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Infinite-dimensional symmetries of a general class of variable coefficient evolution equations in 2+1 dimensions

P. Basarab-Horwath[1], **F. Güngör**[2], **C. Özemir**[3]

[1] Department of Mathematics, Linköping University, S-581 83 Linköping, Sweden

[2] Department of Mathematics, Faculty of Science and Sciences, Istanbul Technical University, 34469 Istanbul, Turkey

[3] Department of Mathematics, Faculty of Science and Letters, Istanbul Technical University, 34469 Istanbul, Turkey

E-mail: `pehor@mai.liu.se`, `gungorf@itu.edu.tr`, `ozemir@itu.edu.tr`

Abstract. We consider generalized KP-Burgers equations and attempt to identify subclasses admitting Virasoro or Kac-Moody type algebras as their symmetries. We give reductions to ODEs constructed from invariance requirement under these infinite-dimensional Lie symmetry algebras and integrate them in cases where it is possible. We also look at the conditions under which the equation passes the Painlevé test and construct some exact solutions by truncation.

1. Introduction

In this paper we give a classification of infinite-dimensional symmetries of the family of equations

$$\begin{aligned}&(u_t + p(y,t)uu_x + q(y,t)u_{xx} + r(y,t)u_{xxx})_x + \sigma(y,t)u_{yy}\\ &+ a(y,t)u_y + b(y,t)u_{xy} + c(y,t)u_{xx} + e(y,t)u_x + f(y,t)u + h(y,t) = 0.\end{aligned} \tag{1.1}$$

with the assumption $p(y,t) \neq 0$, $q(y,t) \neq 0$, $r(y,t) \neq 0$, $\sigma(y,t) \neq 0$. Eq. (1.1) includes generalizations of the 2+1-dimensional Burgers and KP (Kadomtsev-Petviashvili) equations for the choice of $r = 0$ and $q = 0$, respectively. For derivations of constant coefficient KP-Burgers equations in different physical applications we refer, for instance, to Ref. [1, 2]

From group theoretical point of view, the case $q = 0$, $r = r(t)$ was studied in [3], whereas the case $q = q(t)$, $r = 0$ was recently done in [4]. In the former, the equation has a Virasoro algebra for some specific choice of the coefficients. In the latter, this remarkable algebra does not arise. However, both cases contain infinite-dimensional symmetry algebras of Kac-Moody type [5]. In the present work we wish to analyze the situation where both q and r can survive with an additional y-dependence.

We note that for $q(y,t) = 0$ this family of equations is analyzed in [6] in the context of its integrability properties and exact solutions.

2. Canonical Forms and Determining Equations

We shall convert Eq. (1.1) to some canonical form using point transformations preserving the differential structure of the equation. These type of transformations are known as equivalence,

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allowed or form-invariant transformations. It is a straightforward calculation to show that they are given by

$$\begin{aligned} &u(x,y,t) = R(y,t)\tilde{u}(\tilde{x},\tilde{y},\tilde{t}) - \frac{\dot{\alpha}}{\alpha p}x + S_0(y,t), \\ &\tilde{x} = \alpha(t)x + \beta(y,t), \quad \tilde{y} = Y(y,t), \quad \tilde{t} = T(t), \\ &\alpha \neq 0, \quad R \neq 0 \quad Y_y \neq 0, \quad \dot{T} \neq 0 \end{aligned} \tag{2.1}$$

together with some restrictions on $\alpha(t)$ and coefficients of the equation:

$$\Big(\frac{ap_y}{p} + \frac{\sigma p_{yy}}{p} - \frac{2\sigma p_y^2}{p^2} - f\Big)\dot{\alpha} = 0. \tag{2.2}$$

With this transformation coefficient functions are mapped to

$$\begin{aligned} &\tilde{p}(\tilde{y},\tilde{t}) = p(y,t)\frac{R\alpha}{\dot{T}}, \quad \tilde{q}(\tilde{y},\tilde{t}) = q(y,t)\frac{\alpha^2}{\dot{T}}, \quad \tilde{r}(\tilde{y},\tilde{t}) = r(y,t)\frac{\alpha^3}{\dot{T}}, \\ &\tilde{\sigma}(y,t) = \sigma(y,t)\frac{Y_y^2}{\alpha\dot{T}}, \\ &\tilde{a}(\tilde{y},\tilde{t}) = \frac{1}{\alpha\dot{T}}\{aY_y + \sigma Y_{yy} + 2\sigma Y_y\frac{R_y}{R}\}, \\ &\tilde{b}(\tilde{y},\tilde{t}) = \frac{1}{\alpha\dot{T}}\{(b\alpha + 2\sigma\beta_y)Y_y + \alpha Y_t\}, \\ &\tilde{c}(\tilde{y},\tilde{t}) = \frac{1}{\alpha\dot{T}}\{c\alpha^2 + \beta_t\alpha + pS_0\alpha^2 + \sigma\beta_y^2 + b\alpha\beta_y\}, \\ &\tilde{e}(\tilde{y},\tilde{t}) = \frac{1}{\alpha R\dot{T}}\{R\alpha e - R\dot{\alpha} + R_t\alpha + aR\beta_y + \sigma R\beta_{yy} + b\alpha R_y + 2\sigma R_y\beta_y\}, \\ &\tilde{f}(\tilde{y},\tilde{t}) = \frac{1}{\alpha R\dot{T}}(fR + aR_y + \sigma R_{yy}), \\ &\tilde{h}(\tilde{y},\tilde{t}) = \frac{1}{\alpha R\dot{T}}\{h - \frac{\partial}{\partial t}\left(\frac{\dot{\alpha}}{\alpha p}\right) + \frac{1}{p}\left(\frac{\dot{\alpha}}{\alpha}\right)^2 + \sigma S_{0,yy} + aS_{0,y} + fS_0 - e\frac{\dot{\alpha}}{\alpha p} + b\frac{\dot{\alpha}p_y}{\alpha p^2}\}. \end{aligned} \tag{2.3}$$

We can choose $R(y,t) = \dot{T}/(p\alpha)$ to normalize $\tilde{p}(\tilde{y},\tilde{t}) = 1$. With this simplification the condition (2.2) reduces to $\dot{\alpha}f = 0$. If we set

$$Y_y^2 = |\frac{\alpha\dot{T}}{\sigma}|$$

we can normalize $\tilde{\sigma}(\tilde{y},\tilde{t}) = \varepsilon = \mp 1$. Furthermore, choosing $\beta(y,t)$ and $S_0(y,t)$ suitably we can have

$$\tilde{e}(\tilde{y},\tilde{t}) = \tilde{h}(\tilde{y},\tilde{t}) = 0.$$

Therefore, instead of Eq. (1.1) we can equivalently work on

$$\begin{aligned} &(u_t + uu_x + q(y,t)u_{xx} + r(y,t)u_{xxx})_x + \varepsilon u_{yy} \\ &+ a(y,t)u_y + b(y,t)u_{xy} + c(y,t)u_{xx} + f(y,t)u = 0, \quad \varepsilon = \mp 1 \end{aligned} \tag{2.4}$$

which we call the canonical form of (1.1). Allowed transformations for this class are found as

$$\begin{aligned} &\tilde{x} = \alpha(t)x + \beta(y,t), \qquad \tilde{y} = \epsilon\sqrt{\alpha\dot{T}}y + \gamma(t), \quad \epsilon = \mp 1, \\ &\tilde{t} = T(t), \quad u = \frac{\dot{T}}{\alpha}\tilde{u} - \frac{\dot{\alpha}}{\alpha}x + S_0(y,t) \end{aligned} \tag{2.5}$$

with the constraints

$$\dot{\alpha} f = 0, \quad \varepsilon \beta_{yy} + a\beta_y + \alpha \frac{\ddot{T}}{\dot{T}} - 2\dot{\alpha} = 0, \quad \varepsilon S_{0,yy} + aS_{0,y} + fS_0 = \frac{d}{dt}(\frac{\dot{\alpha}}{\alpha}) - (\frac{\dot{\alpha}}{\alpha})^2. \tag{2.6}$$

We shall restrict ourselves to the more manageable case $r_y = 0$ in the rest of the article.

2.1. Canonical class when $r = r(t)$

In this case we can immediately normalize $\tilde{r}(\tilde{t}) = 1$ by choosing $\dot{T} = \alpha^3(t)\, r(t)$ in (2.3). We note that this normalization will lead to an enormous simplicity in the determining equations. We thus consider the canonical family

$$\begin{aligned} &(u_t + uu_x + q(y,t)u_{xx} + u_{xxx})_x + \varepsilon u_{yy} \\ &+ a(y,t)u_y + b(y,t)u_{xy} + c(y,t)u_{xx} + f(y,t)u = 0, \quad \varepsilon = \mp 1 \end{aligned} \tag{2.7}$$

with its allowed transformations

$$\begin{aligned} &\tilde{x} = \alpha(t)x + \beta(y,t), \qquad \tilde{y} = \epsilon\alpha^2 y + \gamma(t), \quad \epsilon = \mp 1, \\ &\tilde{t} = \int \alpha^3(t)dt, \quad u = \alpha^2 \tilde{u} - \frac{\dot{\alpha}}{\alpha}x + S_0(y,t), \end{aligned} \tag{2.8}$$

obeying the constraints

$$\dot{\alpha} f = 0, \quad \varepsilon \beta_{yy} + a\beta_y + \dot{\alpha} = 0, \quad \varepsilon S_{0,yy} + aS_{0,y} + fS_0 = \frac{d}{dt}(\frac{\dot{\alpha}}{\alpha}) - (\frac{\dot{\alpha}}{\alpha})^2. \tag{2.9}$$

2.2. Canonical class when $q = q(t)$, $r = r(t)$, $a = f = 0$

For later convenience we note a further simplification of (2.7) when $q = q(t)$ and $a = f = 0$. If f vanishes identically, we have no condition on $\alpha(t)$ that we can choose $\alpha(t) := q(t)$ to simplify $\tilde{q} = \frac{q\alpha^2}{\dot{T}} = \frac{q}{\alpha} = 1$ in (2.3). Assuming further $a = 0$, we have the canonical family

$$(u_t + uu_x + u_{xx} + u_{xxx})_x + \varepsilon u_{yy} + b(y,t)u_{xy} + c(y,t)u_{xx} = 0, \quad \varepsilon = \pm 1 \tag{2.10}$$

with the equivalence transformations

$$\begin{aligned} &\tilde{x} = x + \beta_1(t)y + \beta_2(t), \qquad \tilde{y} = \epsilon y + \gamma(t), \\ &\tilde{t} = t + T_0, \qquad u = \tilde{u} + S_1(t)y + S_2(t). \end{aligned} \tag{2.11}$$

2.3. The infinitesimal and determining equations

Now we proceed to obtain the symmetry algebra of (2.7). Let us denote the infinitesimal as

$$V = \tau\partial_t + \xi\partial_x + \eta\partial_y + \phi\partial_u. \tag{2.12}$$

Here the coefficients τ, ξ, η, ϕ depend on the variables t, x, y, u. We see that V has the form

$$V = \tau(t)\partial_t + \left(\frac{\dot{\tau}}{3}x + \xi_0(y,t)\right)\partial_x + \left(\frac{2}{3}\dot{\tau}y + \eta_0(t)\right)\partial_y + \left(-\frac{2}{3}\dot{\tau}u + \frac{\ddot{\tau}}{3}x + S(y,t)\right)\partial_u, \tag{2.13}$$

where

$$S(y,t) = -\tau c_t - (\frac{2}{3}\dot{\tau}y + \eta_0)c_y + \xi_{0,t} + b\xi_{0,y} - \frac{2}{3}c\dot{\tau} \tag{2.14}$$

subject to the determining equations

$$\dot\tau(q + 2yq_y) + 3\tau q_t + 3\eta_0 q_y = 0, \tag{2.15}$$
$$3\tau a_t + (2\dot\tau y + 3\eta_0)a_y + 2a\dot\tau = 0, \tag{2.16}$$
$$-3\dot\eta_0 - 2y\ddot\tau + 3\tau b_t + (2\dot\tau y + 3\eta_0)b_y + b\dot\tau - 6\varepsilon\xi_{0,y} = 0, \tag{2.17}$$
$$\ddot\tau + 3a\xi_{0,y} + 3\varepsilon\xi_{0,yy} = 0, \tag{2.18}$$
$$f\ddot\tau = 0, \tag{2.19}$$
$$4f\dot\tau + 3f_t\tau + f_y(2\dot\tau y + 3\eta_0) = 0, \tag{2.20}$$
$$\dddot\tau + 3fS + 3aS_y + 3\varepsilon S_{yy} = 0. \tag{2.21}$$

3. Symmetry Algebras

3.1. Search for the Virasoro symmetries

The set of determining equations (2.16)-(2.21) are almost the same with those obtained in [3]. Equation (2.15) stems from the extra term $q(y,t)u_{xx}$. We are going to try to satisfy Eqs. (2.15)-(2.21) without imposing any condition on $\tau(t)$. This requirement leading to a similar analysis carried out in [3] implies that we must have

$$\xi_0(y,t) = -\frac{\varepsilon}{6}\ddot\tau y^2 + \mu(t)y + \xi(t), \quad \mu(t) = \frac{\varepsilon}{6}\big(b\dot\tau + 3\tau\dot b - 3\dot\eta_0\big), \tag{3.1}$$

where $\tau(t), \xi(t)$ and $\eta_0(t)$ are arbitrary. Using these results in (2.21) we find

$$2\dot\tau(3c_{yy} + yc_{yyy}) + 3\eta_0 c_{yyy} + 3\tau c_{tyy} = 0. \tag{3.2}$$

We have the following splitting:
(i) $c_{tyy} = 0$. This means we have

$$c(y,t) = k_0 y^2 + c_1(t)y + c_0(t) \tag{3.3}$$

with k_0 a constant and (3.2) is satisfied for arbitrary τ if $k_0 = 0$. This is the form of $c(t)$ obtained in [3].
(ii) $3c_{yy} + yc_{yyy} = 0$. We find by integration

$$c(y,t) = \frac{c_2(t)}{y} + c_1(t)y + c_0(t). \tag{3.4}$$

The substitution of $c(y,t)$ into (3.2) results in

$$3c_2\eta_0 - y\tau\dot c_2 = 0. \tag{3.5}$$

$c_2 = 0$ corresponds to the previous case. If $c_2 \neq 0$, we need to have $\eta_0(t) = 0$ and $c_2(t) = l_0$, a constant. We therefore have

$$c(y,t) = \frac{l_0}{y} + c_1(t)y + c_0(t), \tag{3.6}$$

where

$$\eta_0(t) = \begin{cases} \text{free}, & l_0 = 0, \\ 0, & l_0 \neq 0. \end{cases} \tag{3.7}$$

Now Eq. (2.15) has the form

$$\dot\tau(q + 2yq_y) + 3\tau q_t + 3\eta_0 q_y = 0, \tag{3.8}$$

which implies that $q + 2yq_y = 0$ and $q_t = 0$ from which we have $q(y,t) = \frac{q_0}{\sqrt{y}}$ with $q_0 =$ const. and the condition $q_0\eta_0(t) = 0$. From the assumption $q(y,t) \neq 0$ we should have $\eta_0(t) = 0$.

We state the results of this part as theorems.

Theorem 3.1 *The canonical generalized KP-Burgers equation* (2.7) *allows the Virasoro algebra as a symmetry algebra if and only if the coefficients satisfy*

$$a = f = 0, \quad q(y,t) = \frac{q_0}{\sqrt{y}}, \quad b = b(t), \quad c = c_0(t) + c_1(t)y + \frac{l_0}{y}. \tag{3.9}$$

Theorem 3.2 *The canonical generalized KP-Burgers equation*

$$(u_t + uu_x + \frac{q_0}{\sqrt{y}}u_{xx} + u_{xxx})_x + \varepsilon u_{yy} + b(t)u_{xy} + (c_0(t) + c_1(t)y + \frac{l_0}{y})u_{xx} = 0 \tag{3.10}$$

with $\varepsilon = \pm 1$, $l_0 \in \mathbb{R}$ *and* $b(t)$, $c_0(t)$ *and* $c_1(t)$ *arbitrary smooth functions is invariant under an infinite dimensional Lie point symmetry group. Its Lie algebra is realized by vector fields of the form*

$$\hat{V} = T(\tau) + X(\xi), \tag{3.11}$$

where $\tau(t)$ *and* $\xi(t)$ *are arbitrary smooth functions of time and we have*

$$\begin{aligned} T(\tau) &= \tau(t)\partial_t + \frac{1}{6}[3\varepsilon\dot{b}y\tau + (2x + \varepsilon by)\dot{\tau} - \varepsilon\ddot{\tau}y^2]\partial_x \\ &+ \frac{2}{3}\dot{\tau}y\partial_y + \frac{1}{6}\{[-6\dot{c}_0 + 3\varepsilon b\dot{b} + (-6\dot{c}_1 + 3\varepsilon\ddot{b})y]\tau \\ &+ [-4u + \varepsilon b^2 - 4c_0 + 4(\varepsilon\dot{b} - 2c_1)y]\dot{\tau} + (2x - \varepsilon by)\ddot{\tau} - \varepsilon y^2\dddot{\tau}\}\partial_u, \end{aligned} \tag{3.12}$$

$$X(\xi) = \xi(t)\partial_x + \dot{\xi}(t)\partial_u. \tag{3.13}$$

Remark 3.1 *Theorems 1 and 2 are also valid for* $q_0 = 0$, *therefore they should be noted as different canonical classes in [3]. For the case* $q_0 = 0$ *studied in [3], there are three arbitrary functions in the vector field. Two of these functions exactly give rise to the generators* $T(\tau)$ *and* $X(\xi)$. *The last arbitrary function is nothing but* $\eta_0(t)$, *and the above analysis shows that restricting* $\eta_0(t) = 0$ *allows the terms* $q_0 u_{xxx}/\sqrt{y}$ *and* $l_0 u_{xx}/y$ *in* (3.10).

The canonical class (3.10) can further be simplified with the allowed transformations (2.8). Since $a = f = 0$, conditions (2.9) can be formulated as

$$\beta(y,t) = -\frac{\varepsilon\dot{\alpha}}{2}y^2 + \beta_1(t)y + \beta_2(t), \quad S_0(y,t) = \frac{\varepsilon}{2}\chi(t)y^2 + S_1(t)y + S_2(t) \tag{3.14}$$

with $\chi(t) = \frac{d}{dt}(\frac{\dot{\alpha}}{\alpha}) - (\frac{\dot{\alpha}}{\alpha})^2$. Using (2.3) we get

$$\tilde{b}(\tilde{t}) = \frac{1}{\alpha^3}(\epsilon b\alpha^2 + 2\epsilon\varepsilon\alpha\beta_1 + \dot{\gamma}), \tag{3.15}$$

$$\begin{aligned} \tilde{c}(\tilde{y},\tilde{t}) =& \frac{l_0\alpha^{-2}}{y} + \alpha^{-4}(c_0\alpha^2 + \alpha\dot{\beta}_2 + S_2\alpha^2 + \varepsilon\beta_1^2 + b\alpha\beta_1) \\ &+ \alpha^{-4}(c_1\alpha^2 + \alpha\dot{\beta}_1 + S_1\alpha^2 - 2\dot{\alpha}\beta_1 - \varepsilon b\alpha\dot{\alpha})y \\ &+ \alpha^{-4}\varepsilon(-\frac{1}{2}\alpha\ddot{\alpha} + \frac{1}{2}\alpha^2\chi + \dot{\alpha}^2)y^2. \end{aligned} \tag{3.16}$$

Choosing γ appropriately, we can have $\tilde{b}(\tilde{t}) = 0$. If $l_0 \neq 0$, we choose $\alpha^2(t) = |l_0|$ to make the normalization $\tilde{l}_0 = \mp 1$. Coefficient of y^2 vanishes, and there exist functions $S_1(t)$ and $S_2(t)$ such that the coefficients of y^0 and y also vanishes. Similar arguments apply if $l_0 = 0$ in which $\alpha(t)$ is chosen as an arbitrary constant. In summary, we can consider

$$b(t) = c_0(t) = c_1(t) = 0. \tag{3.17}$$

3.2. The function $\eta_0(t)$ free

From (2.15) we see that $\eta_0(t)$ is free only if we have $q_y = 0$, that is, $q = q(t)$. Then (2.15) is easy to solve. We momentarily leave it aside and focus on the other equations. Since (2.16)-(2.21) are the same with those given in [3], we skip the details of the calculations and state that (2.16)-(2.20) can be solved for $\eta_0(t)$ free if

$$\xi_0(y,t) = -\frac{\varepsilon}{6}\ddot{\tau}\, y^2 + \nu(t)y + \xi(t),\ \nu(t) = \frac{\varepsilon}{6}\big(-3\dot{\eta}_0 + 3\tau\dot{b}_0 + 3\eta_0 b_1 + b_0\dot{\tau}\big), \tag{3.18}$$

$$b(y,t) = b_1(t)y + b_0(t), \quad (\tau b_1)^{\cdot} = 0, \quad a(y,t) = f(y,t) \equiv 0. \tag{3.19}$$

Since we have come to the situation that $a = f = 0$, this means the canonical equation (2.10) can be studied instead of (2.7). Furthermore, equivalence transformations (2.11) are valid. If we use $q(t) = 1$ in (2.15), we find that $\tau(t) = \tau_0 =$ const. (2.21) simplifies to

$$\eta_0 c_{yyy} + \tau_0 c_{tyy} = 0. \tag{3.20}$$

Obviously we need to have $c(y,t) = c_0(t) + c_1(t)y + c_2(t)y^2$. Then the only relations to be satisfied are

$$\tau_0\,\dot{b}_1 = 0, \qquad \tau_0\,\dot{c}_2 = 0. \tag{3.21}$$

(i) If $\tau_0 = 0$, we have the symmetry algebra in [3] and state it as a theorem.

Theorem 3.3 *The equation*

$$\begin{aligned}(u_t + uu_x + u_{xx} + u_{xxx})_x + \varepsilon u_{yy} + (b_1(t)y + b_0(t))u_{xy}\\ + (c_2(t)y^2 + c_1(t)y + c_0(t))u_{xx} = 0,\end{aligned} \tag{3.22}$$

where $\varepsilon = \pm 1$ and b_0, b_1, c_0, c_1, c_2 are arbitrary functions of time, is invariant under an infinite-dimensional Lie point symmetry group depending on two arbitrary functions. Its Lie algebra has a Kac-Moody structure and is realized by vector fields of the form

$$\hat{V} = X(\xi) + Y(\eta), \tag{3.23}$$

where $\xi(t)$ and $\eta(t)$ are arbitrary smooth functions of time and

$$X(\xi) = \xi\partial_x + \dot{\xi}\partial_u, \tag{3.24}$$

$$\begin{aligned}Y(\eta) = \frac{\varepsilon}{2}y(-\dot{\eta} + b_1\eta)\partial_x + \eta\partial_y + \{[-2c_2\eta\\ + \frac{\varepsilon}{2}(-\ddot{\eta} + \dot{b}_1\eta + b_1^2\eta)]y - c_1\eta + \frac{\varepsilon}{2}b_0(-\dot{\eta} + b_1\eta)\}\partial_u.\end{aligned} \tag{3.25}$$

It is possible to make

$$b_0(t) = c_0(t) = c_1(t) = 0 \tag{3.26}$$

using transformations (2.11).

Remark 3.2 *This result shows that the Kac-Moody algebra (3.24)-(3.25) of the generalized KP equation [3] and generalized Burgers equation [4] is actually admitted by the KP-Burgers equation.*

(ii) For the case $\tau_0 \neq 0$ we proceed with the case (3.26) and find $b_1(t) = k_1$, $c_2(t) = k_2$ as constants. So, the equation

$$(u_t + uu_x + u_{xx} + u_{xxx})_x + \varepsilon u_{yy} + k_1 y\, u_{xy} + k_2 y^2 u_{xx} = 0 \tag{3.27}$$

has the obvious symmetry $T = \partial_t$ in addition to (3.24) and (3.25).

3.3. One function in symmetry algebra
If we let $\tau(t)$ and $\eta_0(t)$ be constrained by the determining equations, but let some freedom remain in $\xi_0(y,t)$, then we have the following:

Theorem 3.4 *Equation* (2.4) *is invariant under an infinite dimensional Abelian group generated by the vector field*

$$X(\xi) = \xi(t)\partial_x + \dot{\xi}(t)\partial_u \tag{3.28}$$

for $f(y,t) = 0$ and a, b, c, q, r arbitrary.

4. Some Applications

In this Section we apply method of reduction and truncating Painlevé expansion to find exact solutions.

4.1. Reduction for the Virasoro case
The solution invariant under the group $\exp(T(\tau) + X(\xi))$ for $\tau \neq 0$ is given by

$$\begin{aligned} \rho &= x\tau^{-1/3} + \frac{\epsilon}{6}\dot{\tau}\tau^{-4/3}y^2 - \int_0^t \xi(s)\tau(s)^{-4/3}ds, \quad \theta = y^{-3/2}\tau, \\ u &= \tau^{-2/3}w(\rho,\theta) - \frac{\varepsilon}{18}\tau^{-2}y^2(3\tau\ddot{\tau} - 2\dot{\tau}^2) + \frac{x}{3}\frac{\dot{\tau}}{\tau} + \frac{\xi}{\tau}, \end{aligned} \tag{4.1}$$

where w satisfies

$$[w_{\rho\rho\rho} + q_0\theta^{1/3}w_{\rho\rho} + l_0\theta^{2/3}w_\rho + ww_\rho]_\rho + \frac{15\varepsilon}{4}\theta^{7/3}w_\theta + \frac{9\varepsilon}{4}\theta^{10/3}w_{\theta\theta} = 0. \tag{4.2}$$

This equation admits a two-dimensional Lie algebra generated by $X_1 = \partial_\rho$, $X_2 = \rho\partial_\rho - 3\theta\partial_\theta - 2w\partial_w$. The solutions invariant under the subalgebra X_1 are straightforward to obtain. They are given by

$$w(\theta) = w_1 + w_0\theta^{-2/3}$$

which produces solutions

$$u(x,y,t) = w_0\tau^{-4/3}y^{-2/3} + w_1\tau^{-2/3} - \frac{\varepsilon}{18}\tau^{-2}y^2(3\tau\ddot{\tau} - 2\dot{\tau}^2) + \frac{x}{3}\frac{\dot{\tau}}{\tau} + \frac{\xi}{\tau}$$

depending on two arbitrary functions of time $\tau(t)$, $\xi(t)$ and two constants w_0, w_1.

Invariance under X_2 implies that the solution will have the form

$$w(\rho,\theta) = \theta^{2/3}F(z), \quad z = \rho\theta^{1/3}, \tag{4.3}$$

where $F(z)$ satisfies the fourth order ODE

$$F'''' + q_0F''' + (l_0 + \frac{\varepsilon}{4}z^2)F'' + \frac{7\varepsilon}{4}zF' + (FF')' + 2\varepsilon F = 0. \tag{4.4}$$

Unfortunately, we have not been able to integrate it further, neither found a first integral.

We mention that the projective action on the solution u under the subgroup of the full symmetry group

$$C = T(t^2) = t^2\partial_t + \frac{1}{6}(4xt - 2\epsilon y^2)\partial_x + \frac{4}{3}ty\partial_y + \frac{1}{6}(4x - ut)\partial_u \tag{4.5}$$

is given by

$$\begin{aligned}
&\tilde{t} = t(1-\lambda t)^{-1}, \quad \tilde{y} = (1-\lambda t)^{-4/3} y,\\
&\tilde{x} = (1-\lambda t)^{-2/3}(x + \frac{\epsilon}{3}\frac{y^2}{t}) - \frac{\epsilon}{3}\frac{y^2}{t}(1-\lambda t)^{-5/3},\\
&\tilde{u} = (1-\lambda t)^{4/3}\{u + \frac{2}{3}(x + \frac{\epsilon}{3}\frac{y^2}{t})\frac{\lambda}{1-\lambda t} + \frac{\epsilon}{6}\frac{\lambda y^2 t}{(1-\lambda t)^2}(\lambda t - 2)\},
\end{aligned} \tag{4.6}$$

where λ is the group parameter. These type of solution formulas have proved useful in constructing blow-up profiles for evolution equations.

4.2. Reduction for the Kac-Moody case

A solution of (3.22) invariant under $\exp(X(\xi) + Y(\eta))$ will have the form

$$\begin{aligned}
u &= \Big[\frac{\varepsilon}{4}\Big(\dot{b} + b^2 - \frac{\ddot{\eta}}{\eta}\Big) - c\Big]y^2 + \frac{\dot{\xi}}{\eta}\,y + F(z,t),\\
z &= x + \frac{\varepsilon}{4}(-b + \frac{\dot{\eta}}{\eta})y^2 - \frac{\xi}{\eta}\,y.
\end{aligned} \tag{4.7}$$

We assumed the simplification (3.26) and re-labeled $b_1(t) = b(t)$, $c_2(t) = c(t)$. The reduced equation is found to be

$$(F_t + FF_z + F_{zz} + F_{zzz})_z + \varepsilon\frac{\xi^2}{\eta^2}F_{zz} + \frac{1}{2}\Big(\frac{\dot{\eta}}{\eta} - b\Big)F_z - 2\varepsilon c + \frac{1}{2}\Big(\dot{b} + b^2 - \frac{\ddot{\eta}}{\eta}\Big) = 0. \tag{4.8}$$

It is possible to eliminate the term F_{zz} by putting

$$F(z,t) = \tilde{F}(\tilde{z},\tilde{t}), \quad \tilde{z} = z + \beta(t), \quad \tilde{t} = t, \quad \dot{\beta}(t) = -\varepsilon\frac{\xi^2}{\eta^2}. \tag{4.9}$$

Further choosing $\dot{\eta}/\eta = b$ gives rise to the one-dimensional nonhomogeneous KdV-Burgers equation

$$(F_t + FF_z + F_{zz} + F_{zzz})_z = 2\varepsilon c(t). \tag{4.10}$$

4.3. Painlevé property and an exact solution

We checked whether (2.4) has the Painlevé property, using the package PainleveTestV2.m [7] with a Kruskal anzats in x, and we saw that the equation passes the test when

$$\begin{aligned}
&q(y,t) = a(y,t) = f(y,t) = 0, \quad b(y,t) = b_0(t) + b_1(t)y,\\
&r(y,t) = r_0\exp\Big(\frac{2}{3}\int b_1(t)dt\Big), \ \ c(y,t) = \frac{\varepsilon}{3}\Big(\frac{2}{3}b_1^2 + \dot{b}_1\Big)y^2 + c_1(t)y + c_0(t).
\end{aligned} \tag{4.11}$$

Since r is a function of t, it can be normalized to 1, which means that we can make $b_1(t) \to 0$. They are exactly the coefficients of the class which are invariant under the Kac-Moody-Virasoro algebra [3].

According to these results, our canonical equation (3.10) with the Virasoro algebra cannot have the Painlevé property ($q_0 \neq 0$).

We also found that

$$(u_t + uu_x + u_{xx} + u_{xxx})_x + \varepsilon u_{yy} + b(t)yu_{xy} + c(t)y^2u_{xx} = 0 \tag{4.12}$$

fails to have the Painlevé property. In search for any possible exact solutions we truncate the Painlevé series for this equation at the first term and propose a solution of the form

$$u(x,y,t) = \frac{u_0(x,y,t)}{\Phi^2(x,y,t)}, \tag{4.13}$$

where Φ defines the singularity manifold. We substitute (4.13) in (4.12) and require that the coefficients of Φ^j, $j=-6,-5,..,-1$ vanish identically. For $c(t)=0$ it has been possible to solve these system of equations in the form

$$\begin{aligned} u_0 &= -\frac{12k_1^2}{25}\exp\Big[-\frac{2x}{5}-\frac{12t}{125}+2k_2ye^{-\int b(t)dt}+10\varepsilon k_2^2\int e^{-2\int b(t)dt}dt\Big], \\ \Phi &= k_1\exp\Big[-\frac{x}{5}-\frac{6t}{125}+k_2ye^{-\int b(t)dt}+5\varepsilon k_2^2\int e^{-2\int b(t)dt}dt\Big]+k_3, \end{aligned} \tag{4.14}$$

where k_1, k_2, k_3 are arbitrary constants. We checked whether the same approach works for (3.10) with (3.17). It turned out that a solution of the type (4.13) is not possible in this case.

4.4. *The special case* $a=b=c=e=f=h=0$

We finally consider the equation

$$(u_t + p(t)uu_x + q(t)u_{xx} + r(t)u_{xxx})_x + \sigma(t)u_{yy} = 0, \tag{4.15}$$

where all the coefficients are assumed to be nonzero. Allowed transformations are again of the form (2.1), while the coefficients map to

$$\tilde{p}(\tilde{t}) = p(t)\frac{R\alpha}{\dot{T}}, \quad \tilde{q}(\tilde{t}) = q(t)\frac{\alpha^2}{\dot{T}}, \quad \tilde{r}(\tilde{t}) = r(t)\frac{\alpha^3}{\dot{T}}, \quad \tilde{\sigma}(t) = \sigma(t)\frac{Y_y^2}{\alpha\dot{T}}. \tag{4.16}$$

In order that no additional terms appear in the equation, we must have

$$\sigma Y_{yy} = 0, \tag{4.17}$$

$$2\sigma\beta_y Y_y + \alpha Y_t = 0, \tag{4.18}$$

$$\beta_t\alpha + pS\alpha^2 + \sigma\beta_y^2 = 0, \tag{4.19}$$

$$-R\dot{\alpha} + \dot{R}\alpha + \sigma R\beta_{yy} = 0, \tag{4.20}$$

$$-\frac{d}{dt}\left(\frac{\dot{\alpha}}{\alpha p}\right) + \frac{1}{p}\left(\frac{\dot{\alpha}}{\alpha}\right)^2 + \sigma S_{yy} = 0. \tag{4.21}$$

Normalization $\tilde{p}(\tilde{t}) = \tilde{r}(\tilde{t}) = 1$ is again possible if we choose $\dot{T} = \alpha^3 r$ and $R = \alpha^2\frac{r}{p}$. Furthermore, we see that $Y(y,t) = \mu(t)y + \nu(t)$ so that $\tilde{\sigma}$ has no dependence on y. We solve (4.18) for $\beta(y,t)$ and (4.19) for $S(y,t)$ to obtain

$$\beta(y,t) = -\frac{\alpha\dot{\mu}}{4\sigma\mu}y^2 - \frac{\alpha\dot{\nu}}{2\sigma\mu}y + \beta_1(t), \tag{4.22}$$

$$S(y,t) = -\frac{1}{p\alpha^2}(\alpha\beta_t + \sigma\beta_y^2). \tag{4.23}$$

(4.20) is satisfied if $\mu(t) = \mu_0\left(\frac{\alpha r}{p}\right)^2$. There remains Eq. (4.21), which takes the form

$$\frac{\dot{\alpha}}{\alpha}\Big(4\frac{\dot{p}}{p} - 3\frac{\dot{r}}{r} - \frac{\dot{\sigma}}{\sigma}\Big) = \frac{\ddot{p}}{p} - \frac{\ddot{r}}{r} + \frac{\dot{p}^2}{p^2} + 3\frac{\dot{r}^2}{r^2} - 4\frac{\dot{p}\dot{r}}{pr} - \frac{\dot{p}\dot{\sigma}}{p\sigma} + \frac{\dot{r}\dot{\sigma}}{r\sigma}. \tag{4.24}$$

(i) Suppose $\sigma \neq \sigma_0 \frac{p^4}{r^3}$, where σ_0 is an arbitrary constant. Then we can solve $\alpha(t)$ from (4.24) and the normalization process is completed, which means we have obtained the canonical form

$$(u_t + uu_x + q(t)u_{xx} + u_{xxx})_x + \sigma(t)u_{yy} = 0. \tag{4.25}$$

(ii) If $\sigma = \sigma_0 \frac{p^4}{r^3}$, then the left-hand side of (4.24) vanishes and from the right-hand side we have the condition

$$\frac{\ddot{r}}{r} - \frac{\ddot{p}}{p} + 3\frac{\dot{p}^2}{p^2} - 3\frac{\dot{p}\dot{r}}{pr} = 0. \tag{4.26}$$

When we consider the canonical version (4.25), we see that the infinitesimal generator takes the form

$$V = (c_1 t + c_2)\partial_t + (\frac{c_1}{3}x - \frac{\dot{\eta}_0}{2\sigma}\, y + \xi_0)\partial_x + (\frac{2c_1}{3}\, y + \eta_0)\partial_y - \left(\frac{2c_1}{3}\, u + y\frac{d}{dt}\Big(\frac{\dot{\eta}_0}{2\sigma}\Big) - \dot{\xi}_0\right)\partial_u. \tag{4.27}$$

Here $\xi_0(t)$ and $\eta_0(t)$ are arbitrary functions, and we have the following two restricting equations

$$(c_1 t + c_2)\sigma_t = 0, \qquad (c_1 t + c_2)q^3 = q_0, \tag{4.28}$$

where q_0 is an arbitrary constant. Let us consider the case $c_1 = c_2 = 0$, then there is no condition on $q(t)$ and $\sigma(t)$. The symmetry algebra is an infinite-dimensional one generated by the vector field

$$V = X(\xi) + Y(\eta), \tag{4.29}$$

where $X(\xi)$ and $Y(\eta)$ are given by

$$X(\xi) = \xi\partial_x + \dot{\xi}\partial_u, \tag{4.30}$$

$$Y(\eta) = -\frac{\dot{\eta}}{2\sigma}\, y\partial_x + \eta\partial_y - y\frac{d}{dt}\Big(\frac{\dot{\eta}}{2\sigma}\Big)\partial_u. \tag{4.31}$$

Using invariance under $Y(\eta)$, we perform a reduction of (4.25) seeking solutions in the form

$$u = -\frac{y^2}{2\eta}\frac{d}{dt}\Big(\frac{\dot{\eta}}{2\sigma}\Big) + F(z,t), \quad z = x + \frac{\dot{\eta}}{4\sigma\eta}\, y^2, \tag{4.32}$$

and the reduced equation reads

$$(F_t + FF_z + qF_{zz} + F_{zzz})_z + \frac{\dot{\eta}}{2\eta}F_z + \frac{\dot{\eta}\dot{\sigma}}{2\eta\sigma} - \frac{\ddot{\eta}}{2\eta} = 0. \tag{4.33}$$

Choosing $\dot{\eta}(t) = \sigma(t)$ the inhomogeneity is disposed of and we end up with the equation

$$(F_t + FF_z + qF_{zz} + F_{zzz})_z + \tilde{\sigma}(t)F_z = 0, \tag{4.34}$$

where $\tilde{\sigma}(t) = \frac{\sigma(t)}{2\int \sigma(t)\, dt}$. An integration gives a one-dimensional variable coefficient KP-Burgers equation

$$F_t + FF_z + qF_{zz} + F_{zzz} + \tilde{\sigma}(t)F = f(t).$$

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