

Interpolation via symmetric exponential functions

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Abstract. Complex valued functions on the Euclidean space \mathbb{R}^n , symmetric or antisymmetric with respect to the permutation group S_n , are often dealt with in various branches of physics, such as quantum theory or theory of integrable systems. One often needs to approximate such functions with series consisting of various special functions which satisfy nice properties. Questions of uniform convergence of such approximations are crucial for applications. In this article a family of special functions called the symmetric exponential functions are used for such approximation and the uniform convergence of their sums is considered.

1. Introduction. Symmetric exponential functions

The symmetric or antisymmetric functions with respect to the permutation group S_n appear very often, namely in quantum theory or theory of integrable systems. We often need to approximate such functions. In this article, we study approximations with so called symmetric exponential functions, which are defined as permanents of matrices whose entries are exponential functions of one variable. They are introduced in general for any integer n in [4]. Such approximations can be done in three different ways: expansions into the corresponding Fourier series, integral Fourier transforms and multivariate finite Fourier transforms. We deal with finite Fourier transforms and study the uniform convergence of the sum to the original function.

The symmetric and antisymmetric exponential functions [4, 3] are closely related to the symmetric [5], antisymmetric [6] and E -orbit functions [7] and to the corresponding orthogonal polynomials [1]. Through the inclusion of the alternating subgroup $Alt_n \subset S_n$, they are also related to alternating exponential functions [2, 8].

2. Two dimensional symmetric exponential functions

The two dimensional symmetric exponential functions $E_{(k,l)}^+ : \mathbb{R}^2 \rightarrow \mathbb{C}$, with $k, l \in \mathbb{Z}$ and $k \geq l$, are defined by the formula

$$E_{(k,l)}^+(x, y) = \begin{vmatrix} e^{2\pi i k x} & e^{2\pi i k y} \\ e^{2\pi i l x} & e^{2\pi i l y} \end{vmatrix}^+ = e^{2\pi i (kx+ly)} + e^{2\pi i (ky+lx)}.$$

Similar antisymmetric functions can be defined with the help of determinant instead of permanent. Similar construction is possible in each dimension. We consider dimension two for simplicity here.



From the explicit formula we see the symmetry of $E_{(k,l)}^+$ with respect to permutation of $x \leftrightarrow y$,

$$E_{(k,l)}^+(x, y) = E_{(k,l)}^+(y, x),$$

and also with respect to $k \leftrightarrow l$,

$$E_{(k,l)}^+(x, y) = E_{(l,k)}^+(x, y).$$

Therefore we can consider only functions with $k \geq l$.

One has also

$$E_{(k,l)}^+(x + r, y + s) = E_{(k,l)}^+(x, y) \text{ for all } r, s \in \mathbb{Z}.$$

Due to this periodicity and symmetry, the functions $E_{(k,l)}^+$ can be considered on the closure of the so called fundamental domain $F(S_2^{\text{aff}})$ only, where

$$F(S_2^{\text{aff}}) = \{(x, y) \in (0, 1) \times (0, 1) | x > y\}.$$

The functions $E_{(k,l)}^+$ are mutually orthogonal on $F(S_2^{\text{aff}})$, i. e.

$$\int_{F(S_2^{\text{aff}})} E_{(k,l)}^+(x, y) \overline{E_{(m,n)}^+(x, y)} dx dy = G_{kl} \delta_{km} \delta_{ln},$$

$$k, l, m, n \in \mathbb{Z}, \quad k \geq l, \quad m \geq n,$$

where G_{kl} is defined by

$$G_{kl} = \begin{cases} 2 & \text{if } k = l, \\ 1 & \text{otherwise.} \end{cases}$$

Every function $f : \mathbb{R}^2 \rightarrow \mathbb{C}$ which is symmetric and periodic, i. e.

$$f(x, y) = f(y, x), \quad f(x + r, y + s) = f(x, y), \quad x, y \in \mathbb{R}, \quad r, s \in \mathbb{Z},$$

can be under certain conditions expanded in fourier-like series

$$f(x, y) = \sum_{\substack{k, l \in \mathbb{Z} \\ k \geq l}} c_{kl} E_{(k,l)}^+(x, y),$$

where

$$c_{kl} = G_{kl}^{-1} \int_{F(S_2^{\text{aff}})} f(x, y) \overline{E_{(k,l)}^+(x, y)} dx dy.$$

Besides the continuous orthogonality, one has also discrete orthogonality of symmetric exponential functions over the grid of the form

$$L_N = \left\{ \left(\frac{m}{N}, \frac{n}{N} \right) \mid m, n = 0, \dots, N-1; m \geq n \right\}$$

which is given by

$$\sum_{\substack{m, n=0 \\ m \geq n}}^{N-1} G_{mn}^{-1} E_{(m,n)}^+ \left(\frac{m}{N}, \frac{n}{N} \right) \overline{E_{(k',l')}^+ \left(\frac{m}{N}, \frac{n}{N} \right)} = G_{kl} N^2 \delta_{kk'} \delta_{ll'},$$

where $k, l, k', l' \in \{0, \dots, N-1\}$, $k \geq l$, $k' \geq l'$.

Note that for $N = 2M + 1$ odd, discrete orthogonality formulas also hold for the set of functions where $k, l \in \{-M, \dots, M\}$, $k \geq l$.

Now when we have any function $f : L_N \rightarrow \mathbb{C}$ we can compute discrete-like-Fourier transform coefficients

$$\beta_{kl}^+ = \frac{1}{G_{kl}N^2} \sum_{\substack{m,n=0 \\ m \geq n}}^{N-1} G_{mn}^{-1} f\left(\frac{m}{N}, \frac{n}{N}\right) \overline{E_{k,l}^+\left(\frac{m}{N}, \frac{n}{N}\right)},$$

where $k, l = -M, \dots, M$, $k \geq l$. Then we can define the function

$$\psi_N(x, y) = \sum_{\substack{k,l=-M \\ k \geq l}}^M \beta_{kl}^+ E_{(k,l)}^+(x, y).$$

Immediately from orthogonality relation we obtain that this N th interpolation function coincides with f on the grid L_N :

$$\psi_N\left(\frac{m}{N}, \frac{n}{N}\right) = f\left(\frac{m}{N}, \frac{n}{N}\right).$$

Example of grid L_N is depicted on the following figure 1 for $N = 10$.

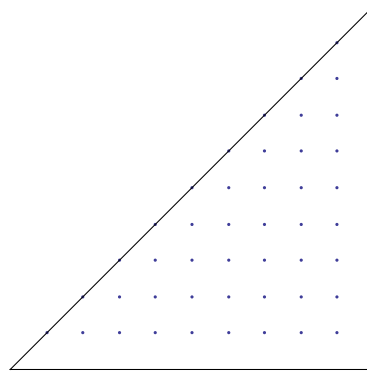


Figure 1. Grid L_{10}

We give a simple example of discrete interpolation. Let us define

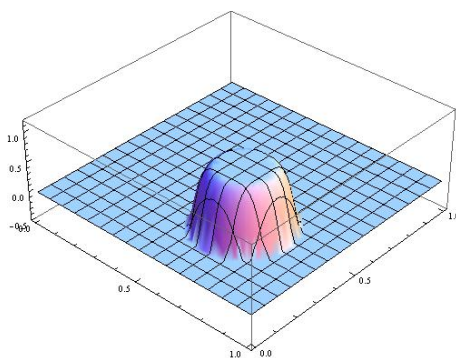
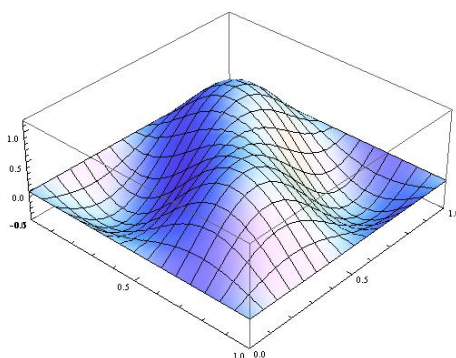
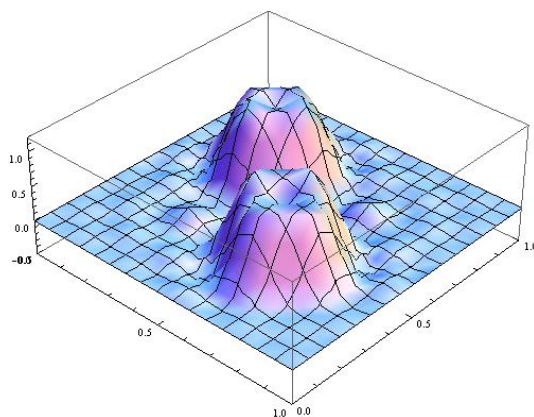
$$f(x, y) = \begin{cases} 0 & \text{if } (x - \frac{2}{3})^2 + (y - \frac{1}{3})^2 \geq \frac{1}{25}, \\ \exp\left(\frac{1}{100\left(\sqrt{(x-\frac{2}{3})^2 + (y-\frac{1}{3})^2} - \frac{1}{10}\right)^2 - 1} + 1\right) & \text{otherwise.} \end{cases}$$

The function f is shown on figure 2.

For $N = 3$ using described procedure we construct

$$\begin{aligned} \psi_3(x, y) = & -\frac{1}{18}(-1)^{5/6}(\sqrt{3} + i)(-1 + e^{2i\pi x}) \\ & (-1 + e^{2i\pi y})e^{-2i\pi(x+y)}(2e^{2i\pi(x+y)} + e^{2i\pi x} + e^{2i\pi y} + 2) \end{aligned}$$

which is shown on figure 3. To see how the sum converges, we show another two finer approximations. For $N = 23$ it is drawn on figure 4, for $N = 55$ we get even better approximation, which is shown on figure 5. Approximation errors of the form $\int_{F(S_2^{\text{aff}})} |f - \psi_N|^2$ of computed functions fall rapidly with increasing N .

**Figure 2.** Function f **Figure 3.** Function ψ_7 **Figure 4.** Function ψ_{23}

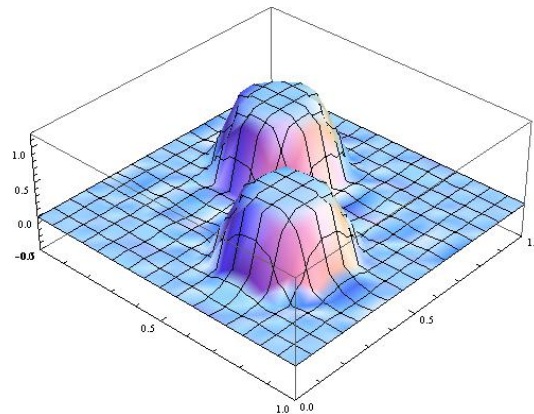
N	Approximation error
3	0.01690682
23	0.00272393
55	0.00016751

Table 1. Approx. errors

3. Uniform convergence

In one dimension, if f is m -times continuously differentiable, $m \geq 2$, and

$$f(x) = \sum_{k \in \mathbb{Z}} c_k e^{2\pi i k x},$$

**Figure 5.** Function ψ_{55}

decay of fourier coefficient is $|c_k| \leq ck^{-m}$, where c is some constant. Therefore, if we define for $N = 2M + 1$

$$\psi_N(x) = \sum_{|k| \leq M} d_k e^{2\pi i k x},$$

where

$$d_k = (f, e^{2\pi i k x})_N,$$

and

$$(f, g)_N = \frac{1}{N} \sum_{j=0}^{N-1} f\left(\frac{j}{N}\right) \overline{g\left(\frac{j}{N}\right)},$$

we obtain

$$|f(x) - \psi_N(x)| \leq \sum_{|k| \leq M} |c_k - d_k| + \sum_{|k| > M} |c_k|,$$

where

$$|c_k - d_k| = |c_k - \sum_{l \in \mathbb{Z}} c_l (e^{2\pi i l x}, e^{2\pi i k x})_N|,$$

which can be simplified due to $(e^{2\pi i l x}, e^{2\pi i k x})_N = 1$ if $k \equiv l \pmod{N}$ and 0 otherwise:

$$|c_k - d_k| \leq \sum_{l \neq 0} |c_{k+lN}| \leq \frac{\text{const}}{N^m}.$$

Because

$$\sum_{|k| > M} |c_k| \leq \frac{\text{const}}{N^{m-1}},$$

we have

$$|f(x) - \psi_N(x)| \leq \frac{\text{const}}{N^{m-1}} \rightarrow 0 \text{ as } N \rightarrow +\infty \text{ if } m \geq 2.$$

Similarly, in n dimensions, if f is m -times continuously differentiable and we have

$$f(x) = \sum_{k \in \mathbb{Z}^n} c_k e^{2\pi i (k, x)},$$

since

$$D^l e^{2\pi i(k,x)} = \prod_{j=1}^n (2\pi i k_j)^{l_j} e^{2\pi i(k,x)}$$

we see that $|\prod_{j=1}^n k_j^{l_j} c_k|$ is bounded for every multi-index l for which $\sum_{j=1}^n l_j \leq m$.

If we now have $2m$ -times continuously differentiable function on \mathbb{R}^2 , symmetric and periodic, expanded in pointwise convergent Fourier series as

$$f(x, y) = \sum_{k \geq l} c_{kl} E_{(k,l)}^+(x, y),$$

we can apply the same procedure. For $N = 2M + 1$ we take

$$\psi_N(x, y) = \sum_{\substack{k,l=-M \\ k \geq l}}^M \beta_{kl}^+ E_{(k,l)}^+(x, y),$$

and study the difference

$$|f(x) - \psi_N(x)|.$$

If we analyze this difference similarly as above, one can see that when m is equal to 2, it converges to zero as $N \rightarrow \infty$.

4. Conclusion

The complex valued functions on the Euclidean space \mathbb{R}^n , symmetric or antisymmetric with respect to the permutation group S_n , are very well suited for approximate of functions with the same symmetry properties, which appear in applications. Uniform convergence happens whenever sufficient smoothness of original approximated function is assured. Our approach was demonstrated on the simplest example of symmetric two dimensional exponential functions, however it is clear that similar analysis can be performed within any families of special functions such as symmetric, antisymmetric, alternating exponential functions, symmetric, antisymmetric, alternating multivariate trigonometric functions, as well as for families of C-, S-, and E-functions of any compact semisimple Lie group (families introduced and studied by Patera, Moody, Klimyk, Hrivnák and others).

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