q-Virasoro algebra at root of unity limit and 2d-4d connection

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Abstract. We propose a limiting procedure in which, starting from the q-lifted version (or K-theoretic five dimensional version) of the (W)AGT conjecture to be assumed, the Virasoro/W block is generated in the r-th root of unity limit in q in the $2d$ side, while the same limit automatically generates the projection of the five dimensional instanton partition function onto that on the ALE space $\mathbb{R}^4/\mathbb{Z}_r$. This proceeding is based on arXiv:1308.2068

1. Introduction

Continuing attention has been paid to the correspondence between the two dimensional conformal block [1] and the instanton sum [2, 3] identified as the partition function of the four dimensional $\mathcal{N} = 2$ supersymmetric gauge theory. The both sides of this correspondence [4, 5] have already been intensively studied for more than a few years and a wealth of such examples has been found by now. One of the central tools for our study is the β-deformed matrix model controlling the integral representation of the conformal block [6, 7, 8, 9, 10, 11, 12, 13, 14] and the use [10] of the formula [15, 16] on multiple integrals. This general correspondence, on the other hand, has stayed as conjectures in most of the examples except the few ones [17, 18, 19, 20] and one of the next steps in the developments would be to obtain efficient understanding among these while avoiding making many conjectures.

We regard the correspondence between q-Virasoro [21, 22, 23, 24, 25, 26]/W block versus five dimensional instanton partition function as a parent one. We propose the following procedure [27] on the orbifolded examples of the correspondence [28, 29, 30, 31, 32, 33, 34, 35, 36, 37]:

(i) assume the q-lifted version (or K-theoretic five dimensional version) of the (W)AGT conjecture
(ii) introduce the limiting procedure $q \to \omega$, where ω is the r-th root of unity.
(iii) apply the same limiting procedure to $Z^{\mathrm{5d}}_{\mathrm{inst}}$, which automatically generates the instanton partition function on ALE space.

We emphasize that, through this limiting procedure (and the assumption (i)), the resulting $2d$ conformal block is guaranteed to agree with the corresponding instanton partition function on ALE space. Our procedure is illustrated in Figure 1.

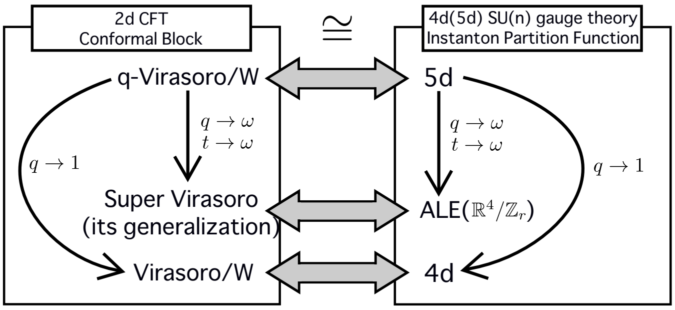

Figure 1. Schematic sketch of our procedure.

2. *q*-Virasoro algebra and root of unit limit

We introduce the q-Virasoro algebra which contains two parameters q and $t = q^\beta$ and consider the $q \to \omega$ limit.

2.1. q-Virasoro algebra

The defining relation of the q-deformed Virasoro algebra [21] is

$$f(z'/z)\mathcal{T}(z)\mathcal{T}(z') - f(z/z')\mathcal{T}(z')\mathcal{T}(z) = \frac{(1-q)(1-t^{-1})}{(1-p)}\Big[\delta(pz/z') - \delta(p^{-1}z/z')\Big], \tag{2.1}$$

with $p = q/t$. Here $\mathcal{T}(z)$ is the q-Virasoro operator and

$$f(z) = \exp\left(\sum_{n=1}^{\infty}\frac{1}{n}\frac{(1-q^n)(1-t^{-n})}{(1+p^n)}z^n\right), \qquad \delta(z) = \sum_{n\in\mathbb{Z}} z^n. \tag{2.2}$$

Using the q-deformed Heisenberg algebra defined by

$$\begin{aligned} [\alpha_n, \alpha_m] &= -\frac{1}{n}\frac{(1-q^n)(1-t^{-n})}{(1+p^n)}\delta_{n+m,0}, \qquad (n \neq 0), \\ [\alpha_n, Q] &= \delta_{n,0}, \end{aligned} \tag{2.3}$$

the q-Virasoro operator is realized as

$$\mathcal{T}(z) =: \exp\left(\sum_{n\neq 0}\alpha_n z^{-n}\right) : p^{1/2}q^{\sqrt{\beta}\alpha_0} + : \exp\left(-\sum_{n\neq 0}\alpha_n (pz)^{-n}\right) : p^{-1/2}q^{-\sqrt{\beta}\alpha_0}. \tag{2.4}$$

Let us introduce two q-deformed free bosons by

$$\widetilde{\varphi}^{(\pm)}(z) = \beta^{\pm 1/2}\, Q + 2\beta^{\pm 1/2}\alpha_0 \log z + \sum_{n\neq 0}\frac{(1+p^{-n})}{(1-\xi_\pm{}^n)}\alpha_n z^{-n}, \tag{2.5}$$

where $\xi_+ = q$, $\xi_- = t$ and define two kinds of deformed screening currents

$$S_\pm(z) =: \mathrm{e}^{\pm\widetilde{\varphi}^{(\pm)}(z)} : . \tag{2.6}$$

It can be checked that $S_\pm$ commute with the q-Virasoro generators up to total q- or t-derivative defined by

$$\frac{\mathrm{d}_{\xi_\pm}}{\mathrm{d}_{\xi_\pm} z} f(z) = \frac{f(z) - f(\xi_\pm z)}{(1-\xi_\pm)z}. \tag{2.7}$$

Deformed screening charges [25] are defined by the Jackson integral

$$Q^{+}_{[a,b]} = \int_a^b \mathrm{d}_q z\, S_+(z), \qquad Q^{-}_{[a,b]} = \int_a^b \mathrm{d}_t z\, S_-(z), \tag{2.8}$$

over a suitably chosen integral domain $[a, b]$.

2.2. $q \to -1$, $t \to -1$ limit

We consider a $q \to -1$ limit. Simultaneously, $t \to -1$ limit is taken. Let us realize this limit by

$$q = -\mathrm{e}^{-(1/\sqrt{\beta})h}, \qquad t = -\mathrm{e}^{-\sqrt{\beta}h}, \qquad p = q/t = \mathrm{e}^{Q_E h}, \qquad h \to +0, \tag{2.9}$$

where $Q_E = \sqrt{\beta} - 1/\sqrt{\beta}$. Choosing the branch of logarithms of q and t appropriately, we see that β takes the following rational value:

$$\beta = \frac{k_- + 1/2}{k_+ + 1/2} = \frac{2k_- + 1}{2k_+ + 1}, \tag{2.10}$$

where $k_\pm$ are non-negative integers.

First, we consider the $q \to -1$ limit of the q-bosons (2.5). Let us decompose them into "even" and "odd" parts:

$$\widetilde{\varphi}^{(\pm)}(z) = \widetilde{\varphi}^{(\pm)}_{\text{even}}(z) + \widetilde{\varphi}^{(\pm)}_{\text{odd}}(z), \tag{2.11}$$

where

$$\begin{aligned} \widetilde{\varphi}^{(\pm)}_{\text{even}}(z) &:= \beta^{\pm 1/2} Q + \beta^{\pm 1/2}\alpha_0 \log(z^2) + \sum_{n\neq 0} \frac{1 + p^{-2n}}{1 - {\xi_\pm}^{2n}} \alpha_{2n} z^{-2n}, \\ \widetilde{\varphi}^{(\pm)}_{\text{odd}}(z) &:= \sum_{n\in\mathbb{Z}} \frac{1 + p^{-2n-1}}{1 - {\xi_\pm}^{2n+1}} \alpha_{2n+1} z^{-2n-1}. \end{aligned} \tag{2.12}$$

In the $q \to -1$ limit of the q-bosons, we obtain two free bosons, $\phi(w)$ and $\varphi(w)$:

$$\begin{aligned} \widetilde{\varphi}^{(\pm)}_{\text{even}}(z) &= \beta^{\pm 1/2}\phi(w) + O(h), \\ \widetilde{\varphi}^{(\pm)}_{\text{odd}}(z) &= \varphi(w) + O(h), \end{aligned} \tag{2.13}$$

where $w = z^2$ and

$$\phi(w) = Q + a_0 \log w - \sum_{n\neq 0} \frac{a_n}{n} w^{-n}, \tag{2.14}$$

$$\varphi(w) = \sum_{n\in\mathbb{Z}} \frac{\tilde{a}_{n+1/2}}{n + 1/2} w^{-n-1/2}, \tag{2.15}$$

with the following standard commutation relations of boson and twisted boson oscillators:

$$[a_n, a_m] = n\delta_{n+m,0}, \qquad [a_n, Q] = \delta_{n,0}, \tag{2.16}$$

$$[\tilde{a}_{n+1/2}, \tilde{a}_{-m-1/2}] = (n + 1/2)\delta_{n,m}. \tag{2.17}$$

Next, we consider the $q \to -1$ limit of the screening currents (2.6). Using the limit of q-bosons (2.13), we obtain

$$\lim_{q\to -1} S_+(z) =: \mathrm{e}^{\sqrt{\beta}\phi(w)} \mathrm{e}^{\varphi(w)} :, \qquad \lim_{q\to -1} S_-(z) =: \mathrm{e}^{-(1/\sqrt{\beta})\phi(w)} \mathrm{e}^{-\varphi(w)} : . \tag{2.18}$$

We can construct the two fermion,

$$\psi(w) \equiv \frac{\mathrm{i}}{2\sqrt{2w}}\Big(:\mathrm{e}^{\varphi(w)}: - :\mathrm{e}^{-\varphi(w)}:\Big), \qquad \widehat{\psi}(w) \equiv \frac{1}{2\sqrt{2w}}\Big(:\mathrm{e}^{\varphi(w)}: + :\mathrm{e}^{-\varphi(w)}:\Big), \tag{2.19}$$

and see that $\psi(w)$ (resp. $\widehat{\psi}(w)$) is an NS fermion (resp. an R fermion) on the w-plane. The screening charge together with the regularization factor becomes

$$\lim_{q\to -1} \frac{\mathrm{i}}{\sqrt{2}}(1+q)Q^{+}_{[a,b]} = \int_{a^2}^{b^2} \mathrm{d}w\, \psi(w) :\mathrm{e}^{\sqrt{\beta}\phi(w)}: \equiv Q^{(+)}_{[a^2,b^2]}, \tag{2.20}$$

which is the screening charge for the superconformal block [38, 39]. Here we have ignored the subtlety due to the zero-mode and negative power terms. To be more precise, the "limit" (2.20) includes a kind of projection imposed by hand.

Let us choose a q-deformed Vertex operator $V_\alpha(z)$ as

$$V_\alpha(z) =: \mathrm{e}^{\Phi_\alpha(z)}:, \tag{2.21}$$

where

$$\Phi_\alpha(z) = \alpha\, Q + 2\alpha\, \alpha_0 \log z + \sum_{n\neq 0} \frac{q^{-n}(1-q^{2\sqrt{\beta}\alpha|n|})}{(1-q^{-|n|})(1-t^{-n})}\alpha_n z^{-n}. \tag{2.22}$$

We restrict the parameter α to take values corresponding to those of the primary fields of the minimal theories in the NS sector:

$$\alpha = \alpha_{r,s} = -\left(\frac{1-r}{2}\right)\frac{1}{\sqrt{\beta}} + \left(\frac{1-s}{2}\right)\sqrt{\beta}, \tag{2.23}$$

where

$$1 \le r \le 2k_-, \qquad 1 \le s \le 2k_+, \qquad r-s \in 2\mathbb{Z}. \tag{2.24}$$

The $q \to -1$ limit of the deformed vertex operator (2.21) is given by

$$\lim_{q\to -1} V_{\alpha_{r,s}}(z) =: \mathrm{e}^{\alpha_{r,s}\phi(w)}: \quad \text{for } L_{r,s} \text{ even}, \tag{2.25}$$

where

$$L_{r,s} \equiv (2k_+ + 1)\sqrt{\beta}\alpha_{r,s} = -k_+(1-r) + (1-s)k_- + \left(\frac{r-s}{2}\right) \in \mathbb{Z}. \tag{2.26}$$

Note that $:\mathrm{e}^{\alpha_{r,s}(w)}:$ is exactly equal to the Coulomb gas representation of the bosonic primary field in the NS sector with scaling dimension

$$\Delta_{\alpha_{r,s}} = \frac{1}{2}\alpha_{r,s}(\alpha_{r,s} - Q_E) = -\frac{1}{8}Q_E^2 + \frac{1}{8}\left(-\frac{r}{\sqrt{\beta}} + s\sqrt{\beta}\right)^2. \tag{2.27}$$

2.3. $q \to -1$ limit of q-Virasoro generators and $N = 1$ superconformal algebra

In this subsection, we consider directly the $q \to -1$ limit of the generating function $\mathcal{T}(z)$ of the q-Virasoro generators (2.4). In this limit, we can see that the fermionic currents

$$\begin{aligned} G(w) &= \psi(w)\partial\phi(w) + Q_E\, \partial\psi(w), \\ \widehat{G}(w) &= \widehat{\psi}(w)\partial\phi(w) + Q_E\, \partial\widehat{\psi}(w), \end{aligned} \tag{2.28}$$

appear [27]. Through OPE, we obtain the stress tensors $T(w)$ and $\hat{T}(w)$. Then, $T(w)$ and $G(w)$ (resp. $\hat{T}(w)$ and $\hat{G}(w)$) obey the $N = 1$ superconformal algebra in the NS (resp. R) sector. The central charge is $c = \frac{3}{2}\hat{c}$ where

$$\hat{c} = 1 - 2Q_E^2 = 1 - 2\left(\sqrt{\beta} - \frac{1}{\sqrt{\beta}}\right)^2 = 1 - \frac{8(k_- - k_+)^2}{(2k_+ + 1)(2k_- + 1)}. \tag{2.29}$$

Recall that the $N = 1$ minimal superconformal models have the central charge

$$\hat{c} = 1 - \frac{2(m' - m)^2}{mm'}. \tag{2.30}$$

Therefore, $\hat{c}$ (2.29) corresponds to an $N = 1$ minimal model such that m and m' are both positive odd integers: $m = 2k_+ + 1$ and $m' = 2k_- + 1$. Without loss of generality, we can take $m' > m$, i.e., $k_- > k_+$. The unitary minimal models [40] occur when $k_- = k_+ + 1$ and $k_+ \geq 1$.

2.4. r-th root of unity limit

The limiting procedure is realized by

$$q = e^{2\pi i/r}\, e^{-(1/\sqrt{\beta})h}, \qquad t = e^{2\pi i/r}\, e^{-\sqrt{\beta}h}, \qquad p = q/t = e^{Q_E h}, \qquad h \to +0. \tag{2.31}$$

Let us decompose the q-deformed free bosons (2.5) into two parts:

$$\widetilde{\varphi}^{(\pm)}(z) = \widetilde{\varphi}_0^{(\pm)}(z) + \widetilde{\varphi}_R^{(\pm)}(z), \tag{2.32}$$

where

$$\begin{aligned} \widetilde{\varphi}_0^{(\pm)}(z) &= \beta^{\pm 1/2} Q + \frac{2}{r}\beta^{\pm 1/2}\alpha_0 \log z^r + \sum_{n\neq 0} \frac{(1 + p^{-nr})}{(1 - \xi_\pm^{nr})} \alpha_{nr}\, z^{-nr}, \\ \widetilde{\varphi}_R^{(\pm)}(z) &= \sum_{\ell=1}^{r-1} \sum_{n\in\mathbb{Z}} \frac{(1 + p^{-nr-\ell})}{1 - \xi_\pm^{nr+\ell}} \alpha_{nr+\ell}\, z^{-nr-\ell}. \end{aligned} \tag{2.33}$$

In the $q \to e^{2\pi i/r}$ limit, we have

$$\lim_{h\to+0} \widetilde{\varphi}_0^{(\pm)}(z) = \sqrt{\frac{2}{r}}\, \beta^{\pm 1/2}\phi(w), \qquad \lim_{h\to+0} \widetilde{\varphi}_R^{(\pm)}(z) = \sqrt{\frac{2}{r}}\, \varphi(w), \tag{2.34}$$

where $w = z^r$ and

$$\phi(w) = Q_0 + a_0 \log w - \sum_{n\neq 0} \frac{a_n}{n} w^{-n}, \tag{2.35}$$

$$\varphi(w) = \sum_{\ell=1}^{r-1} \varphi^{(\ell)}(w), \qquad \varphi^{(\ell)}(w) = \sum_{n\in\mathbb{Z}} \frac{\tilde{a}_{n+\ell/r}}{n + \ell/r} w^{-n-\ell/r}. \tag{2.36}$$

The oscillators obey the following commutation relations:

$$[a_m, a_n] = m\delta_{m+n,0}, \qquad [a_n, Q_0] = \delta_{n,0}, \tag{2.37}$$

$$[\tilde{a}_{n+\ell/r}, \tilde{a}_{-m-\ell'/r}] = (n + \ell/r)\delta_{m,n}\delta_{\ell,\ell'}. \tag{2.38}$$

The screening currents (2.6) turn into

$$\lim_{q\to \mathrm{e}^{2\pi \mathrm{i}/r}} S_\pm(z) =: \exp\left(\pm\sqrt{\frac{2}{r}}\beta^{\pm 1/2}\phi(w)\right) :: \exp\left(\pm\sqrt{\frac{2}{r}}\varphi(w)\right) : . \tag{2.39}$$

Using the vertex operators $\mathrm{e}^{\pm\sqrt{2/r}\,\varphi(w)}$, we can introduce the following fields

$$\Psi_\pm^{(\ell)}(w) := \frac{1}{r^{3/2}w^{1-(1/r)}}\sum_{k=0}^{r-1}\mathrm{e}^{2\pi \mathrm{i}k\ell/r} : \exp\left(\pm\sqrt{\frac{2}{r}}\varphi(\mathrm{e}^{2\pi \mathrm{i}k}w)\right) : . \tag{2.40}$$

Note that for $w_1 \to w_2$,

$$\langle 0|\Psi_\pm^{(\ell_1)}(w_1)\Psi_\mp^{(\ell_2)}(w_2)|0\rangle = \frac{\delta_{\ell_1+\ell_2,r}}{(w_1-w_2)^{2(1-(1/r))}}\left\{1+O\Big((w_1-w_2)^2\Big)\right\}. \tag{2.41}$$

These fields $\Psi_\pm^{(\ell)}(w)$ are analogue of the parafermion current with scaling dimension $\Delta_1 = 1-(1/r)$.

3. $SU(n)$ conformal/W block and q-deformation

The q-deformed W algebra at roots of unity itself will not be exploited here for our study of the q-lifted version of (W)AGT conjecture. For the conformal block, there is a simple recipe for the q-lift (q-deformation) without explicitly treating the q-W algebra.

3.1. *integral representation of the SU(n) Vir/W block*

In this subsection, we review the conformal block of W_n algebra. Let us consider the conformal field theory with the central charge c

$$c = (n-1)\big(1-n(n+1)Q_E^2\big), \tag{3.1}$$

associated with the $A_{n-1} = \mathfrak{sl}(n)$ Lie algebra. Let $\mathfrak{h}$ be the Cartan subalgebra of A_{n-1} and $\mathfrak{h}^*$ its dual.

The four-point conformal block of the chiral vertex operators can be expressed in terms of free fields. Let $\phi(z)$ be an $\mathfrak{h}$-valued free chiral boson with correlation functions:

$$\begin{gathered}\langle\phi_a(z)\phi_b(w)\rangle = (e_a,e_b)\log(z-w), \quad \phi_a(z) = \langle e_a,\phi(z)\rangle, \\ e_a \in \mathfrak{h}^* : \text{ a simple root of } A_{n-1}, \quad a = 1,\cdots,n-1.\end{gathered} \tag{3.2}$$

Here $(\cdot,\cdot)$ is the symmetric bilinear form on $\mathfrak{h}^*$ and $\langle\cdot,\cdot\rangle$ is the natural pairing between $\mathfrak{h}^*$ and $\mathfrak{h}$. With $(e_a,e_a)=2$, $C_{ab} := (e_a,e_b)$ is the Cartan matrix of the A_{n-1} algebra.

The free-field representation of the conformal block is then given by

$$\begin{aligned}&\mathcal{F}(c,\Delta_I,\Delta_i|\Lambda) \\ &= \left\langle V_{(1/\sqrt{\beta})\alpha_1}(0)V_{(1/\sqrt{\beta})\alpha_2}(\Lambda)V_{(1/\sqrt{\beta})\alpha_3}(1)V_{(1/\sqrt{\beta})\alpha_4}(\infty)\prod_{a=1}^{n-1}\mathcal{Q}_a{}^{N_a}\widetilde{\mathcal{Q}}_a{}^{\widetilde{N}_a}\right\rangle,\end{aligned} \tag{3.3}$$

with $\alpha_i \in \mathfrak{h}^*$ $(i=1,2,3,4)$. The vertex operator is defined by $V_\alpha(z) =: \mathrm{e}^{\langle\alpha,\phi(z)\rangle} :$ $(\alpha\in\mathfrak{h}^*)$ and two kinds of screening charges are inserted:

$$\mathcal{Q}_a = \int_0^\Lambda \mathrm{d}z\, V_{\sqrt{\beta}e_a}(z), \qquad \widetilde{\mathcal{Q}}_a = \int_1^\infty \mathrm{d}z\, V_{\sqrt{\beta}e_a}(z). \tag{3.4}$$

The four points are set to $z_1 = 0$, $z_2 = \Lambda$, $z_3 = 1$, $z_4 = \infty$. Hence Λ is the cross ratio.

By using the Wick's theorem and changing the integration variables from $z_I^{(a)}$ ($I = 1, 2, \dots, N_a + \widetilde{N}_a$) to $x_I^{(a)}$ ($I = 1, 2, \dots, N_a$) and $y_J^{(a)}$ ($J = 1, 2, \dots, \widetilde{N}_a$), defined by

$$\Lambda x_I^{(a)} := z_I^{(a)}, \qquad \frac{1}{y_J^{(a)}} := z_{N_a+J}^{(a)}, \tag{3.5}$$

we obtain the following Selberg type multiple integral [10]:

$$\mathcal{F} = \Lambda^{\Delta_I - \Delta_1 - \Delta_2}(1-\Lambda)^{(\alpha_2,\alpha_3)/\beta} Z_S(\Lambda), \tag{3.6}$$

where

$$\begin{aligned} Z_S(\Lambda) := \int_{[0,1]^N} \mathrm{d}x \left(\Delta_{A_{n-1}}(x)\right)^\beta \prod_{a=1}^{n-1}\prod_{I=1}^{N_a} (x_I^{(a)})^{(\alpha_1,e_a)}(1-x_I^{(a)})^{(\alpha_2,e_a)} \\ \times \int_{[0,1]^{\widetilde{N}}} \mathrm{d}y \left(\Delta_{A_{n-1}}(y)\right)^\beta \prod_{b=1}^{n-1}\prod_{J=1}^{N_b} (y_J^{(b)})^{(\alpha_4,e_b)}(1-y_J^{(b)})^{(\alpha_3,e_b)}\, F(x,y|\Lambda). \end{aligned} \tag{3.7}$$

Here $N := \sum_a N_a$, $\widetilde{N} := \sum_a \widetilde{N}_a$,

$$F(x,y|\Lambda) := \prod_{a=1}^{n-1}\left\{\prod_{I=1}^{N_a}(1-\Lambda x_I^{(a)})^{(\alpha_3,e_a)}\prod_{J=1}^{\widetilde{N}_a}(1-\Lambda y_J^{(a)})^{(\alpha_2,e_a)}\right\}\prod_{a=1}^{n-1}\prod_{b=1}^{n-1}\prod_{I=1}^{N_a}\prod_{J=1}^{\widetilde{N}_b}(1-\Lambda x_I^{(a)}y_J^{(b)})^{\beta C_{ab}}, \tag{3.8}$$

$$\mathrm{d}x = \prod_{a=1}^{n-1}\prod_{I=1}^{N_a}\mathrm{d}x_I^{(a)}, \qquad \mathrm{d}y = \prod_{a=1}^{n-1}\prod_{J=1}^{\widetilde{N}_a}\mathrm{d}y_J^{(a)}, \tag{3.9}$$

$$\Delta_{A_{n-1}}(x) = \prod_{a=1}^{n-1}\prod_{I<J}^{N_a}|x_I^{(a)} - x_J^{(a)}|^2 \prod_{b=1}^{n-2}\prod_{I=1}^{N_b}\prod_{J=1}^{N_{b+1}}|x_J^{(b+1)} - x_I^{(b)}|^{-1}, \tag{3.10}$$

$$\Delta_{A_{n-1}}(y) = \prod_{a=1}^{n-1}\prod_{I<J}^{\widetilde{N}_a}|y_I^{(a)} - y_J^{(a)}|^2 \prod_{b=1}^{n-2}\prod_{I=1}^{\widetilde{N}_b}\prod_{J=1}^{\widetilde{N}_{b+1}}|y_J^{(b+1)} - y_I^{(b)}|^{-1}. \tag{3.11}$$

Let us rewrite (3.7) as

$$Z_S(\Lambda) = Z_s(0)\left\langle\!\left\langle F(x,y|\Lambda)\right\rangle_+\right\rangle_-. \tag{3.12}$$

Here $\langle\cdots\rangle_+$ (resp. $\langle\cdots\rangle_-$) is the average over the A_{n-1} Selberg integral normalized as $\langle 1\rangle_\pm = 1$.

The AGT relation implies that $Z_S(\Lambda)/Z_S(0)$ is equal to the instanton part of the Nekrasov partition function of $\mathcal{N} = 2$ $SU(n)$ gauge theory with $N_f = 2n$ fundamental matters.

3.2. q-deformation

In order to study the five-dimensional version of AGT conjecture, we need a q-deformed conformal block. There is a simple recipe [42, 47] to obtain a q-deformation of the Selberg-type multiple integral which is given by the following replacements:

$$\int_0^1 \mathrm{d}x_I \;\;\to\;\; \int_0^1 \mathrm{d}_q x_I, \tag{3.13}$$

and

$$
\begin{aligned}
(1-x_I)^{\beta_2-1} &\quad\to\quad \prod_{k=0}^{\beta_2-1}(1-q^k x_I),\\
\prod_{1\le I<J\le n}(x_I-x_J)^{2\beta} &\quad\to\quad \prod_{1\le I\neq J\le n}\prod_{k=0}^{\beta-1}(x_I-q^k x_J).
\end{aligned}
\tag{3.14}
$$

By this replacement, we obtain the q-deformation of the conformal block (3.12):

$$
\begin{aligned}
Z_S^{(q)} = \Bigg\langle\!\!\Bigg\langle \prod_{a=1}^{n-1}\left\{\prod_{I=1}^{N_a}\prod_{i=0}^{v_{a-}-1}(1-\Lambda x_I^{(a)}q^i)\prod_{J=1}^{\widetilde{N}_a}\prod_{j=0}^{v_{a+}-1}(1-\Lambda y_J^{(a)}q^j)\right\}\times\\
\times\prod_{a,b=1}^{n-1}\prod_{\ell=0}^{\beta-1}\prod_{I=1}^{N_a}\prod_{J=1}^{\widetilde{N}_a}(1-\Lambda x_I^{(a)}y_J^{(b)}q^\ell)^{C_{ab}}\Bigg\rangle_{N+,q}\Bigg\rangle_{\widetilde{N}-,q},
\end{aligned}
\tag{3.15}
$$

where

$$
\Big\langle f(x)\Big\rangle_{N\pm,q} = \frac{1}{S_{N,q}}\left(\prod_{I=1}^{N}\int_0^1 \mathrm{d}_q x_I\right)\prod_{I=1}^{N}x_I^{u_{a\pm}}\prod_{i=1}^{v_{a\pm}-1}(1-x_Iq^i)\prod_{1\le I\neq J\le N}\prod_{i=1}^{\beta-1}(x_I-q^ix_J)f(x)
\tag{3.16}
$$

Rearranging the integrand of eq.(3.15), we obtain

$$
Z_S^{(q)} = \langle\langle I_S^{(q)}\rangle_{N+,q}\rangle_{\widetilde{N}-,q},
\tag{3.17}
$$

where

$$
I_S^{(q)} = \exp\left\{-\sum_{k=1}^{\infty}[\beta]_{q^k}\frac{\Lambda^k}{k}\sum_{a=1}^{n}\left[\left(r_k^{(a)}+\frac{[v_{a+}]'_{q^k}}{[\beta]_{q^k}}\right)\left(\tilde{r}_k^{(a)}+\frac{[v_{a-}]'_{q^k}}{[\beta]_{q^k}}\right)-\frac{[v_{a+}]'_{q^k}}{[\beta]_{q^k}}\frac{[v_{a-}]'_{q^k}}{[\beta]_{q^k}}\right]\right\}.
\tag{3.18}
$$

Here we have introduced the q-number

$$
[a]_q := \frac{1-q^a}{1-q},
\tag{3.19}
$$

and

$$
[v_{a-}]'_{q^k} := -\sum_{s=1}^{a-1}[v_{s-}]_{q^k}, \qquad [v_{(n-a)+}]'_{q^k} := \sum_{s=1}^{a}[v_{(n-s)+}]_{q^k},
$$

with

$$
[v_{1-}]'_{q^k} = [v_{n+}]'_{q^k} = 0.
$$

Let λ be a partition, λ' be its conjugate and $a_\lambda(s)$ (resp. $\ell_\lambda(s)$) be the arm length (resp. leg length) at $s=(i,j)\in\lambda$:

$$
a_\lambda(s) = a_\lambda(i,j) = \lambda_i - j, \qquad \ell_\lambda(s) = \ell_\lambda(i,j) = \lambda'_j - i.
\tag{3.20}
$$

Using the Cauchy identity for the Macdonald polynomials $P_\lambda(x;q,t)$,

$$\sum_\lambda \frac{C_\lambda}{C'_\lambda} P_\lambda(x;q,t) P_\lambda(y;q,t) = \exp\left\{ \sum_{k=1}^{\infty} [\beta]_{q^k} \frac{1}{k} p_k \tilde{p}_k \right\}, \tag{3.21}$$

where

$$\frac{C_\lambda}{C'_\lambda} = \prod_{s\in\lambda} \frac{[a_\lambda(s)+\beta\ell_\lambda(s)+1]_q}{[a_\lambda(s)+\beta\ell_\lambda(s)+\beta]_q}, \tag{3.22}$$

we obtain a Λ-expansion of the q-deformed conformal block into a basis given by products of the Macdonald polynomials:

$$Z_S^{(q)} = \sum_{k=0}^{\infty} \Lambda^k \sum_{\substack{\vec{Y} \\ |\vec{Y}|=k}} \prod_{a=1}^{n} \frac{C_{Y_a}}{C'_{Y_a}} \left\langle \prod_{a=1}^{n} P_{Y_a}\left(-r_k^{(a)} - \frac{[v_{a+}]'_{q^k}}{[\beta]_{q^k}} \right) \right\rangle_{N+} \times$$

$$\times \left\langle \prod_{a=1}^{n} P_{Y_a}\left(\tilde{r}_k^{(a)} + \frac{[v_{a-}]'_{q^k}}{[\beta]_{q^k}} \right) \right\rangle_{\widetilde{N}-}. \tag{3.23}$$

According to AGT conjecture at q-lifted case, this q-deformed conformal block is identified with the five dimensional instanton partition function. In Section 2, we have seen that the deformed screening charges and vertex operators reduce to those of the superconformal model and its generalization. We will see that the ALE instanton partition function can be obtained from the five dimensional instanton partition function by taking the root of unity limit in next section. Therefore, once the dictionary is found, AGT conjecture at q-lifted case provides the equality between the block and ALE instanton partition function.

4. ALE ($\mathbb{R}^4/\mathbb{Z}_r$) instanton partition function

4.1. brief review of 4d $SU(n)$ instanton partition function on ALE

The instanton partition function of four dimensional $\mathcal{N}=2$ supersymmetric $SU(n)$ gauge theory on $\mathbb{R}^4$ (with Ω-deformation) can be calculated by the method of localization. The fixed points of the torus action $U(1)^2 \times U(1)^n$ is labeled by an n-tuple of Young diagrams $\vec{Y} = (Y_\alpha)_{\alpha=1,2,\cdots,n}$. For the fixed points corresponding to the k-instanton, $|\vec{Y}| = \sum_\alpha |Y_\alpha| = k$. Here we denote by $|Y_\alpha|$ the number of boxes carried by Y_α.

Suppose that the torus action is generated by $(\epsilon_1, \epsilon_2, a_1, \cdots, a_n)$. Then the weight of an individual box $(i,j) \in Y_\alpha$ of Young diagram is given by $a_\alpha + (i-1)\epsilon_1 + (j-1)\epsilon_2$. Here $\epsilon_1 = \sqrt{\beta} g_s$ and $\epsilon_2 = -g_s/\sqrt{\beta}$. Therefore,

$$g_s^2 = -\epsilon_1\epsilon_2, \qquad \beta = -\frac{\epsilon_1}{\epsilon_2}. \tag{4.1}$$

Let us consider the instanton on $\mathbb{R}^4/\mathbb{Z}_r$. The $\mathbb{Z}_r$ action is [45, 48]

$$\epsilon_1 \to \epsilon_1 - \frac{2\pi \mathrm{i}}{r}, \quad \epsilon_2 \to \epsilon_2 + \frac{2\pi \mathrm{i}}{r}, \quad a_\alpha \to a_\alpha + q_\alpha \frac{2\pi \mathrm{i}}{r}. \tag{4.2}$$

From the weight associated with the torus action, we see that $\mathbb{Z}_r$-charges are assigned to each Young diagram and its boxes. The charge of each box (i,j) of the Young diagram Y_α is

$$q_{\alpha,(i,j)} = q_\alpha - (i-1) + (j-1) \pmod r, \qquad \alpha = 1,2,\cdots,n, \tag{4.3}$$

and q_α is regarded as the $\mathbb{Z}_r$-charge of each Young diagram Y_α.

4.2. more on the labeling of ALE instantons
Let us consider the case where the 1st Chern class vanishes. This condition leads to

$$n_\ell - 2k_\ell + k_{\ell+1} + k_{\ell-1} = 0. \tag{4.4}$$

Here n_ℓ is the number of Young diagram $\{Y_\alpha\}$ such that $\mathbb{Z}_r$-charge $q_\alpha = \ell$ and k_ℓ is the total number of the boxes such that the $\mathbb{Z}_r$-charge $q_{\alpha,(i,j)} = \ell$. Of course they satisfy $n = \sum_\ell n_\ell$ and $k = \sum_\ell k_\ell$. When (4.4) is satisfied, the 2nd Chern number is k/r and the ALE partition function is schematically written as

$$Z^{\mathbb{R}^4/\mathbb{Z}_r} = \sum_{k=0}^{\infty} (\Lambda')^{k/r} \sum_{|\vec{Y}|=k} \mathcal{A}_{\vec{Y}} \Big|_{\substack{\text{1st Chern}=0 \\ \mathbb{Z}_r\text{charge of each factor} = 0}} . \tag{4.5}$$

Here $\mathcal{A}_{\vec{Y}}$ represents the contribution of the fixed point $\vec{Y}$.

4.3. limiting procedure from 5d instanton partition function
The five dimensional instanton partition function is given by given by [46, 25]

$$Z^{\mathbb{R}^5} = \sum_{k=0}^{\infty} \widetilde{\Lambda}^k \sum_{|\vec{Y}|=k} \widetilde{\mathcal{A}}_{\vec{Y}}, \tag{4.6}$$

$$\widetilde{\mathcal{A}}_{\vec{Y}} = \frac{\prod_{\alpha=1}^n \prod_{k=1}^n f_{Y_\alpha}^{q+}(m_k + a_\alpha) f_{Y_\alpha}^{q-}(m_{k+n} + a_\alpha)}{\prod_{\alpha,\alpha'}^n g_{Y_\alpha Y_{\alpha'}}^q (a_\alpha - a_{\alpha'})}, \tag{4.7}$$

$$g_{YW}^q(x) = \prod_{(i,j)\in Y} [x + \beta \ell_Y(i,j) + a_W(i,j) + \beta]_q [-x - \beta\ell_Y(i,j) - a_W(i,j) - 1]_q, \tag{4.8}$$

$$f_Y^{q\pm}(x) = \prod_{(i,j)\in Y} [\pm x \mp \beta(i-1) \pm (j-1)]_q. \tag{4.9}$$

The parameter m_i $(i = 1, \cdots, 2n)$ is related to the five dimensional fundamental mass m_i^{5d} by

$$m_i = m_i^{5d} + \frac{1}{2}(1-\beta). \tag{4.10}$$

Eq. (4.2) can be read off as the shift in q as well as in t [49]

$$q = \mathrm{e}^{\epsilon_2} \to \omega\, q, \quad t = \mathrm{e}^{-\epsilon_1} \to \omega\, t, \quad q^{\frac{a_\alpha}{\epsilon_2}} = \mathrm{e}^{a_\alpha} \to \omega^{q_\alpha} q^{\frac{a_\alpha}{\epsilon_2}}, \qquad \omega = \mathrm{e}^{2\pi \mathrm{i}/r}. \tag{4.11}$$

Here we have rescaled $a_\alpha \to a_\alpha/\epsilon_2$. If we take the limit $q \to 1$ subsequently, we expect that this is equivalent to taking the $\mathbb{Z}_r$ orbifold projection on four dimensional space. On the other hand, this limit is equal to the root of unity limit of q and t and the five dimensional Nekrasov partition function reduces to that on $\mathbb{R}^4/\mathbb{Z}_r$. In what follows, let us realize this procedure as

$$q = \omega\, \mathrm{e}^{h\epsilon_2}, \quad t = \omega\, \mathrm{e}^{-h\epsilon_1}, \quad q^{\frac{a_\alpha}{\epsilon_2}} = \omega^{q_\alpha} \mathrm{e}^{h a_\alpha}, \quad q^{\frac{m_i}{\epsilon_2}} = \mathrm{e}^{h m_i}, \qquad h \to +0. \tag{4.12}$$

Here we have rescaled m_i by m_i/ϵ_2.

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4.4. obtaining ALE instanton partition function

We have established the method of obtaining the $SU(n)$ instanton partition function on $\mathbb{R}^4/\mathbb{Z}_r$ from the five dimensional partition function. The appropriate factors to remove the extra coefficients which appear in the limiting process have been also obtained. In the following, $\widetilde{\mathcal{A}}^{(q_\alpha)}_{\vec{Y}}$ represents that each Young diagram Y_α is assigned the $\mathbb{Z}_r$-charge q_α. The results are as follows:

- $SU(2)$ and $r=2$ case

$$Z^{\mathbb{R}^4/\mathbb{Z}_2}_{SU(2)} = \sum_{k=0}^{\infty} \left(\Lambda'\right)^k Z^{(2)}_k + \sum_{k=0}^{\infty} \left(\Lambda'\right)^{k+1/2} Z^{(2)}_{k+1/2}, \tag{4.13}$$

where

$$Z^{(2)}_{k+1/2} := \lim_{h\to 0} \frac{h^2}{2^2} \sum_{|A|+|B|=2k+1} \widetilde{\mathcal{A}}^{(1,1)}_{AB} \tag{4.14}$$

$$Z^{(2)}_{k} := \lim_{h\to 0} \sum_{|A|+|B|=2k} \widetilde{\mathcal{A}}^{(0,0)}_{AB}. \tag{4.15}$$

- $SU(2)$ and general r case

$$Z^{\mathbb{R}^4/\mathbb{Z}_r}_{SU(2)} = \sum_{q_a=0}^{\lfloor \frac{r}{2} \rfloor} \sum_{k=0}^{\infty} \left(\Lambda'\right)^{k+\frac{q_a(r-q_a)}{r}} Z^{(r)}_{k+\frac{q_a(r-q_a)}{r}}, \tag{4.16}$$

where

$$Z^{(r)}_{k+\frac{q_a(r-q_a)}{r}} := \lim_{h\to 0} \Xi_{q_a} \sum_{\substack{|A|+|B| \\ =rk+q_a(r-q_a)}} \widetilde{\mathcal{A}}^{(q_a,-q_a)}_{AB}. \tag{4.17}$$

Here

$$\Xi_0 = 1, \qquad \Xi_1 = h^2 \frac{1}{(1-\omega)(1-\omega^{-1})}, \tag{4.18}$$

$$\Xi_i = h^{2i} \frac{\prod_{k=1}^{i-1}(1-\omega^k)^{2i-3k}(1-\omega^{-k})^{2i-3k}}{\prod_{l=1}^{i}(1-\omega^{i+l-1})(1-\omega^{-(i+l-1)})}, \qquad 2 \le i \le \left\lfloor \frac{r}{2} \right\rfloor. \tag{4.19}$$

- $SU(n)$ and general r case

$$Z^{\mathbb{R}^4/\mathbb{Z}_r}_{SU(n)} = \sum_{q_\alpha} \sum_{k=0}^{\infty} \left(\Lambda'\right)^{k+\frac{d}{r}} Z^{(r)}_{k+\frac{d}{r}}, \tag{4.20}$$

where

$$Z^{(r)}_{k+\frac{d}{r}} := \lim_{h\to 0} \Xi_{q_\alpha} \sum_{\substack{|\vec{Y}| \\ =rk+d}} \widetilde{\mathcal{A}}^{(q_\alpha)}_{\vec{Y}}. \tag{4.21}$$

Here

$$\Xi_{q_\alpha} = h^{2\sum_{\alpha=1}^{n-1} \alpha q_\alpha} \xi_{q_\alpha}(\omega)\xi_{q_\alpha}(\omega^{-1}), \tag{4.22}$$

$$
\begin{aligned}
\xi_{q_\alpha}(\omega) = &\prod_{a<b}^{n-1}\prod_{j=1}^{q_b-1} \frac{(1-\omega^j)^{2(q_b-j)}}{(1-\omega^{\sum_{i=a}^{b-1} q_i+j})^{q_b-j}(1-\omega^{\sum_{i=a}^{b-1} q_i+j})^{q_b-j}} \frac{1}{(1-\omega^{\sum_{i=a}^{b-1} q_i})^{q_b}} \\
&\times \prod_{a=1}^{n-1}\prod_{j=1}^{q_a-1} \frac{(1-\omega^j)^{q_a-j}(1-\omega^{\sum_{i=a+1}^{n-1} q_i+j})^{q_a-j}}{(1-\omega^{\sum_{i=1}^{a-1} q_i+j})^{j}(1-\omega^{\sum_{i=1}^{n-1} q_i+j})^{q_a-j}} \frac{1}{(1-\omega^{\sum_{i=1}^{a} q_i})^{q_a}} \\
&\times \prod_{a=1}^{n-1}\prod_{j=1}^{q_{a+1}+\cdots q_{n-1}} \frac{(1-\omega^j)^{q_a}}{(1-\omega^{\sum_{i=1}^{a} q_i+j})^{q_a}} \\
&\times \prod_{a+2\le b}^{n-1}\prod_{j=1}^{q_b-1} \frac{(1-\omega^j)^{j}(1-\omega^{\sum_{i=a+1}^{b-1} q_i+j})^{q_b-j}}{(1-\omega^{q_a-q_b+j})^{j}(1-\omega^{\sum_{i=k}^{l-1} q_i-j})^{j}} \prod_{j=0}^{q_{a+1}+\cdots+q_{b-1}-q_b} \frac{(1-\omega^{q_b+j})^{q_b}}{(1-\omega^{q_a+j})^{q_b}}. \quad (4.23)
\end{aligned}
$$

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