[X^{m-1}Y^{n-1}] \\
&\quad + p\sum_{0\le i\le m-2} \langle X^{m-2-i}[a_x^- - ba_y^-, X^iY^n]\phi, \phi\rangle \\
&= (m-1)E[X^{m-2}Y^n] + nbE[X^{m-1}Y^{n-1}] \\
&\quad + p\sum_{0\le j<i\le m-2} \langle X^{m-2-i}X^{i-1-j}[a_x^- - ba_y^-, X]X^jY^n\phi, \phi\rangle \\
&\quad + p\sum_{0\le i\le m-2}\sum_{0\le k\le n-1} \langle X^{m-2-i}X^iY^{n-1-k}[a_x^- - ba_y^-, Y]Y^k\phi, \phi\rangle.
\end{aligned}$$

Using the simple facts that $X^{m-2-i}X^{i-1-j} = X^{m-3-j}$ and $X^{m-2-i}X^i = X^{m-2}$, we obtain:

$$\begin{aligned} E[X^mY^n] &= (m-1)E[X^{m-2}Y^n] + nbE[X^{m-1}Y^{n-1}] \\ &\quad +p \sum_{0\le j<i\le m-2} \langle X^{m-3-j}[a_x^- - ba_y^-, X]X^jY^n\phi, \phi\rangle \\ &\quad +(m-1)p \sum_{0\le k\le n-1} \langle X^{m-2}Y^{n-1-k}[a_x^- - ba_y^-, Y]Y^k\phi, \phi\rangle. \end{aligned}$$

Let us observe that:

$$\begin{aligned} [a_x^- - ba_y^-, X] &= [a_x^-, X] - b[a_y^-, X] \\ &= I + p(a_x^- - ba_y^-) - b^2I \\ &= (1-b^2)I + p(a_x^- - ba_y^-) \end{aligned}$$

and

$$\begin{aligned} [a_x^- - ba_y^-, Y] &= [a_x^-, Y] - b[a_y^-, Y] \\ &= bI - b\{I + q(a_y^- - ba_x^-)\} \\ &= -bq(a_y^- - ba_x^-). \end{aligned}$$

Thus, we obtain:

$$\begin{aligned} E[X^mY^n] &= (m-1)E[X^{m-2}Y^n] + nbE[X^{m-1}Y^{n-1}] \\ &\quad +p \sum_{0\le j<i\le m-2} \langle X^{m-3-j}\{(1-b^2)I + p(a_x^- - ba_y^-)\}X^jY^n\phi, \phi\rangle \\ &\quad -(m-1)bpq \sum_{0\le k\le n-1} \langle X^{m-2}Y^{n-1-k}(a_y^- - ba_x^-)Y^k\phi, \phi\rangle \\ &= (m-1)E[X^{m-2}Y^n] + nbE[X^{m-1}Y^{n-1}] \\ &\quad +(1-b^2)\binom{m-1}{2}pE[X^{m-3}Y^n] \\ &\quad +p^2 \sum_{0\le j<i\le m-2} \langle X^{m-3-j}(a_x^- - ba_y^-)X^jY^n\phi, \phi\rangle \\ &\quad -(m-1)bpqR_2, \end{aligned}$$

where

$$R_2 := \sum_{0\le k\le n-1} \langle X^{m-2}Y^{n-1-k}(a_y^- - ba_x^-)Y^k\phi, \phi\rangle.$$

□

Lemma 3.2 *Using the notations from the Lemma 3.1, we have:*

$$R_2 = \frac{1-b^2}{q^2}E\left[X^{m-2}\left\{(Y+qI)^n - Y^n - nqY^{n-1}\right\}\right].$$

Proof. Swapping $a_y^- - ba_x^-$ and Y^k, we get:

$$\begin{aligned}
R_2 &= \sum_{0\le k\le n-1} \langle X^{m-2}Y^{n-1-k}[a_y^- - ba_x^-, Y^k]\phi,\phi\rangle \\
&= \sum_{0\le l<k\le n-1} \langle X^{m-2}Y^{n-1-k}Y^{k-1-l}[a_y^- - ba_x^-, Y]Y^l\phi,\phi\rangle \\
&= \sum_{0\le l<k\le n-1} \langle X^{m-2}Y^{n-2-l}\{(1-b^2)I + q(a_y^- - ba_x^-)\}Y^l\phi,\phi\rangle \\
&= (1-b^2)\binom{n}{2}E[X^{m-2}Y^{n-2}] \\
&\quad +q\sum_{0\le l<k\le n-1} \langle X^{m-2}Y^{n-2-l}(a_y^- - ba_x^-)Y^l\phi,\phi\rangle.
\end{aligned}$$

We repeat this procedure, swapping now $a_y^- - ba_x^-$ and Y^l, and so on, and obtain in the end that:

$$\begin{aligned}
R_2 &= (1-b^2)\left\{\binom{n}{2}E[X^{m-2}Y^{n-2}] + \binom{n}{3}qE[X^{m-2}Y^{n-3}] + \cdots \right. \\
&\qquad \left. \cdots + \binom{n}{n}q^{n-2}E[X^{m-2}Y^0]\right\} \\
&= \frac{1-b^2}{q^2}E\left[X^{m-2}\left\{(Y+qI)^n - Y^n - nqY^{n-1}\right\}\right].
\end{aligned}$$

□

Lemma 3.3 *If X and Y satisfy the commutation relationships (1)–(8), then for any m and n non–negative integers, with $m \geq 1$, we have:*

$$\begin{aligned}
& E[X^mY^n] \\
= \ & (m-1)E[X^{m-2}Y^n] + nbE[X^{m-1}Y^{n-1}] \\
& +\frac{1-b^2}{p}E\left[\left\{(X+pI)^{m-1} - X^{m-1} - (m-1)pX^{m-2}\right\}Y^n\right] \\
& -\frac{b(1-b^2)}{q}E\left[\left\{(X+pI)^{m-1} - X^{m-1}\right\}\left\{(Y+qI)^n - Y^n - nqY^{n-1}\right\}\right].
\end{aligned} \tag{20}$$

Proof. Using Lemmas 3.1 and 3.2, we get:

$$\begin{aligned}
& E[X^mY^n] \\
= \ & (m-1)E[X^{m-2}Y^n] + nbE[X^{m-1}Y^{n-1}] \\
& +(1-b^2)\binom{m-1}{2}pE[X^{m-3}Y^n] \\
& +p^2\sum_{0\le j<i\le m-2} \langle X^{m-3-j}(a_x^- - ba_y^-)X^jY^n\phi,\phi\rangle \\
& -\frac{b(1-b^2)}{q}E\left[(m-1)pX^{m-2}\left\{(Y+qI)^n - Y^n - nqY^{n-1}\right\}\right].
\end{aligned}$$

We do now a similar procedure as in the proof of Lemma 3.1, by swapping first $(a_x^- - ba_y^-)$ and X^jY^n. In this way we arrive at a new sum R_3 which is similar to R_2. We compute R_3 in the same way that we calculated R_2 in the proof of Lemma 3.2, and so on, obtaining in the end that:

$$\begin{aligned}
& E[X^mY^n] \\
= \; & (m-1)E[X^{m-2}Y^n] + nbE[X^{m-1}Y^{n-1}] \\
& +(1-b^2)\left\{\binom{m-1}{2}pE[X^{m-3}Y^n] + \binom{m-1}{3}p^2E[X^{m-4}Y^n] + \cdots \right. \\
& \qquad \left. \cdots + \binom{m-1}{m-1}p^{m-2}E[X^0Y^n]\right\} \\
& -\frac{b(1-b^2)}{q}E\left[\binom{m-1}{1}pX^{m-2}\left\{(Y+qI)^n - Y^n - nqY^{n-1}\right\}\right] \\
& -\frac{b(1-b^2)}{q}E\left[\binom{m-1}{2}p^2X^{m-3}\left\{(Y+qI)^n - Y^n - nqY^{n-1}\right\}\right] \\
& \cdots \\
& -\frac{b(1-b^2)}{q}E\left[\binom{m-1}{m-1}p^{m-1}X^0\left\{(Y+qI)^n - Y^n - nqY^{n-1}\right\}\right] \\
= \; & (m-1)E[X^{m-2}Y^n] + nbE[X^{m-1}Y^{n-1}] \\
& +\frac{1-b^2}{p}E\left[\left\{(X+pI)^{m-1} - X^{m-1} - (m-1)pX^{m-2}\right\}Y^n\right] \\
& -\frac{b(1-b^2)}{q}E\left[\left\{(X+pI)^{m-1} - X^{m-1}\right\}\left\{(Y+qI)^n - Y^n - nqY^{n-1}\right\}\right].
\end{aligned}$$

□

4. Joint left X − right Y Laplace transform of X and Y

In this section, we compute the joint Laplace transform of the random variables X and Y from the previous section. We start first by presenting some important bounds for the joint moments of X and Y.

Lemma 4.1 *Let $X = a_x^- + a_x^0 + a_x^+$ and $Y = a_y^0 + a_y^0 + a_y^+$ be joint quantum decompositions of two random variables, whose terms satisfy the commutation relationships (1)–(8), where b, p, p', q, and q' are real numbers, such that $|b| < 1$, $p' = -pb$, and $q' = -qb$. We assume that $E[X] = E[Y] = 0$. Then there exists a positive constant C, such that, for any m and n non–negative integers, we have:*

$$|E[X^mY^n]| \quad \leq \quad Cm!n!. \tag{21}$$

Proof. Indeed, if we define, for all non–negative integers m and n,

$$C_{m,n} \quad := \quad \max\{|E[X^iY^j]|/(i!j!) \mid 0 \leq i \leq m, 0 \leq j \leq n\}, \tag{22}$$

then the sequence $\{C_{m,n}\}_{m\geq 0, n\geq 0}$ is left and right increasing, in the sense that if we fix n, then it is increasing with respect to m, and vice–versa. We have:

$$\begin{aligned}
& \left|E\left[\left\{(X+pI)^{m-1} - X^{m-1} - (m-1)pX^{m-2}\right\}Y^n\right]\right| \\
\leq \; & (m-1)!n!\sum_{i=2}^{m-1}\frac{|E[X^{m-1-i}Y^n]|}{(m-1-i)!n!}\cdot\frac{p^i}{i!} \\
\leq \; & (m-1)!n!C_{m-3,n}\sum_{i=2}^{m-1}\frac{p^i}{i!} \\
\leq \; & (e^p - 1 - p)\,C_{m-3,n}(m-1)!n!.
\end{aligned}$$

Similarly, we have:

$$\begin{aligned} & \left|E\left[\left\{(X+pI)^{m-1}-X^{m-1}\right\}\left\{(Y+qI)^{n}-Y^{n}-nqY^{n-1}\right\}\right]\right| \\ \leq\ & \left(e^{p}-1\right)\left(e^{q}-1-q\right)C_{m-2,n-2}(m-1)!n!. \end{aligned}$$

It follows now from our recursive formula (20) that:

$$\begin{aligned} & |E[X^mY^n]| \\ \leq\ & (m-1)C_{m-2,n}(m-2)!n! + nbC_{m-1,n-1}(m-1)!(n-1)! \\ & + \frac{1-b^2}{|p|}\left(e^p-1-p\right)C_{m-3,n}(m-1)!n! \\ & + \frac{|b|(1-b^2)}{|q|}\left(e^p-1\right)\left(e^q-1-q\right)C_{m-2,n-2}(m-1)!n!. \end{aligned}$$

Since the sequence $\{C_{m,n}\}_{m\geq 0,n\geq 0}$ is left and right increasing, we conclude now that there is a constant K depending on b, p, and q, such that:

$$|E[X^mY^n]| \ \leq\ KC_{m-1,n}(m-1)!n!,$$

for all $m \geq 1$, $n \geq 0$. Dividing both sides of this inequality by $m!n!$, we obtain:

$$\frac{|E[X^mY^n]|}{m!n!} \ \leq\ \frac{K}{m}C_{m-1,n},$$

for all $m \geq 1$ and $n \geq 0$. Thus, we have:

$$\begin{aligned} C_{m,n} &= \max\left\{C_{m-1,n}, \left|E[X^mY^0]\right|/(m!0!), \ldots, |E[X^mY^n]|/(m!n!)\right\} \\ &\leq \max\left\{C_{m-1,n}, \frac{K}{m}C_{m-1,0}, \ldots, \frac{K}{m}C_{m-1,n}\right\} \\ &= \max\left\{C_{m-1,n}, \frac{K}{m}C_{m-1,n}\right\} \\ &= C_{m-1,n}, \end{aligned}$$

for all $m \geq \lfloor K \rfloor + 1$ and $n \geq 0$. On the other hand we have $C_{m,n} \geq C_{m-1,n}$, for all $m \geq 1$ and all $n \geq 0$. Thus, the sequences $\{C_{m,0}\}_{m\geq 1}$, $\{C_{m,1}\}_{m\geq 1}$, $\ldots$ are uniformly stationary, in the sense that there is natural number m_0, that is the same for all $n \geq 0$, such that $C_{m_0,n} = C_{m_0+1,n} = C_{m_0+2,n} = \cdots$.

On the other hand, since X and Y are symmetric operators, we have:

$$\begin{aligned} E[Y^nX^m] &= \langle Y^nX^m\phi, \phi\rangle \\ &= \langle \phi, X^mY^n\phi\rangle \\ &= E[X^mY^n]. \end{aligned}$$

Thus a similar argument will imply now that there exists an $n_0 \geq 1$, such that for all $m \geq 0$, $C_{m,n_0} = C_{m,n_0+1} = C_{m,n_0+2} = \cdots$. Since the sequence $\{C_{m,n}\}_{m\geq 0,n\geq 0}$ is left and right increasing, we conclude now that, for all $m \geq 0$ and all $n \geq 0$, we have $C_{m,n} \leq C_{m_0,n_0}$. Thus, for all $m \geq 0$ and $n \geq 0$, $|E[X^mY^n]| \leq C_{m_0,n_0}m!n!$. □

We are now ready to compute the joint Laplace transform of X and Y.

Theorem 4.2 *Let $X = a_x^- + a_x^0 + a_x^+$ and $Y = a_y^0 + a_y^0 + a_y^+$ be joint quantum decompositions of two random variables, whose terms satisfy the commutation relationships (1)–(8), where b, p, p', q, and q' are real numbers, such that $|b| < 1$, $p' = -pb$, and $q' = -qb$. We assume that $E[X] = E[Y] = 0$. Then, for all $(s,\, t) \in (-1,\, 1)^2$, the series:*

$$E\left[e^{sX}e^{tY}\right] := \sum_{m\geq 0}\sum_{n\geq 0}\frac{s^m t^n E[X^m Y^n]}{m!n!} \tag{23}$$

is absolutely convergent, and the joint left X–right Y Laplace transform *is:*

$$\begin{aligned} E\left[e^{sX}e^{tY}\right] &= e^{\frac{s^2+2bst+t^2}{2}} \\ &\quad\times e^{\frac{1-b^2}{p^2}\left(e^{ps}-1-ps-\frac{1}{2}p^2s^2\right)} e^{\frac{1-b^2}{q^2}\left(e^{qt}-1-qt-\frac{1}{2}q^2t^2\right)} \\ &\quad\times e^{-\frac{b(1-b^2)}{pq}(e^{ps}-1-ps)(e^{qt}-1-qt)}. \end{aligned} \tag{24}$$

If $p = 0$ or $q = 0$, then formula (24) makes sense by taking the limit as $p \to 0$ or $q \to 0$ in its right–hand side. Moreover, there exist two commuting random variables U and V, such that $(X,\, Y) \equiv (U,\, V)$, if and only if $bpq = 0$. If $bpq = 0$, then there exist an invertible 2×2 matrix T, a vector $(u,\, v)$ in $\mathbb{R}^2$, and two independent classic random variables X' and Y', that are either Gaussian or Poisson, such that $(X,\, Y) \equiv (X',\, Y')T + (u,\, v)$.

Proof. It follows from Lemma 4.1 that the series $\sum_{m\geq 0, n\geq 0} s^m t^n E[X^m Y^n]/m!n!$ is absolutely convergent for all s and t complex numbers, such that $|s| < 1$ and $|t| < 1$. We denote this series by $E[e^{sX}e^{tY}]$, to be in agreement with the classic commutative case. Let $\varphi(s,\, t) := E[e^{sX}e^{tY}]$. It follows from Weierstrass M–test that the double series, from the definition of $\varphi(s,\, t)$, can be differentiated term by term, with respect to both s and t, for $|s| < 1$ and $|t| < 1$. If we multiply both sides of the recursive relation (20) by $s^{m-1}t!/[(m-1)!n!]$ and sum up the resulting relations from $m = 1$ to ∞, and from $n = 0$ to ∞, then we obtain:

$$\begin{aligned} \frac{\partial\varphi}{\partial s}(s,t) &= s\varphi(s,t) + bt\varphi(s,t) \\ &\quad+\frac{1-b^2}{p}(e^{ps}-1-ps)\varphi(s,t) \\ &\quad-\frac{b(1-b^2)}{q}(e^{ps}-1)(e^{qt}-1-qt)\varphi(s,t). \end{aligned} \tag{25}$$

In particular, for $t = 0$, we conclude that the function $g(s) := E[e^{sX}]$ satisfies the differential equation:

$$\frac{dg}{ds}(s) = sg(s) + \frac{1-b^2}{p}\left(e^{ps}-1-ps\right)g(s).$$

Since $g(0) = 1$, we must have:

$$g(s) = e^{\frac{s^2}{2}+\frac{1-b^2}{p^2}\left(e^{ps}-1-ps-\frac{p^2s^2}{2}\right)}.$$

Similarly, we can see that:

$$E[e^{tY}] = e^{\frac{t^2}{2}+\frac{1-b^2}{q^2}\left(e^{qt}-1-qt-\frac{q^2t^2}{2}\right)}.$$

Solving now the differential equation, in the variable s, (25), we obtain:

$$\begin{aligned}\varphi(s,t) &= K(t)e^{\frac{s^2}{2}}e^{bst}e^{\frac{1-b^2}{p^2}\left(e^{ps}-1-ps-\frac{p^2s^2}{2}\right)}\\ &\quad\times e^{-\frac{b(1-b^2)}{pq}(e^{ps}-1-ps)\left(e^{qt}-1-qt\right)}.\end{aligned}$$

Setting $s=0$ in this equation and using the fact that $\varphi(0,t)=E[e^{tY}]$, we conclude that:

$$K(t) = e^{\frac{t^2}{2}+\frac{1-b^2}{q^2}\left(e^{qt}-1-qt-\frac{q^2t^2}{2}\right)}.$$

Thus, we have:

$$\begin{aligned}E[e^{sX}e^{tY}] &= e^{\frac{t^2}{2}+\frac{1-b^2}{q^2}(e^{qt}-1-qt-\frac{q^2t^2}{2})}\\ &\quad\times e^{\frac{s^2}{2}}e^{bst}e^{\frac{1-b^2}{p^2}(e^{ps}-1-ps-\frac{p^2s^2}{2})}\\ &\quad\times e^{-\frac{b(1-b^2)}{pq}(e^{ps}-1-ps)(e^{qt}-1-qt)}\\ &= e^{\frac{s^2+2bst+t^2}{2}}\\ &\quad\times e^{\frac{1-b^2}{p^2}(e^{ps}-1-ps-\frac{p^2s^2}{2})}\\ &\quad\times e^{\frac{1-b^2}{q^2}(e^{qt}-1-qt-\frac{q^2t^2}{2})}\\ &\quad\times e^{-\frac{b(1-b^2)}{pq}(e^{ps}-1-ps)(e^{qt}-1-qt)}.\end{aligned}$$

$\square$

5. The existence of the operators X and Y

We close the paper by showing that, for any $|b|<1$, and any real numbers p and q, we can construct two random variables satisfying the commutation relationships from the previous theorem. Let b, p, and q be fixed, such that $|b|<1$. Let U and V be two jointly Gaussian classic (commutative) random variables, with mean 0, variance 1, and covariance b, defined on the same probability space $(\Omega,\mathcal{F},P)$. We have $U=a_u^-+a_u^+$ and $V=a_v^-+a_v^+$, $[a_u^-,a_u^+]=[a_v^-,a_v^+]=I$ and $[a_u^-,a_v^+]=[a_v^-,a_u^+]=E[UV]I=bI$. See [5], [6], and [8]. Let $\mathcal{A}$ be the unital algebra generated by a_u^-, a_u^+, a_v^-, and a_v^+. This is an algebra of operators that are densely defined on the Hilbert space $H:=L^2(\Omega,\sigma(U,V),P)$, where $\sigma(U,V)$ represents the smallest sub–sigma–algebra of $\mathcal{F}$, with respect to which both U and V are measurable functions. Inside the algebra $\mathcal{A}$, let us consider the following operators:

$$X := a_u^- + \left[\alpha_1 a_u^+a_u^- + \beta_1(a_u^+a_v^- + a_v^+a_u^-) + \gamma_1 a_v^+a_v^-\right] + a_u^+ \tag{26}$$

and

$$Y := a_v^- + \left[\alpha_2 a_v^+a_v^- + \beta_2(a_v^+a_u^- + a_u^+a_v^-) + \gamma_2 a_u^+a_u^-\right] + a_v^+, \tag{27}$$

where α_i, β_i, and γ_i, $1\le i\le 2$, are real constants that will be determined such that X and Y satisfy the commutation relationships (1)–(8). The coefficients of $a_x^+a_y^-$ and $a_y^+a_x^-$ are the same in the definitions of X and Y, β_1 and β_2, respectively, to ensure that X and Y are both symmetric operators. Let $\phi:=1\in H$, be the unit vector with respect to which the

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noncommutative expectation will be computed. The terms of a joint quantum decomposition of X and Y (relative to $\mathcal{A}$), are:

$$\begin{aligned}
a_x^- &= a_u^-\,, \\
a_x^0 &= \alpha_1 a_u^+ a_u^- + \beta_1(a_u^+ a_v^- + a_v^+ a_u^-) + \gamma_1 a_v^+ a_v^-\,, \\
a_x^+ &= a_u^+\,, \\
a_y^- &= a_v^-\,, \\
a_y^0 &= \alpha_2 a_v^+ a_v^- + \beta_2(a_v^+ a_u^- + a_u^+ a_v^-) + \gamma_2 a_u^+ a_u^-\,, \\
a_y^+ &= a_v^+\,.
\end{aligned}$$

It is now clear that (1), (2), (3), (4) hold. Also, because in the formulas of a_x^0 and a_y^0, the annihilation operators a_u^- and a_v^- are always to the right of the creation operators a_u^+ and a_v^+, we conclude that the conditions $E[X] = 0$ and $E[Y] = 0$ hold. To compute the commutators $[a_x^-, a_x^0]$, $[a_x^-, a_y^0]$, $[a_y^-, a_x^0]$, and $[a_y^-, a_y^0]$, we apply again the product rule for commutators. For example, one commutator involved in the computation of $[a_x^-, a_x^0]$ is $[a_u^-, a_v^+ a_u^-]$, which can be computed as follows:

$$\begin{aligned}
[a_u^-, a_v^+ a_u^-] &= [a_u^-, a_v^+]a_u^- + a_v^+[a_u^-, a_u^-] \\
&= bIa_u^- + 0 \\
&= ba_x^-\,.
\end{aligned}$$

Doing all these types of commutators, we obtain:

$$\begin{aligned}
\left[a_x^-, a_x^0\right] &= (\alpha_1 + b\beta_1)a_x^- + (\beta_1 + b\gamma_1)a_y^-\,, \\
\left[a_x^-, a_y^0\right] &= (\gamma_2 + b\beta_2)a_x^- + (\beta_2 + b\alpha_2)a_y^-\,, \\
\left[a_y^-, a_x^0\right] &= (\beta_1 + b\alpha_1)a_x^- + (\gamma_1 + b\beta_1)a_y^-\,, \\
\left[a_y^-, a_y^0\right] &= (\beta_2 + b\gamma_2)a_x^- + (\alpha_2 + b\beta_2)a_y^-\,.
\end{aligned}$$

If we want the commutation relationships (5), (6), (7), and (8) to hold, then identifying the coefficients of a_x^- and a_y^-, we should have:

$$\begin{aligned}
\alpha_1 + b\beta_1 &= p, \\
\beta_1 + b\gamma_1 &= -pb, \\
\gamma_2 + b\beta_2 &= 0, \\
\beta_2 + b\alpha_2 &= 0, \\
\beta_1 + b\alpha_1 &= 0, \\
\gamma_1 + b\beta_1 &= 0, \\
\beta_2 + b\gamma_2 &= -qb, \\
\alpha_2 + b\beta_2 &= q.
\end{aligned}$$

This is a system of eight equations and six unknowns, that can be split up into two sub–systems of four equations and three unknowns each. Each of these two sub–systems is consistent and has a unique solution:

$$(\alpha_1, \beta_1, \gamma_1) = \frac{p}{1 - b^2}(1, -b, b^2)$$

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and

$$(\alpha_2, \beta_2, \gamma_2) \quad = \quad \frac{q}{1-b^2}(1, -b, b^2).$$

Thus we have proven the existence of two noncommutative random variables X and Y, for which the commutation relationships (1)–(8) and the condition $E[X] = E[Y] = 0$ hold.

The left X–right Y Laplace transform $E[e^{sX}e^{tY}]$ is a generating function only for the joint moments of the form $E[X^m Y^n]$, $m \geq 0$, $n \geq 0$. Because X and Y do not commute, we also need to compute other Laplace transforms like $E[e^{s_1X}e^{t_1Y}e^{s_2X}e^{t_2Y}]$, $E[e^{s_1X}e^{t_1Y}e^{s_2X}e^{t_2Y}e^{s_3X}e^{t_3Y}]$ and so on, to generate the other joint moments like $E[X^{m_1}Y^{n_1}X^{m_2}Y^{n_2}]$, $E[X^{m_1}Y^{n_1}X^{m_2}Y^{n_2}X^{m_3}Y^{n_3}], \ldots$. Our commutator method can be applied to compute these other Laplace transforms, but the calculations are more complicated.

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