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Generating functions for tensor product decomposition

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Abstract. The paper deals with the tensor product decomposition problem. Tensor product decompositions are of great importance in the quantum physics. A short outline of the state of the art for the of semisimple Lie groups is mentioned. The generality of generating functions is used to solve tensor products. The corresponding generating function is rational. The feature of this technique lies in the fact that the decompositions of all tensor products of all irreducible representations are solved simultaneously. Obtaining the generating function is a difficult task in general. We propose some changes to an algorithm using Patera-Sharp character generators to find this generating function, which simplifies the whole problem to simple operations over rational functions.

1. Introduction

Generating functions have been usefully applied in many areas of mathematics and mathematical physics as powerful and general tools. In the representation theory of continuous and discrete groups, they provide techniques superior to other existing ones. The distinguishing feature is that they solve an infinity of problems of a given type at the same time.

One important application of generating functions is computation of restricted representations of a group. Restriction forms a representation of a subgroup from a representation of the whole group. Often, the restricted representation is simpler to understand. Rules for decomposing the restriction of an irreducible representation of the group into irreducible representations of the subgroup are of great importance in physics. For example, in the case of explicit symmetry breaking, the symmetry group of the problem is reduced from the whole group to one of its subgroups. In quantum mechanics, this reduction in symmetry appears as a splitting of degenerate states into multiplets, as in the Stark or Zeeman effect.

Branching rules for classical groups are well known. Generating functions have been used to decompose characters of semisimple Lie groups obtained by restriction from overgroups or by tensor products of irreducible representations. [6, 1] In the paper of A. Cohen and G. M. Ruitenburg [2] is proved that these generating functions are always rational, and an example of derivation of the generating function for restricted representations of $\boldsymbol{G}_2$ to $\boldsymbol{A}_2$ is provided. Nevertheless, the proposed algorithm, which is efficient to compute branching rules, turns out to be rather ponderous to decompose tensor products of representations.

We propose some change to a more suitable algorithm to compute tensor products decompositions. The algorithm is based on character generators [7]

$$X(A) = \sum_{\lambda\in\Lambda^+} \operatorname{ch} V(\lambda)\cdot A^{\lambda}, \tag{1}$$

the generating functions of characters of representations. The distinguishing feature of our algorithm lies in the fact that we do not use generating functions only to solve some recurrence, but we propose an operation over character generators to obtain a generating function for tensor products decompositions. This "*generatingfunctionologist*" [8] approach turns out to be simple and straightforward.

The paper is divided into three sections. In the first section, basics of semisimple Lie groups and their representations are summarized; general properties of tensor product decompositions are mentioned. In the second section, a generating function for tensor product decompositions is outlined. In the last section, the method to obtain the generating function is demonstrated on the example of $SU(3)$.

2. Tensor product decomposition

Let $\boldsymbol{G}$ be a semisimple connected complex Lie group with the Lie algebra $\mathfrak{g}$ of rank n. Let $\boldsymbol{T}\subset\boldsymbol{B}$ is a maximal torus of $\boldsymbol{G}$ with the Lie algebra $\mathfrak{t}$, where $\boldsymbol{B}$ is a fixed Borel subgroup of $\boldsymbol{G}$. Let $W = W_{\boldsymbol{G}}$ be the associated Weyl group. Let $\Lambda = \Lambda(\boldsymbol{T})$ denote the character group of $\boldsymbol{T}$, i.e., the group of all the algebraic group morphisms $\boldsymbol{T}\mapsto\mathbb{C}$. The Weyl group W acts on Λ. There is a straightforward identification between elements of Λ and weights $\lambda\in\mathfrak{t}^*$, which make a lattice in $\mathfrak{t}^*$. Therefore, we will identify Λ and the lattice of the weights in $\mathfrak{t}^*$. Let $R = R_{\mathfrak{g}}\subset\mathfrak{t}^*$ be the set of roots of $\mathfrak{g}$ with respect to the Cartan subalgebra $\mathfrak{t}$, and let R^+ be the set of positive roots. Let $\Delta = \{\alpha_1,\ldots,\alpha_n\}\subset R^+$ be the set of simple roots, $\{\alpha_1^\vee,\ldots,\alpha_n^\vee\}\subset\mathfrak{t}$ the corresponding simple coroots. For any $1\le j\le n$, the fundamental weight $\omega_j\in\Lambda$ is defined by

$$\omega_j(\alpha_i^\vee) = \delta_{j,i},\ \forall\, 1\le i\le n. \tag{2}$$

An element $\lambda\in\Lambda$ is called dominant if $\lambda(\alpha_i^\vee)\ge 0$ for all the simple coroots $\alpha_i^\vee$. Let Λ^+ denote the set of all the dominant characters. There is an isomorphism between the set of isomorphism classes of irreducible finite-dimensional representations of $\boldsymbol{G}$ and Λ^+ via the highest weight of the irreducible representation. For $\lambda\in\Lambda^+$, we denote by $V(\lambda)$ the corresponding irreducible representation of the highest weight λ. The W-orbit of any $\lambda\in\Lambda$ contains a unique element in Λ^+. We also have the shifted action of W on Λ via $w*\lambda = w(\lambda+\rho)-\rho$, where ρ is the half sum of positive roots.

Let $\mathbb{Z}[\Lambda(\boldsymbol{G})] = \{e^{\lambda}\,|\,\lambda\in\Lambda(\boldsymbol{G})\}$ be a ring of formal exponentials and their linear combinations with coefficients in the ring $\mathbb{Z}$ with the multiplication rule $e^{\lambda}\cdot e^{\mu} = e^{\lambda+\mu}$, for a group $\boldsymbol{G}$.

An algebraic character $\operatorname{ch} V(\lambda)$, $\lambda\in\Lambda^+$, of a compact Lie group $\boldsymbol{G}$ can be obtained by the Weyl character formula

$$\operatorname{ch} V(\lambda) = \frac{\sum_{w\in W}\operatorname{sgn}(w)\cdot e^{w(\lambda+\rho)}}{\sum_{w\in W}\operatorname{sgn}(w)\cdot e^{w(\rho)}}, \tag{3}$$

where the elements e^{λ} are from the ring $\mathbb{Z}[\Lambda(\boldsymbol{G})]$.

For a semisimple connected complex Lie group $\boldsymbol{G}$ with the Lie algebra $\mathfrak{g}$, the irreducible finite-dimensional representations of $\boldsymbol{G}$ are parametrized by the set Λ^+ of dominant characters of $\boldsymbol{T}$, where $\boldsymbol{T}$ is a maximal torus of $\boldsymbol{G}$ with the Lie algebra $\mathfrak{t}$. By the complete reducibility, for any $\lambda,\mu\in\Lambda^+$, the tensor product $V(\lambda)\otimes V(\mu)$ decomposes as

$$V(\lambda)\otimes V(\mu) = \bigoplus_{\nu\in\Lambda^+} m_{\lambda,\mu}^{\nu} V(\nu), \tag{4}$$

where $m^{\nu}_{\lambda,\mu}$ are the tensor product multiplicities, which denote the multiplicity of $V(\nu)$ in the tensor product $V(\lambda)\otimes V(\mu)$. We say that $V(\nu)$ occurs in $V(\lambda)\otimes V(\mu)$ if $m^{\nu}_{\lambda,\mu}>0$.

One of the major goals of the tensor product decomposition is to determine all the components of $V(\lambda)\otimes V(\mu)$ together with their multiplicities. In general, this is a very difficult task. [5]

3. Generating function for tensor products decomposition

3.1. Restricted representations

Let $\boldsymbol{G}$ be a semisimple connected complex Lie group and $\boldsymbol{H}$ its connected subgroup. Let $\boldsymbol{T}$ and $\boldsymbol{S}$ are maximal tori in $\boldsymbol{G}$ resp. $\boldsymbol{H}$, such that $\boldsymbol{S}\subset\boldsymbol{T}$. The straightforward algorithm to compute a decomposition $(V(\mu)|V(\lambda)\downarrow\boldsymbol{H})$ of the restricted representation $V(\lambda)$ of $\boldsymbol{G}$ into irreducible representations $V(\mu)$ of the subgroup $\boldsymbol{H}$ is to determine the set of all weights of the $\boldsymbol{G}$-module $V(\lambda)$, then to compute their restrictions to $\boldsymbol{S}$, and then to decompose them into $\boldsymbol{H}$-modules.

Let $r:\Lambda(\boldsymbol{G})\mapsto\Lambda(\boldsymbol{H})$ be a linear restriction of the weights of $\boldsymbol{T}$ to the weights of $\boldsymbol{S}$. By choosing appropriate Borel subgroups, we may assume that $r(\alpha)\notin R^{-}_{\boldsymbol{H}}$ for $\alpha\in R^{+}_{\boldsymbol{G}}$. Let $R_{\boldsymbol{0}}=\{\alpha\in R_{\boldsymbol{G}}\,|\,r(\alpha)=0\}$ and $R^{+}_{\boldsymbol{0}}=R_{\boldsymbol{0}}\cap R^{+}_{\boldsymbol{G}}$. Let $W_{\boldsymbol{0}}$ be the subgroup of $W_{\boldsymbol{G}}$ generated by $R_{\boldsymbol{0}}$. Let $W=W_{\boldsymbol{G}}/W_{\boldsymbol{0}}$; each coset of W has a unique representative in $W_{\boldsymbol{G}}$ of a minimal length.

Let $A=r(R^{+}_{\boldsymbol{G}}\backslash\{0\})$, let L be the lattice of non-negative integral linear combinations of A. The Kostant's partition function p_A on L is defined in the following way:[4] for $\beta\in L$, let $p_A(\beta)$ be the number of all different linear combinations of elements of A giving the element β, i.e., if $a=|A|$, then $p_A(\beta)=|\{(k_1,\dots,k_a)\,|\,\beta=\sum_{i=1}^{a}k_i\alpha_i,k_i\in\mathbb{Z}\}|$, where $|M|$ is the cardinality of M.

For $\lambda\in\Lambda^{+}(\boldsymbol{G})$ and $\mu\in\Lambda^{+}(\boldsymbol{H})$ holds Kostant's formula[3]

$$(V(\mu)|V(\lambda)\downarrow\boldsymbol{H})=\sum_{w\in W}\det(w)\cdot D_{\boldsymbol{H}}(w(\lambda+\rho_{\boldsymbol{G}}))\cdot p_A[r(w(\lambda+\rho_{\boldsymbol{G}}))-(\mu+r(\lambda+\rho_{\boldsymbol{G}}))],\quad(5)$$

where

$$D_{\boldsymbol{H}}(\nu)=\prod_{\alpha\in R^{+}_{\boldsymbol{H}}}\frac{(\nu,\alpha)}{(\rho_{\boldsymbol{H}},\alpha)},\quad\text{for }\nu\in\Lambda^{+}(\boldsymbol{H}),$$

is the Weyl dimensional polynomial. The corresponding generating function $P^{\boldsymbol{G}}_{\boldsymbol{H}}$ for restricted representations is obtained[2] by multiplying (5) by A^{λ} from the ring $\mathbb{Z}[\Lambda^{+}(\boldsymbol{G})]$, $X^{\mu}\in\mathbb{Z}[\Lambda^{+}(\boldsymbol{H})]$ and by summing over all $\lambda\in\Lambda^{+}(\boldsymbol{G})$, $\mu\in\Lambda^{+}(\boldsymbol{H})$.

$$P^{\boldsymbol{G}}_{\boldsymbol{H}}(A,X)=\sum_{\lambda\in\Lambda^{+}(\boldsymbol{G})}\sum_{\mu\in\Lambda^{+}(\boldsymbol{H})}(V(\mu)|V(\lambda)\downarrow\boldsymbol{H})A^{\lambda}X^{\mu}.\quad(6)$$

3.2. Tensor product of representations as restricted representation

To compute the decomposition of a tensor product of two irreducible representations $V(\lambda)\otimes V(\mu)$ of a Lie group $\boldsymbol{H}$ it is possible to consider representations of the group $\boldsymbol{H}$ as restricted representations of the diagonally embedded subgroup isomorphic to $\boldsymbol{H}$ with respect to the overgroup $\boldsymbol{G}=\boldsymbol{H}\times\boldsymbol{H}$. We can identify $\Lambda^{+}(\boldsymbol{G})$ with $\Lambda^{+}(\boldsymbol{H})\times\Lambda^{+}(\boldsymbol{H})$. The tensor product multiplicities for the Lie group $\boldsymbol{H}$ then fulfill the equality $m^{\nu}_{\lambda,\mu}=(V(\nu)|\,V(\lambda)\otimes V(\mu)\downarrow\boldsymbol{H})$, where $\mu,\nu,\lambda\in\Lambda^{+}(\boldsymbol{H})$.

Let $\mathfrak{g}$ and $\mathfrak{h}$ are the Lie algebras of $\boldsymbol{G}$ resp. $\boldsymbol{H}$, let $\mathfrak{s}$ resp. $\mathfrak{t}$ are the corresponding Cartan subalgebras. We construct the linear restriction r in the following way. Let $\pi:\mathfrak{h}\mapsto\mathfrak{g}$ be the projection of $\mathfrak{h}$ into $\mathfrak{g}$, which is identical on $\mathfrak{h}$. It follows that π is identical on $\mathfrak{t}$ as the projection into $\mathfrak{s}$. The restriction $r:\mathfrak{s}^{*}\mapsto\mathfrak{t}^{*}$ is defined by $r(\xi)=\xi\circ\pi$ for $\xi\in\mathfrak{s}^{*}$. It holds that $r:\Lambda(\boldsymbol{G})\mapsto\Lambda(\boldsymbol{H})$.

Following the equation (6), the generating function $P_{\boldsymbol{H}}^{\boldsymbol{H}\times\boldsymbol{H}}$ can be obtained by multiplying the multiplicities $m_{\lambda,\mu}^{\nu}$ by $A^{\lambda}, B^{\mu}, X^{\nu} \in \mathbb{Z}[\Lambda(\boldsymbol{H})]$ and by summing over all $\mu, \nu, \lambda \in \Lambda(\boldsymbol{H})$.

$$P_{\boldsymbol{H}}^{\boldsymbol{H}\times\boldsymbol{H}}(A,B,X) = \sum_{\lambda,\mu,\nu\in\Lambda^{+}(\boldsymbol{H})} (V(\nu)|V(\lambda)\otimes V(\mu)\downarrow \boldsymbol{H}) A^{\lambda}B^{\mu}X^{\nu}, \tag{7}$$

where the multiplicity $(V(\nu)|V(\lambda)\otimes V(\mu)$ is computed via Kostant's formula (5).

4. An algorithm for tensor products decomposition

The above mentioned algorithm[2] to obtain a generating function for tensor products decomposition, in fact, use an advantage of generating functions only partially. The essence lies in Kostant's formula (5), which is a recurrence relation in the weights of Λ^{+}. The authors use the generating functions method to solve this recurrence.

We use an algorithm which employs the Patera-Sharp character generators (1). The distinguishing feature of our algorithm lies in the fact that we do not use the generating function only to solve some recurrence, but we use an operation over character generators, which are generating functions themselves, to produce a new generating function for the tensor products decomposition.

Let $\boldsymbol{G}$ be a semisimple connected complex Lie group with the Lie algebra $\mathfrak{g}$. As mentioned above, the irreducible finite-dimensional representations of $\boldsymbol{G}$ are parametrized by the set Λ^{+} of the dominant characters of a maximal torus $\boldsymbol{T}$. Let $\lambda, \mu \in \Lambda^{+}$; by the complete reducibility theorem (4) and orthogonality relations, we have the following relation for the characters of the corresponding representations

$$\operatorname{ch} V(\lambda)\cdot \operatorname{ch} V(\mu) = \sum_{\nu\in\Lambda^{+}} m_{\lambda,\mu}^{\nu} \operatorname{ch} V(\nu). \tag{8}$$

The equation (8) is the key for the following considerations. The characters on the right hand side of (8) can be computed by Weyl character formula (3). Nevertheless, all the characters are completely determined by the highest weight $\nu \in \Lambda^{+}$ of the corresponding representation $V(\nu)$. Therefore, all the terms on the right hand side of (8) can be simply substituted by the expression e^{ν} without an information loss on the presence of the representation $V(\nu)$.

Using Weyl character formula (3), the expression e^{ν} can be obtained from the character $\operatorname{ch} V(\nu)$, easily. First of all, the character $\operatorname{ch} V(\nu)$ is to be multiplied by the denominator of the Weyl character $\sum_{w\in W} \operatorname{sgn}(w)e^{w(\rho)}$, then by $e^{-\rho}$, where ρ is the half sum of the positive roots of $\boldsymbol{H}$. At last, the annihilating operator $\mathfrak{D}$,

$$\mathfrak{D}\left(\sum_{i,j,\ldots,k=-\infty}^{\infty} a_{i,j,\ldots,k}x^{i}y^{j}\ldots z^{k}\right) = \sum_{i,j,\ldots,k=0}^{\infty} a_{i,j,\ldots,k}x^{i}y^{j}\ldots z^{k},$$

which annihilates all the terms with negative powers, is to be applied.

If the whole sequence of the steps is applied to equation (8), obtain

$$\mathfrak{D}\left[\left(e^{-\rho}\sum_{w\in W}\operatorname{sgn}(w)e^{w(\rho)}\right)\operatorname{ch} V(\lambda)\cdot \operatorname{ch} V(\mu)\right] = \sum_{\nu\in\Lambda^{+}} m_{\lambda,\mu}^{\nu} e^{\nu}. \tag{9}$$

If equation (9) is multiplied by $A^{\lambda}, B^{\mu} \in \mathbb{Z}[\Lambda(\boldsymbol{H})]$ and summed over all $\lambda, \mu \in \Lambda^{+}$, the generating function for the tensor product decomposition $P_{\boldsymbol{H}}^{\boldsymbol{H}\times\boldsymbol{H}}$ is obtained.

$$\mathfrak{D}\left[\left(e^{-\delta}\sum_{w\in W}\operatorname{sgn}(w)e^{w(\delta)}\right)X(A)\cdot X(B)\right] = \sum_{\lambda,\mu\in\Lambda^{+}}\sum_{\nu\in\Lambda^{+}} m_{\lambda,\mu}^{\nu} e^{\nu}A^{\lambda}B^{\mu}. \tag{10}$$

The object on the right hand side of (10) is the generating function $P_{\boldsymbol{H}}^{\boldsymbol{H}\times\boldsymbol{H}}$; on the left hand side appear the character generators $X(A)$ and $X(B)$.

As can be seen, the tensor products decomposition generating function can be obtained by simple calculations over the characters generators. Let us mention two examples for the groups $SU(2)$ and $SU(3)$, respectively.

4.1. SU(2) case

The weight lattice of the group $SU(2)$ is $\Lambda = \mathbb{Z}\omega$, where ω is the single fundamental weight. Then the character generator for $SU(2)$ is of the form

$$X(A,x) = \sum_{m=0}^{\infty} \operatorname{ch} V(m\omega) A^{m\omega} = \sum_{n=0}^{\infty} \frac{x^{m+1} - x^{-m-1}}{x - x^{-1}} A^m = \frac{1}{(1-Ax)(1-Ax^{-1})}, \tag{11}$$

where we denote $x = e^{\omega}$ and $A = A^{\omega}$, for the elements of the ring $\mathbb{Z}[\Lambda]$, for simplicity.

The generating function for tensor product decomposition $P_{SU(2)}^{SU(2)\times SU(2)}$ is obtained simple by multiplying $X(A,x)X(B,x)$ by $x^{-1}(x - x^{-1})$, which corresponds to $e^{-\rho}\sum_{w\in W}\operatorname{sgn}(w)e^{w(\rho)}$, and by applying the operator $\mathfrak{D}$.

$$P_{SU(2)}^{SU(2)\times SU(2)}(A,B,x) = \mathfrak{D}\left[x^{-1}(x-x^{-1})X(A,x)X(B,x)\right] = \frac{1}{(1-Ax)(1-AB)(1-Bx)}. \tag{12}$$

4.2. SU(3) case

The weight lattice of $SU(3)$ is $\Lambda = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$, where $\{\omega_1, \omega_2\}$ are the fundamental weights. The character generator is computed

$$X(A,B,x,y) = \sum_{m,n=0}^{\infty} \operatorname{ch} V(m\omega_1 + n\omega_2) A^{m\omega_1} B^{n\omega_2} = \\ \frac{x^2y^2(1-AB)}{(1-Ax)(1-By)(y-A)(x-B)(x-Ay)(y-Bx)}, \tag{13}$$

where again $x = e^{\omega_1}, y = e^{\omega_2}$ and $A = A^{\omega_1}, B = B^{\omega_2}$, for simplicity.

The generating function for tensor product decomposition $P_{SU(3)}^{SU(3)\times SU(3)}$ is obtained simple by multiplying $X(A,B,x,y)X(R,S,x,y)$ by the term

$$(x^{-1}y^{-1})\left(xy - x^{-1}y^2 - x^2y^{-1} - x^{-1}y^{-1} + x^{-2}y + xy^{-2}\right), \tag{14}$$

which corresponds to $e^{-\rho}\sum_{w\in W}\operatorname{sgn}(w)e^{w(\rho)}$, and by applying the operator $\mathfrak{D}$. We obtain

$$P_{SU(3)}^{SU(3)\times SU(3)}(A,B,R,S,x,y) = \\ \frac{1-ABRSxy}{(AS-1)(BR-1)(Ax-1)(By-1)(Rx-1)(Sy-1)(ARy-1)(BSx-1)}. \tag{15}$$

Let us describe this process in more detail how to obtain (15). Let us denote the product $X(A,B,x,y)X(R,S,x,y)$ multiplied by factor (14) by $g(x)$, i. e.

$$g(x) = \frac{xy(AB-1)(RS-1)\left(x^4y - x^3y^3 - x^3 + xy^4 + xy - y^3\right)}{d(x)},$$

where

$$d(x) = (Ax-1)(A-y)(B-x)(By-1)(Rx-1)(R-y)\times \\ (x-S)(Sy-1)(x-Ay)(Bx-y)(x-Ry)(Sx-y).$$

We now perform a partial fractions decomposition of g, obtaining

$$\begin{aligned} g(x) = &-\frac{A^2\left(ARSy^3-Ay^3-RSy+y\right)}{(A-R)(AS-1)(Ax-1)(By-1)(R-y)(Sy-1)(Ay-B)(ARy-1)(Ay-S)}- \\ &\frac{y\left(A^2RSy^2-A^2y^2-ARS+A\right)}{(A-R)(AS-1)(By-1)(R-y)(Sy-1)(Ay-B)(ARy-1)(Ay-S)(x-Ay)}- \\ &\frac{B^2RSy-B^2y-BRSy^3+By^3}{(A-y)(BR-1)(B-S)(x-B)(R-y)(Sy-1)(B-Ay)(B-Ry)(BS-y)}+ \\ &\frac{B^2y\left(BRS-B-RSy^2+y^2\right)}{(y-A)(BR-1)(B-S)(R-y)(Sy-1)(B-Ay)(B-Ry)(BS-y)(Bx-y)}+ \\ &\frac{R^2\left(ABRy^3-ABy-Ry^3+y\right)}{(A-R)(A-y)(BR-1)(By-1)(Rx-1)(Sy-1)(ARy-1)(Ry-B)(Ry-S)}+ \\ &\frac{y\left(ABR^2y^2-ABR-R^2y^2+R\right)}{(A-R)(A-y)(BR-1)(By-1)(Sy-1)(ARy-1)(Ry-B)(Ry-S)(x-Ry)}- \\ &\frac{ABS^2y-ABSy^3-S^2y+Sy^3}{(AS-1)(A-y)(B-S)(By-1)(y-R)(x-S)(S-Ay)(BS-y)(S-Ry)}- \\ &\frac{S^2y\left(ABS-ABy^2-S+y^2\right)}{(AS-1)(A-y)(B-S)(By-1)(y-R)(S-Ay)(BS-y)(S-Ry)(Sx-y)}. \end{aligned}$$

We now look at the terms which can be thrown away immediatelly. The first one, for example, cannot be thrown away because it contains the term $(Ax-1)$ in the denominator and this leads to positive powers of x. The second one can be thrown away, because the term $(x-Ay)$ can lead to negative powers of x. Similar way we examine all terms, obtaining the result consisting of four terms only, denoted by $\tilde{g}(y)$:

$$\begin{aligned} \tilde{g}(y) = &-\frac{A^2\left(ARSy^3-Ay^3-RSy+y\right)}{(A-R)(AS-1)(Ax-1)(By-1)(R-y)(Sy-1)(Ay-B)(ARy-1)(Ay-S)}+ \\ &\frac{B^2y\left(BRS-B-RSy^2+y^2\right)}{(y-A)(BR-1)(B-S)(R-y)(Sy-1)(B-Ay)(B-Ry)(BS-y)(Bx-y)}+ \\ &\frac{R^2\left(ABRy^3-ABy-Ry^3+y\right)}{(A-R)(A-y)(BR-1)(By-1)(Rx-1)(Sy-1)(ARy-1)(Ry-B)(Ry-S)}- \\ &\frac{S^2y\left(ABS-ABy^2-S+y^2\right)}{(AS-1)(A-y)(B-S)(By-1)(y-R)(S-Ay)(BS-y)(S-Ry)(Sx-y)}. \end{aligned}$$

Now comes the decomposition to the partial fractions concerning the variable y. Generally, such decomposition is possible via solving system of linear equations. We obtain the result consisting of 18 terms:

$$
\begin{aligned}
\tilde{g}(y) =& \frac{R^2A^3}{(A-R)(AR-B)(BR-1)(AR-S)(AS-1)(Ax-1)(ARy-1)}+\\
&\frac{RA^2}{(A-R)(AR-B)(BR-1)(AR-S)(AS-1)(Ax-1)(y-R)}-\\
&\frac{R^3A^2}{(A-R)(AR-B)(BR-1)(AR-S)(AS-1)(Rx-1)(ARy-1)}+\\
&\frac{S^2A^2}{(A-R)(AR-S)(S-B)(AS-1)(A-BS)(Ax-1)(Sy-1)}-\\
&\frac{R^2A}{(A-R)(AR-B)(BR-1)(AR-S)(AS-1)(Rx-1)(y-A)}-\\
&\frac{S^2A}{(A-R)(B-S)(AR-S)(AS-1)(A-BS)(A-Sx)(y-A)}+\\
&\frac{AB^2RS-AB^2}{(A-R)(AR-B)(BR-1)(B-S)(AS-1)(A-BS)(A-Bx)(y-A)}-\\
&\frac{B^2R}{(A-R)(AR-B)(BR-1)(B-S)(R-BS)(R-Bx)(y-R)}+\\
&\frac{ABRS^2-RS^2}{(A-R)(BR-1)(B-S)(AR-S)(AS-1)(R-BS)(R-Sx)(y-R)}-\\
&\frac{BS}{(BR-1)(B-S)(AS-1)(BS-A)(BS-R)(B-x)(y-BS)}+\\
&\frac{BS}{(BR-1)(B-S)(AS-1)(BS-A)(BS-R)(S-x)(y-BS)}-\\
&\frac{B(RS-1)\left(Bx^3-x\right)}{(BR-1)(B-S)(S-x)(Ax-1)(Bx-A)(Bx-R)(Rx-1)(BSx-1)(y-Bx)}+\\
&\frac{(AB-1)S\left(Sx^3-x\right)}{(B-S)(AS-1)(B-x)(Ax-1)(Rx-1)(Sx-A)(Sx-R)(BSx-1)(y-Sx)}+\\
&\frac{B^2\left(A^2RS-A^2\right)}{(A-R)(AR-B)(BR-1)(B-S)(AS-1)(A-BS)(Ax-1)(By-1)}-\\
&\frac{B^2R^2}{(A-R)(AR-B)(BR-1)(B-S)(R-BS)(Rx-1)(By-1)}+\\
&\frac{B^3S^2}{(BR-1)(B-S)(AS-1)(BS-A)(BS-R)(BSx-1)(By-1)}+\\
&\frac{\left(ABR^2-R^2\right)S^2}{(A-R)(BR-1)(B-S)(AR-S)(AS-1)(R-BS)(Rx-1)(Sy-1)}-\\
&\frac{B^2S^3}{(BR-1)(B-S)(AS-1)(BS-A)(BS-R)(BSx-1)(Sy-1)}.
\end{aligned}
$$

We apply the same procedure to the above sum. The first term cannot be thrown away because of the $(ARy-1)$ in the denominator. The second one is thrown because of $(y-R)$, etc. Finally, we get sum of eight terms, namely

$$\begin{aligned}
g_1 =& \frac{A^3R^2}{(A-R)(AS-1)(Ax-1)(BR-1)(AR-B)(AR-S)(ARy-1)} + \\
& \frac{B^2\left(A^2RS-A^2\right)}{(A-R)(AS-1)(Ax-1)(BR-1)(B-S)(By-1)(AR-B)(A-BS)} - \\
& \frac{A^2R^3}{(A-R)(AS-1)(BR-1)(Rx-1)(AR-B)(AR-S)(ARy-1)} + \\
& \frac{A^2S^2}{(A-R)(AS-1)(Ax-1)(S-B)(Sy-1)(A-BS)(AR-S)} + \\
& \frac{B^3S^2}{(AS-1)(BR-1)(B-S)(By-1)(BS-A)(BS-R)(BSx-1)} - \\
& \frac{B^2R^2}{(A-R)(BR-1)(B-S)(By-1)(Rx-1)(AR-B)(R-BS)} - \\
& \frac{B^2S^3}{(AS-1)(BR-1)(B-S)(Sy-1)(BS-A)(BS-R)(BSx-1)} + \\
& \frac{S^2\left(ABR^2-R^2\right)}{(A-R)(AS-1)(BR-1)(B-S)(Rx-1)(Sy-1)(AR-S)(R-BS)}.
\end{aligned}$$

Summing these terms up, we get the desired result (15).

5. Conclusions

The feature of the obtained generating functions is the generality, i.e., they solve all the tensor products decomposition problems for a given group at the same time. In other words, “they are equivalent to the table of all tensor product decompositions”. [7]

For example, the generating function $P^{SU(3)\times SU(3)}_{SU(3)}$ for the group $SU(3)$ computed in subsection 4.2 has the following meaning. The symbols A, B resp. R, S correspond to the first resp. second factorrepresentation. A term A^aB^b corresponds to the representation $V(a\omega_1+b\omega_2)$, a term R^rS^s corresponds to the representation $V(r\omega_1+s\omega_2)$, a term x^my^n corresponds to the representation $V(m\omega_1+n\omega_2)$. Then, the presence of a term $A^aB^bR^rS^sx^my^n$ in the expansion of $G(A, B, R, S, x, y)$ into a series implies that the tensor product of the representations $V(a\omega_1+b\omega_2)$ and $V(r\omega_1+s\omega_2)$ contains the representation $V(m, n)$, i.e.,

$$V(a\omega_1+b\omega_2)\otimes V(r\omega_1+s\omega_2)\supset V(m\omega_1+n\omega_2).$$

Using the Patera-Sharp character generators, we can simply obtain the generating functions for the tensor products decomposition for all semisimple Lie groups. These character generators are always rational functions.[2] Therefore, the algorithm used in the section 4 involves only operations over rational functions. The advantage of the algorithm against other methods is in its simultaneous simplicity and generality.

An unsolved task, which should be studied in the future, is the action of the annihilating operator $\mathfrak{D}$. Generally, it is not obvious, how it acts on a general rational function. This should be the direction of the following investigations.

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