XXIst International Conference on Integrable Systems and Quantum Symmetries (ISQS21) IOP Publishing
Journal of Physics: Conference Series **474** (2013) 012015 doi:10.1088/1742-6596/474/1/012015

Integrable Hamiltonian systems generated by antisymmetric matrices

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Abstract. We construct a family of integrable systems generated by the Casimir functions of Lie algebra of skew-symmetric matrices, where the Lie bracket is deformed by a symmetric matrix.

1. Introduction

Let $\mathcal{A}(n)$ be the vector space of antisymmetric $n \times n$ matrices and $Sym(n)$ be the vector space of symmetric $n \times n$ matrices. The $(\mathcal{A}(n), [\cdot,\cdot]_S)$ is a Lie algebra with the S–bracket defined by a deformeded commutator

$$[X,Y]_S = XSY - YSX \tag{1}$$

for fixed $S \in Sym(n)$ and $X, Y \in \mathcal{A}(n)$, (see [4, 5]).

In this paper we construct the family of integrable systems — a hierarchy generated by the Casimir functions on the dual of Lie algebra $\mathcal{A}(n)$. We prove that the integrals of this family of Hamiltonian systems are in involution.

The idea of considering these systems comes from [1]. In this paper we present more general case, which reduces to the case considering in [1] if we put that the matrix $S = \mathbf{1}$. Also in [3] the authors studied similar systems in the complex setting and for matrices with a different internal structure.

2. Hierarchy generated by Casimir functions

We identify $\mathcal{A}(n)$ with its dual $\mathcal{A}^*(n) \cong \mathcal{A}(n)$ using natural non-degenerate pairing by trace of the product

$$\langle X, \rho \rangle = \mathrm{Tr}(\rho X), \quad \rho \in \mathcal{A}^*(n), \quad X \in \mathcal{A}(n). \tag{2}$$

We shall write a general element $\mathcal{A}(n)$ as

$$X = \begin{pmatrix} A & B \\ -B^\top & C \end{pmatrix}, \tag{3}$$

where $A \in \mathcal{A}(2)$, $C \in \mathcal{A}(n-2)$ and $B \in \mathrm{Mat}_{2\times(n-2)}(\mathbb{R})$. Having Lie algebra $(\mathcal{A}(n), [\cdot,\cdot]_S)$ one defines the Lie-Poisson bracket on $C^\infty(\mathcal{A}(n))$ by

$$\{f,g\}_S = \mathrm{Tr}\left(X \left[\frac{\partial f}{\partial X}, \frac{\partial g}{\partial X} \right]_S \right), \quad f,g \in C^\infty(\mathcal{A}(n)), \tag{4}$$

where

$$\frac{\partial f}{\partial X} = \begin{pmatrix} \frac{\partial f}{\partial A} & \frac{\partial f}{\partial B} \\ -\frac{\partial f}{\partial B^\top} & \frac{\partial f}{\partial C} \end{pmatrix}, \tag{5}$$

In order to obtain the second Poisson bracket ("frozen" Poisson bracket) we fix the element $X_0 \in \mathcal{A}(n)$ and put

$$\{f,g\}_{FS} = \operatorname{Tr}\left(X_0 \left[\frac{\partial f}{\partial X}, \frac{\partial g}{\partial X}\right]_S\right), \quad f,g \in C^\infty(\mathcal{A}(n)). \tag{6}$$

It is a general fact that the Lie-Poisson bracket and frozen bracket are compatible in the sense that their linear combination

$$\alpha\{\cdot,\cdot\}_S + \beta\{\cdot,\cdot\}_{FS} \tag{7}$$

is also a Poisson bracket.

We shall choose

$$X_0 = \frac{1}{2}\begin{pmatrix} A_0 & 0 \\ 0 & 0 \end{pmatrix}, \quad S = \begin{pmatrix} S_1 & S_3 \\ S_3^\top & S_2 \end{pmatrix}, \tag{8}$$

where A_0 is 2×2 matrix defined by

$$A_0 := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{9}$$

After simple calculation we show that the Poisson bracket (6) can be written in the form

$$\begin{aligned}\{f,g\}_{FS} = \operatorname{Tr}&\left(\left(\frac{\partial g}{\partial A}A_0\frac{\partial f}{\partial B} - \frac{\partial f}{\partial A}A_0\frac{\partial g}{\partial B}\right)S_3^\top\right) + \\ &+ \operatorname{Tr}\left(\frac{\partial f}{\partial B^\top}A_0\frac{\partial g}{\partial B}S_2\right).\end{aligned} \tag{10}$$

Basic assumption. From now we put the block $\boldsymbol{S}_3$ equal to zero ($\boldsymbol{S}_3 \equiv 0$). After reducing to this case we obtain

$$\{f,g\}_{FS} = \operatorname{Tr}\left(\frac{\partial f}{\partial B^\top}A_0\frac{\partial g}{\partial B}S_2\right). \tag{11}$$

Thus, we can think of this bracket as being defined on $C^\infty(\mathrm{Mat}_{2\times(n-2)}(\mathbb{R}))$ (thus $\mathcal{A}(n) \to \mathrm{Mat}_{2\times(n-2)}(\mathbb{R})$ is injective smooth Poisson map).

In the case when $\det S \neq 0$ the Casimir functions for the Lie–Poisson bracket (4) are given by

$$C_k(X) = \frac{1}{2k}\operatorname{Tr}(XS^{-1})^{2k}, \quad k = 1,2,\dots \tag{12}$$

see [5]. For the degenerate case when $S_1 \equiv 0$ we know only some Casimir functions of the following form

$$C_k(X) = \frac{1}{k}\operatorname{Tr}\left(B^\top B S_2^{-1}\right)^k, \quad k = 1,2,\dots, \tag{13}$$

(see [2] for the case $S_2 = 1$). In this case the Lie-Poisson bracket (4) can be rewritten in the form

$$\begin{aligned}\{f,g\}_S =& 2\operatorname{Tr}\left(\frac{\partial f}{\partial B^\top}A\frac{\partial g}{\partial B}S_2 + \frac{\partial g}{\partial C}C\frac{\partial f}{\partial C}S_2\right) + \\ &+ 2\operatorname{Tr}\left(\left(\frac{\partial f}{\partial B^\top}B\frac{\partial g}{\partial C} - \frac{\partial g}{\partial B^\top}B\frac{\partial f}{\partial C}\right)S_2\right).\end{aligned} \tag{14}$$

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Now, we show that the functions given by (13) are Casimir functions for the bracket (14). Since the derivative of C_k is

$$\frac{\partial C_k}{\partial B} = 2BS_2^{-1}\left(B^\top BS_2^{-1}\right)^{k-1}, \tag{15}$$

$$\frac{\partial C_k}{\partial B^\top} = 2S_2^{-1}\left(B^\top BS_2^{-1}\right)^{k-1}B^\top, \tag{16}$$

$$\frac{\partial C_k}{\partial C} = 0, \tag{17}$$

we have

$$\begin{aligned}\{C_k, C_l\}_S = \mathrm{Tr}\left(\frac{\partial C_k}{\partial B^\top}A\frac{\partial C_l}{\partial B}S_2\right) = \\ = 4\,\mathrm{Tr}\left(S_2^{-1}\left(B^\top BS_2^{-1}\right)^{k-1}B^\top ABS_2^{-1}\left(B^\top BS_2^{-1}\right)^{l-1}S_2\right) = \\ = 4\,\mathrm{Tr}\left(B^\top ABS_2^{-1}\left(B^\top BS_2^{-1}\right)^{k+l-2}\right) = 0 = \{C_k, C_l\}_{FS},\end{aligned} \tag{18}$$

because the matrix $B^\top AB$ is antisymmetric and $S_2^{-1}\left(B^\top BS_2^{-1}\right)^{k+l-2}$ is symmetric. Moreover we have the following proposition.

Proposition 1 *The Casimir functions C_k defined by (12) or (13) for the Lie-Poisson bracket (4) considered as functions of B are in involution with respect to the frozen bracket (11).*

Proof 1 *Since the derivative of C_k given by (12) is*

$$\frac{\partial C_k}{\partial B} = -2P_+\left(XS^{-1}\right)^{2k-1}P_-, \tag{19}$$

$$\frac{\partial C_k}{\partial B^\top} = 2P_-\left(S^{-1}X\right)^{2k-1}P_+, \tag{20}$$

where P_+, P_- are the orthogonal projectors given, in block matrix notation, by

$$P_+ = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & 0 \end{pmatrix}, \quad P_- = \begin{pmatrix} 0 & 0 \\ 0 & \mathbf{1} \end{pmatrix}. \tag{21}$$

After a direct calculation we obtain

$$\begin{aligned}\{C_k, C_l\}_{FS} = \mathrm{Tr}\left(\frac{\partial C_k}{\partial B^\top}A\frac{\partial C_l}{\partial B}S_2\right) = \\ = -4\,\mathrm{Tr}\left(P_-\left(S^{-1}X\right)^{2k-1}P_+A_0P_+\left(XS^{-1}\right)^{2l-1}P_-S_2\right) = \\ = -4\,\mathrm{Tr}\left(P_-\left(XS^{-1}\right)^{2k-2}XP_+A_0P_+\left(XS^{-1}\right)^{2l-1}P_-\right) = \\ = -4\,\mathrm{Tr}\left(\left(XS^{-1}\right)^{2k-2}XP_+A_0P_+\left(XS^{-1}\right)^{2l-1}\right) + \\ + 4\,\mathrm{Tr}\left(P_+\left(XS^{-1}\right)^{2k-2}XP_+A_0P_+\left(XS^{-1}\right)^{2l-1}P_+\right) = \\ = -4\,\mathrm{Tr}\left(\left(XS^{-1}\right)^{k+l-2}XP_+A_0P_+X\left(S^{-1}X\right)^{k+l-2}S^{-1}\right) = 0.\end{aligned} \tag{22}$$

Above vanishes because in the first term we have a product of three antisymmetric 2×2 matrices which is also antisymmetric and in the second term we have a product of an antisymmetric matrix $\left(XS^{-1}\right)^{k+l-2}XP_+A_0P_+X\left(S^{-1}X\right)^{k+l-2}$ and symmetric matrix S^{-1}.

The proof of the involution of the functions (13) with respect to the frozen Poisson bracket (11), was given before this proposition.

Proposition 2 *The smooth functions $\delta_k : \mathrm{Mat}_{2\times(n-2)}(\mathbb{R}) \to \mathbb{R}$ defined by*

$$\delta_k(B) = \mathrm{Tr}\left(BS_2^{-1}\left(CS_2^{-1}\right)^{2k-1}B^\top A_0\right) \tag{23}$$

are in involution with respect to the frozen Poisson bracket (11)

$$\{\delta_k, \delta_l\}_{FS} = 0. \tag{24}$$

Proof 2 *Since the derivative of δ_k is*

$$\frac{\partial \delta_k}{\partial B} = 2A_0BS_2^{-1}\left(CS_2^{-1}\right)^{2k-1}, \tag{25}$$

$$\frac{\partial \delta_k}{\partial B^\top} = 2S_2^{-1}\left(CS_2^{-1}\right)^{2k-1}B^\top A_0, \tag{26}$$

we have

$$\begin{aligned}\{\delta_k, \delta_l\}_{FS} &= \mathrm{Tr}\left(\frac{\partial \delta_k}{\partial B^\top}A_0\frac{\partial \delta_l}{\partial B}S_2\right) = \\ &= 4\,\mathrm{Tr}\left(S_2^{-1}\left(CS_2^{-1}\right)^{2k-1}B^\top A_0A_0A_0BS_2^{-1}\left(CS_2^{-1}\right)^{2l-1}S_2\right) = \\ &= -4\,\mathrm{Tr}\left(B^\top A_0BS_2^{-1}\left(CS_2^{-1}\right)^{2(k+l)-2}\right) = 0,\end{aligned} \tag{27}$$

because the matrix $B^\top A_0B$ is antisymmetric and $S_2^{-1}\left(CS_2^{-1}\right)^{2(k+l)-2}$ is symmetric.

Proposition 3 *Assume that $S_1 = 1$. Then the functions δ_k and C_l (given by (12)), $k, l = 1, 2, \dots$, are in involution with respect to the frozen Poisson bracket (11)*

$$\{\delta_k, C_l\}_{FS} = 0. \tag{28}$$

Proof 3 *First, we show that δ_1 commutes with C_k given by (12)*

$$\begin{aligned}\{\delta_1, C_k\}_{FS} &= \mathrm{Tr}\left(\frac{\partial \delta_1}{\partial B^\top}A_0\frac{\partial C_k}{\partial B}\right) = \\ &= -4\,\mathrm{Tr}\left(S_2^{-1}CS_2^{-1}B^\top A_0A_0P_+\left(XS^{-1}\right)^{2k-1}P_-S_2\right) = \\ &= 4\,\mathrm{Tr}\left(CS_2^{-1}B^\top P_+\left(XS^{-1}\right)^{2k-1}P_-\right) = \\ &= -4\,\mathrm{Tr}\left(P_-\left(XS^{-1}\right)P_-XP_+\left(XS^{-1}\right)^{2k-1}P_-\right) = \\ &= -4\,\mathrm{Tr}\left(\left(XS^{-1}\right)P_-XP_+\left(XS^{-1}\right)^{2k-1}\right) + \\ &+ 4\,\mathrm{Tr}\left(P_+\left(XS^{-1}\right)P_-XP_+\left(XS^{-1}\right)^{2k-1}P_+\right) = \\ &= -4\,\mathrm{Tr}\left(P_-XP_+\left(XS^{-1}\right)^{2k}\right) + \\ &- 4\,\mathrm{Tr}\left(P_-XP_+\left(XS^{-1}\right)^{2k-1}\left(P_-XP_+\right)^\top S^{-1}\right) = \\ &= -4\,\mathrm{Tr}\left(XP_+\left(XS^{-1}\right)^{2k}\right) + \\ &+ 4\,\mathrm{Tr}\left(P_+XP_+\left(XS^{-1}\right)^{2k}P_+\right) = \\ &= -4\,\mathrm{Tr}\left(P_+\left(XS^{-1}\right)^{2k}XP_+\right) = 0,\end{aligned} \tag{29}$$

because $P_+\left(XS^{-1}\right)^{2k}XP_+$ is antisymmetric, P_+XP_+ is antisymmetric and $P_+\left(XS^{-1}\right)^{2k}P_+$ is symmetric. Second, the functions δ_k and C_l satisfy the following recursion formula

$$\{\delta_k, C_l\}_{FS} = \{\delta_{k-1}, C_{l+1}\}_{FS}. \tag{30}$$

Thus the relation (28) is valid for any k.

Proposition 4 *The functions δ_k and C_l (given by (13)), $k, l = 1, 2, \ldots$, are in involution with respect to the frozen Poisson bracket (11)*

$$\{\delta_k, C_l\}_{FS} = 0. \tag{31}$$

Proof 4 *For the functions C_k given by (13) we have*

$$\begin{aligned}\{\delta_k, C_l\}_{FS} &= \mathrm{Tr}\left(\frac{\partial \delta_k}{\partial B^\top} A_0 \frac{\partial C_l}{\partial B}\right) = &(32)\\ &= 4\,\mathrm{Tr}\left(S_2^{-1}\left(CS_2^{-1}\right)^{2k-1} B^\top A_0 A_0 B S_2^{-1}\left(B^\top B S_2^{-1}\right)^{l-1} S_2\right) = &\\ &= -4\,\mathrm{Tr}\left(\left(CS_2^{-1}\right)^{2k} CS_2^{-1}\left(B^\top B S_2^{-1}\right)^{l}\right) = 0, &(33)\end{aligned}$$

because the matrix $\left(CS_2^{-1}\right)^{2k}C$ is antisymmetric and $S_2^{-1}\left(B^\top B S_2^{-1}\right)^{l}$ is symmetric.

We obtain a hierarchy of Hamilton's equations generated by Hamiltonians C_k given by (12) or (13) with respect the frozen Poisson bracket (11)

$$\frac{\partial B}{\partial t_k} = A_0 \frac{\partial C_k}{\partial B} S_2, \quad k = 1, 2, \ldots \tag{34}$$

Example 1 *In this example we consider the case when X is 5×5-matrix which we denote*

$$X = \begin{pmatrix} 0 & a & p_1 & p_2 & p_3 \\ -a & 0 & q_1 & q_2 & q_3 \\ -p_1 & -q_1 & 0 & -c_3 & c_2 \\ -p_2 & -q_2 & c_3 & 0 & -c_1 \\ -p_3 & -q_3 & -c_2 & c_1 & 0 \end{pmatrix} \tag{35}$$

and matrix S is degenerate, that mean $S_1 = 0$ and

$$S_2 = \begin{pmatrix} e_1 & 0 & 0 \\ 0 & e_2 & 0 \\ 0 & 0 & e_3 \end{pmatrix}. \tag{36}$$

The frozen Poisson bracket in this case is

$$\begin{aligned}\{f, g\}_{FS}(p_i, q_i) =& e_1\left(\frac{\partial f}{\partial p_1}\frac{\partial g}{\partial q_1} - \frac{\partial f}{\partial q_1}\frac{\partial g}{\partial p_1}\right) + e_2\left(\frac{\partial f}{\partial p_2}\frac{\partial g}{\partial q_2} - \frac{\partial f}{\partial q_2}\frac{\partial g}{\partial p_2}\right) + \\ &+ e_3\left(\frac{\partial f}{\partial p_3}\frac{\partial g}{\partial q_3} - \frac{\partial f}{\partial q_2}\frac{\partial g}{\partial p_3}\right).\end{aligned} \tag{37}$$

The integrals in involution are

$$C_1 = \left(S_2^{-1}\vec{p}\right)\cdot\vec{p} + \left(S_2^{-1}\vec{q}\right)\cdot\vec{q}, \tag{38}$$

$$C_2 = \frac{1}{2}C_1^2 - \left(\left(S_2^{-1}\vec{q}\right)\times\left(S_2^{-1}\vec{p}\right)\right)\cdot\left(\vec{q}\times\vec{p}\right), \tag{39}$$

$$\delta_1 = -2\vec{C}\cdot\left(\left(S_2^{-1}\vec{q}\right)\times\left(S_2^{-1}\vec{p}\right)\right). \tag{40}$$

Hamilton's equations for the Hamiltonian C_1 *are*

$$\frac{\partial\vec{p}}{\partial t} = 2\vec{q}, \tag{41}$$

$$\frac{\partial\vec{q}}{\partial t} = -2\vec{p}. \tag{42}$$

Hamilton's equations for the Hamiltonian C_2 *are*

$$\frac{\partial\vec{p}}{\partial t} = 2\left(C_1\vec{q} + \left(S_2^{-1}\vec{p}\right)\times\left(\vec{p}\times\vec{q}\right)\right), \tag{43}$$

$$\frac{\partial\vec{q}}{\partial t} = 2\left(-C_1\vec{p} + \left(S_2^{-1}\vec{q}\right)\times\left(\vec{p}\times\vec{q}\right)\right). \tag{44}$$

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