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Representations of a_∞ and d_∞ with central charge 1 on the single neutral fermion Fock space $F^{\otimes \frac{1}{2}}$

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Abstract. We construct a new representation of the infinite rank Lie algebra a_∞ with central charge $c = 1$ on the Fock space $F^{\otimes \frac{1}{2}}$ of a single neutral fermion. We show that $F^{\otimes \frac{1}{2}}$ is a direct sum of irreducible integrable highest weight modules for a_∞ with central charge $c = 1$. We prove that as a_∞ modules $F^{\otimes \frac{1}{2}}$ is isomorphic to the Fock space $F^{\otimes 1}$ of the charged free fermions. As a corollary we obtain the decompositions of certain irreducible highest weight modules for d_∞ with central charge $c = \frac{1}{2}$ into irreducible highest weight modules for d_∞ with central charge $c = 1$.

1. Introduction

Our motivation for this paper was to better understand the various boson-fermion correspondences and their connection with the representation theory of certain infinite dimensional Lie algebras. The first to write on the topic of the relationship between a boson-fermion correspondence and representation theory were Date, Jimbo, Kashiwara, Miwa in [DKM81], [DJKM81a] and I. Frenkel in [Fre81]. Since then many attempts have been made to understand the boson-fermion correspondences as nothing else but an isomorphism of infinite dimensional Lie algebra modules. In his seminal paper I. Frenkel wrote: "The Boson-Fermion correspondence is nothing else but the **canonical** isomorphism between two realizations of the same representation of the affine Lie algebra $\hat{D}(2l)$ and in particular of its subalgebra $\widehat{gl}(l)$" (page 317 of [Fre81]). In the language that became commonly used later many had translated this to mean that the boson-fermion correspondence of type A is **just** an isomorphism between the vertex (the bosonic) and the spinor (the fermionic) realizations of the standard modules of $a_\infty = \widehat{gl}_\infty$ (the label "type A" derives from the $a_\infty = \widehat{gl}_\infty$, and is intended to distinguish this correspondence from the boson-fermion correspondence of type B for example, [DJKM81b]). This point of view of course had to be amended, as the charged free fermion Fock space $F^{\otimes 1}$ underlying the fermionic side of the boson-fermion correspondence of type A is actually an infinite direct sum of irreducible standard modules of $a_\infty = \widehat{gl}_\infty$ (for details on $F^{\otimes 1}$ see e.g. [Fre81], [KR87], [Kac98], [Wan99a], as well as Remark 3.2 in this paper). Starting with I. Frenkel's work in [Fre81], and later, the boson-fermion correspondence of type A (and the correspondence of type B) was related to different kinds of Howe-type dualities ([KWY98], [Wan99a], [Wan99b]). For instance, in [Wan99a] Wang wrote that "the $(GL_1, \hat{D})$ -duality in Theorem 5.3 is essentially the celebrated boson-fermion correspondence" ($\hat{D}$ denotes the universal central extension of the Lie algebra of differential operators on the circle, sometimes also labeled by $W_{1+\infty}$). But what we contend is that the boson-fermion correspondences are more than just isomorphisms between certain Lie algebra modules: a boson-fermion correspondence is first and foremost an isomorphism between two different chiral field theories, one fermionic (expressible in terms of free fermions and their descendants), the other bosonic (expressible in terms of exponentiated bosons). In fact, as I. Frenkel was careful to summarize in Theorem II.4.1 of his very influential paper [Fre81], "the canonical" isomorphism of the two $o(2l)$-current algebra modules in the bosonic and the fermionic Fock spaces **follows** from the boson-fermion correspondence (in fact that is what makes the isomorphism canonical), but not vice versa. But what we will show is that although the isomorphism of Lie algebra representations (and indeed various dualities) follow from a boson-fermion correspondence, the isomorphism as Lie algebra modules is **not equivalent** to a boson-fermion correspondence. To do that we consider a **single** neutral fermion Fock space $F^{\otimes \frac{1}{2}}$ and show that as modules for the Lie algebra a_∞, $F^{\otimes \frac{1}{2}} \cong F^{\otimes 1}$. Of course, it is known that even as super vertex algebras, and certainly as modules for the Lie algebras a_∞, d_∞ with central charge $c = 1$, as well as other affine Lie algebras, $F^{\otimes 1} \cong F^{\otimes \frac{1}{2}} \otimes F^{\otimes \frac{1}{2}}$. This fact is often and extensively used in many papers on vertex algebras, and it was once again I. Frenkel who used it first in [Fre81] in connection to representation theory. But, the representations of a_∞, d_∞ and other affine algebras on $F^{\otimes 1} \cong F^{\otimes \frac{1}{2}} \otimes F^{\otimes \frac{1}{2}}$ that are known in the literature do not reduce to representations on each of the $F^{\otimes \frac{1}{2}}$ factors. What was known is that $F^{\otimes \frac{1}{2}}$ is a representation of the Lie algebra d_∞ with central charge $c = \frac{1}{2}$ (this is one of the explanations for the label $\frac{1}{2}$ in $F^{\otimes \frac{1}{2}}$, the other being that $F^{\otimes \frac{1}{2}}$ is only a "half-infinite" Fock space, as opposed to $F^{\otimes 1}$). This is then what we do in this paper: First, we build a fermionic (spinor) representation of a_∞ with central charge $c = 1$ on $F^{\otimes \frac{1}{2}}$. Next we show how this representation decomposes into irreducible highest weight modules, which ultimately shows that as modules for the Lie algebra a_∞ with central charge $c = 1$, $F^{\otimes \frac{1}{2}} \cong F^{\otimes 1}$. This shows that it is not the a_∞-module structure that distinguishes these spaces—$F^{\otimes \frac{1}{2}}$ and $F^{\otimes 1}$ are identical as vector spaces, or even as a_∞ Lie algebra modules. The difference is in the vertex algebra structure (field theory) on $F^{\otimes \frac{1}{2}}$, versus the vertex algebra structure on $F^{\otimes 1}$. The field theory on $F^{\otimes 1}$ is local in the usual sense (at $z = w$, or as we can refer to it, 1-point local, see Definition 2.2); or more precisely $F^{\otimes 1}$ has a **super vertex algebra** structure (see e.g. [Kac98], [LL04], [FBZ04] for a precise definition of a super vertex algebra). On the other hand, even though $F^{\otimes \frac{1}{2}}$ has a super vertex algebra structure, this super vertex algebra structure is not enough to produce the new representations that we obtain below– to do that we at the minimum need to introduce 2-point locality (i.e., the fields we consider on $F^{\otimes \frac{1}{2}}$ are allowed to be multi-local, at both $z = w$ and $z = -w$). More precisely, there is a **twisted vertex algebra** structure on $F^{\otimes \frac{1}{2}}$ (see [Ang12], [ACJ13] for a precise definition of a twisted vertex algebra). This shows that the type of vertex algebra structure on $F^{\otimes 1}$ versus $F^{\otimes \frac{1}{2}}$ is of great importance, in particular the set of points of locality is a necessary part of the data describing any boson-fermion correspondence.

The outlay of the paper is as follows. First, we recall the necessary definitions and technical tools in Section 2. In Section 3 we introduce the infinite Lie algebras that we will work with, and the Fock space $F^{\otimes \frac{1}{2}}$ and its different gradings. Next we show that $F^{\otimes \frac{1}{2}}$ is a module for the Lie algebra a_∞ with central charge $c = 1$, and by restriction for the Lie algebra d_∞ with central charge $c = 1$. Next we show that each homogeneous component of $F^{\otimes \frac{1}{2}}$ is a highest weight module for a_∞ with central charge $c = 1$, which is moreover irreducible. That allows us to show that $F^{\otimes \frac{1}{2}}$ is completely reducible and to show its decomposition in terms of irreducible modules for a_∞ with central charge $c = 1$. Hence we can compare and conclude that as a_∞ modules with central charge $c = 1$ $F^{\otimes \frac{1}{2}} \cong F^{\otimes 1}$. Finally as a corollary we obtain the decomposition of certain $c = \frac{1}{2}$ modules for d_∞ in terms of irreducible highest weight d_∞ with central charge $c = 1$.

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2. Notation and background

We work over the field of complex numbers $\mathbb{C}$.

The mathematical definitions of a field in a chiral quantum field theory and normal ordered products of fields are well known, they can be found for instance in [FLM88], [FHL93], [Kac98], [LL04] and others, we include them for completeness:

Definition 2.1 (Field) *A field $a(z)$ on a vector space V is a series of the form*

$$a(z) = \sum_{n\in\mathbf{Z}} a_{(n)} z^{-n-1}, \quad a_{(n)} \in \operatorname{End}(V), \quad \textit{such that} \ \ a_{(n_v)}v = 0 \ \ \textit{for any } v \in V,\ n_v \gg 0. \tag{2.1}$$

Denote

$$a(z)_- := \sum_{n\geq 0} a_n z^{-n-1}, \quad a(z)_+ := \sum_{n<0} a_n z^{-n-1}. \tag{2.2}$$

Definition 2.2 *([ACJ13])* **(N-point local fields)** *Let ϵ be a primitive Nth root of unity. We say that a field $a(z)$ on a vector space V is* **even** *and N-point self-local at $1, \epsilon, \epsilon^2, \ldots, \epsilon^{N-1}$, if there exist $n_0, n_1, \ldots, n_{N-1} \in \mathbb{Z}_{\geq 0}$ such that*

$$(z-w)^{n_0}(z-\epsilon w)^{n_1}\cdots(z-\epsilon^{N-1}w)^{n_{N-1}}[a(z), a(w)] = 0. \tag{2.3}$$

In this case we set the **parity** *$p(a(z))$ of $a(z)$ to be 0.*
We set $\{a, b\} := ab + ba$. We say that a field $a(z)$ on V is N-point self-local at $1, \epsilon, \epsilon^2, \ldots, \epsilon^{N-1}$ and **odd** *if there exist $n_0, n_1, \ldots, n_{N-1} \in \mathbb{Z}_{\geq 0}$ such that*

$$(z-w)^{n_0}(z-\epsilon w)^{n_1}\cdots(z-\epsilon^{N-1}w)^{n_{N-1}}\{a(z), a(w)\} = 0. \tag{2.4}$$

In this case we set the **parity** *$p(a(z))$ to be 1. For brevity we will just write $p(a)$ instead of $p(a(z))$.*
Finally, if $a(z), b(z)$ are fields on V, we say that $a(z)$ and $b(z)$ are N-point mutually local at $1, \epsilon, \epsilon^2, \ldots, \epsilon^{N-1}$ if there exist $n_0, n_1, \ldots, n_{N-1} \in \mathbb{Z}_{\geq 0}$ such that

$$(z-w)^{n_0}(z-\epsilon w)^{n_1}\cdots(z-\epsilon^{N-1}w)^{n_{N-1}}\Big(a(z)b(w) - (-1)^{p(a)p(b)}b(w)a(z)\Big) = 0. \tag{2.5}$$

Definition 2.3 (Normal ordered product) *Let $a(z), b(z)$ be fields on a vector space V. Define*

$$: a(z)b(w) := a(z)_+ b(w) + (-1)^{p(a)p(b)} b(w) a_-(z). \tag{2.6}$$

One calls this the "normal ordered product" of $a(z)$ and $b(w)$.

Remark 2.4 Let $a(z), b(z)$ be any fields on a vector space V. Then
$: a(z)b(\lambda z) :$ and $: a(\lambda z)b(z) :$ are well defined fields on V for any $\lambda \in \mathbb{C}^*$.

For a rational function $f(z, w)$, with poles only at $z = 0$, $z = \epsilon^i w,\ 0 \leq i \leq N-1$, we denote by $i_{z,w} f(z, w)$ the expansion of $f(z, w)$ in the region $|z| \gg |w|$ (the region in the complex z plane outside of all the points $z = \epsilon^i w,\ 0 \leq i \leq N-1$), and correspondingly for $i_{w,z} f(z, w)$. The mathematical background of the well-known and often used (both in physics and in mathematics) notion of Operator Product Expansion (OPE) of product of two fields for the case of usual locality ($N = 1$) has been established for example in [Kac98], [LL04]. The following lemma extended the mathematical background to the case of N-point locality:

Lemma 2.5 *([ACJ13])* **(Operator Product Expansion (OPE) of N-point local fields)** *Let $a(z)$, $b(w)$ be N-point mutually local fields on a vector space V. Then exists fields $c_{jk}(w)$, $j = 0, \dots, N-1; k = 0, \dots, n_j - 1$, such that we have*

$$a(z)b(w) = i_{z,w} \sum_{j=0}^{N-1} \sum_{k=0}^{n_j-1} \frac{c_{jk}(w)}{(z - \epsilon^j w)^{k+1}} + :a(z)b(w):. \tag{2.7}$$

We call the fields $c_{jk}(w)$, $j = 0, \dots, N-1; k = 0, \dots, n_j - 1$ OPE coefficients. We will write the above OPE as

$$a(z)b(w) \sim \sum_{j=0}^{N-1} \sum_{k=0}^{n_j-1} \frac{c_{jk}(w)}{(z - \epsilon_j w)^{k+1}}. \tag{2.8}$$

The $\sim$ signifies that we have only written the singular part, and also we have omitted writing explicitly the expansion $i_{z,w}$, which we do acknowledge tacitly. Often also the following notation is used for short:

$$\lfloor ab \rfloor = a(z)b(w) - :a(z)b(w): = [a(z)_-, b(w)], \tag{2.9}$$

i.e., the *contraction* of any two fields $a(z)$ and $b(w)$ is in fact also the $i_{z,w}$ expansion of the singular part of the OPE of the two fields $a(z)$ and $b(w)$.

The OPE expansion of the product of two fields is very convenient, as it completely determines in a very compact manner the commutation relations between the modes of the two fields, and we will use it extensively in what follows. In particular, extending of the OPEs to the case of N-point local fields allows us to extend and use Wick's Theorem for N-point local fields:

Theorem 2.6 (Wick's Theorem, [BS83], [Hua98] or [Kac98]) *Let $a(z)$, $b(w)$ be N-point mutually local fields on a vector space V, satisfying*

(i) $[\lfloor a^i(z) b^j(w) \rfloor, c^k(x)_\pm] = [\lfloor a^i b^j \rfloor, c^k(x)_\pm] = 0$, *for all i, j, k and $c^k(x) = a^k(z)$ or $c^k(x) = b^k(w)$.*

(ii) $[a^i(z)_\pm, b^j(w)_\pm] = 0$ *for all i and j.*

Then

$$:a^1(z) \cdots a^M(z)::b^1(w) \cdots b^N(w): = \sum_{s=0}^{\min(M,N)} \sum_{\substack{i_1 < \cdots < i_s, \\ j_1 \neq \cdots \neq j_s}} \pm \lfloor a^{i_1} b^{j_1} \rfloor \cdots \lfloor a^{i_s} b^{j_s} \rfloor :a^1(z) \cdots a^M(z) b^1(w) \cdots b^N(w):_{(i_1, \dots, i_s; j_1, \dots, j_s)}.$$

Here the subscript $(i_1, \dots, i_s; j_1, \dots, j_s)$ means that those factors $a^i(z)$, $b^j(w)$ with indices $i \in \{i_1, \dots, i_s\}$, $j \in \{j_1, \dots, j_s\}$ are to be omitted from the product $:a^1 \cdots a^M b^1 \cdots b^N:$ and when $s = 0$ we do not omit any factors. The plus or minus sign is determined as follows: each permutation of an adjacent odd field changes the sign.

3. The Fock space $F^{\otimes \frac{1}{2}}$ and representations of a_∞ and d_∞ with central charge 1

We recall the definitions and notation for the Fock space $F^{\otimes \frac{1}{2}}$ and the double-infinite rank Lie algebras a_∞ and d_∞ as in [Fre81], [DJKM81a], [Kac90], [Wan99b]; in particular we follow the notation of [Wan99b], [Wan99a].

Consider a single odd self-local field $\phi^D(z)$, which we index in the form $\phi^D(z) = \sum_{n \in \mathbb{Z} + \frac{1}{2}} \phi^D_n z^{-n-\frac{1}{2}}$. The OPE of $\phi^D(z)$ is given by

$$\phi^D(z)\phi^D(w) \sim \frac{1}{z - w}. \tag{3.1}$$

This OPE completely determines the commutation relations between the modes ϕ_n^D, $n \in \mathbb{Z} + \frac{1}{2}$:

$$\{\phi_m^D, \phi_n^D\} = \phi_m^D \phi_n^D + \phi_n^D \phi_m^D = \delta_{m,-n} 1. \tag{3.2}$$

and so the modes generate a Clifford algebra Cl_D. The field $\phi^D(z)$ is usually called a "neutral fermion field". Now Cl_D has basis consisting of 1 and the products $\phi_{i_1}^D \phi_{i_2}^D \cdots \phi_{i_k}^D$ where $i_1 < i_2 < \cdots < i_k$, $i_j \in \mathbb{Z} + 1/2$. We introduce a $\mathbb{Z}$-grading dg on Cl_D by defining the following degree of a basis element:

$$dg(1) = 0,$$
$$dg(\phi^D_{n_k - \frac{1}{2}} \dots \phi^D_{n_2 - \frac{1}{2}} \phi^D_{n_1 - \frac{1}{2}}) = \#\{i = 1, 2, \dots, k |\ n_i = \text{odd}\} - \#\{i = 1, 2, \dots, k |\ n_i = \text{even}\}.$$

Lemma 3.1 *The $\mathbb{Z}$-grading of Cl_D is an algebra grading. Furthermore, the operation left multiplication by an element $\phi_n^D \in CL_D$ for any $n \in \mathbb{Z} + 1/2$ is a homogenous operator on CL_D, of degree 1 if $n = 2k + 1/2$, and of degree -1 if $n = 2k - 1/2$ for some integer k.*

Proof: For any pair ϕ_m, ϕ_n $(m \neq n)$ we claim $dg(\phi_m \phi_n) = dg(\phi_n \phi_m)$. If $m \neq -n$ then $\phi_m \phi_n = -\phi_n \phi_m$, and one of these expressions appears in the given basis so determines the degree of both $\phi_m \phi_n$ and $\phi_n \phi_m$. If $m = -n$, $\phi_m \phi_n = 1 - \phi_n \phi_m$, and again, either the left or right hand side is a sum of basis vectors, since $dg(1) = 0$ the degree of $\phi_m \phi_n$ and $\phi_n \phi_m$ again agree. Thus the Clifford algebra relation (3.2) is compatible with the definition of dg. Now it is obvious from the definition of the grading that as an operator left multiplication by $\phi_n^D \in CL_D$ is a homogeneous operator of the given degree. □

The Fock space of the field $\phi^D(z)$ is the highest weight module of Cl_D with vacuum vector $|0\rangle$, so that $\phi_n^D |0\rangle = 0$ for $n > 0$. It is denoted by $F^{\otimes \frac{1}{2}}$ (see e.g. [DJKM81b], [FFR91], [KW94], [Wan99a], [Wan99b], [KWY98]). $F^{\otimes \frac{1}{2}}$ has basis

$$\{\phi^D_{-n_k - \frac{1}{2}} \dots \phi^D_{-n_2 - \frac{1}{2}} \phi^D_{-n_1 - \frac{1}{2}} |0\rangle, |0\rangle \big|\ n_k > \dots > n_2 > n_1 \geq 0, n_i \in \mathbb{Z}, i = 1, 2, \dots, k\} \tag{3.3}$$

The space $F^{\otimes \frac{1}{2}}$ has a $\mathbb{Z}_2$ grading given by $k\ mod\ 2$,

$$F^{\otimes \frac{1}{2}} = F_{\bar{0}}^{\otimes \frac{1}{2}} \oplus F_{\bar{1}}^{\otimes \frac{1}{2}},$$

where $F_{\bar{0}}^{\otimes \frac{1}{2}}$ (resp. $F_{\bar{1}}^{\otimes \frac{1}{2}}$) denote the even (resp. odd) components of $F^{\otimes \frac{1}{2}}$. This $\mathbb{Z}_2$ grading can be extended to a $\mathbb{Z}_{\geq 0}$ grading $\tilde{L}$, called "length", by setting

$$\tilde{L}(\phi^D_{-n_k - \frac{1}{2}} \dots \phi^D_{-n_2 - \frac{1}{2}} \phi^D_{-n_1 - \frac{1}{2}} |0\rangle) = k. \tag{3.4}$$

The space $F^{\otimes \frac{1}{2}}$ can be given a super vertex algebra structure, as is known from e.g. [FFR91], [KW94], [Kac98].

The $\mathbb{Z}$ grading dg on Cl_D induces a $\mathbb{Z}$ grading dg on $F^{\otimes \frac{1}{2}}$ by assigning $dg(|0\rangle) = 0$ and

$$\begin{aligned} dg(\phi^D_{-n_k - \frac{1}{2}} \dots \phi^D_{-n_2 - \frac{1}{2}} \phi^D_{-n_1 - \frac{1}{2}} |0\rangle) = \#\{i = 1, 2, \dots, k |\ n_i = \text{odd}\} \\ - \#\{i = 1, 2, \dots, k |\ n_i = \text{even}\}. \end{aligned} \tag{3.5}$$

Denote the space of homogenous elements of degree $dg = n \in \mathbb{Z}$ by $F_{(n)}^{\otimes \frac{1}{2}}$, hence as vector spaces we have

$$F^{\otimes \frac{1}{2}} = \oplus_{n \in \mathbb{Z}} F_{(n)}^{\otimes \frac{1}{2}}. \tag{3.6}$$

Introduce also the special vectors $v_n \in F^{\otimes \frac{1}{2}}_{(n)}$ defined by

$$v_0 = |0\rangle \in F^{\otimes \frac{1}{2}}_{(0)}; \tag{3.7}$$

$$v_n = \phi^D_{-2n+1-\frac{1}{2}} \cdots \phi^D_{-3-\frac{1}{2}} \phi^D_{-1-\frac{1}{2}} |0\rangle \in F^{\otimes \frac{1}{2}}_{(n)}, \quad \text{for } n > 0; \tag{3.8}$$

$$v_{-n} = \phi^D_{-2n+2-\frac{1}{2}} \cdots \phi^D_{-2-\frac{1}{2}} \phi^D_{-\frac{1}{2}} |0\rangle \in F^{\otimes \frac{1}{2}}_{(-n)}, \quad \text{for } n > 0. \tag{3.9}$$

Note that the vectors $v_n \in F^{\otimes \frac{1}{2}}_{(n)}$ have minimal length $\tilde{L} = |n|$ among the vectors within $F^{\otimes \frac{1}{2}}_{(n)}$, and they are in fact the unique (up-to a scalar) vectors minimizing the length $\tilde{L}$, such that the index n_k is minimal too.

The Lie algebra $\bar{a}_\infty$ (sometimes denoted $\bar{gl}_\infty$ or just $\mathfrak{gl}$, see for instance [Wan99a], [Wan99b], [KWY98]) is the Lie algebra of infinite matrices

$$\bar{a}_\infty = \{(a_{ij}) |\ i, j \in \mathbb{Z},\ a_{ij} = 0 \text{ for} |i - j| \gg 0\}. \tag{3.10}$$

As usual denote the elementary matrices by E_{ij}.

The algebra a_∞ (often denoted also by $\widehat{gl}_\infty$ or $\widehat{\mathfrak{gl}}$) is a central extension of $\bar{a}_\infty$ by a central element c, $a_\infty = \bar{a}_\infty \oplus \mathbb{C}c$, with cocycle given by

$$C(A, B) = Trace([J, A]B), \tag{3.11}$$

where the matrix $J = \sum_{i \leq 0} E_{ii}$. In particular

$$\begin{aligned} &C(E_{ij}, E_{ji}) = -C(E_{ji}, E_{ij}) = 1, \quad \text{if } i \leq 0,\ \ j \geq 1 \\ &C(E_{ij}, E_{kl}) = 0 \quad \text{ in all other cases.} \end{aligned}$$

The commutation relations for the elementary matrices in a_∞ are

$$[E_{ij}, E_{kl}] = \delta_{jk} E_{il} - \delta_{li} E_{kj} + C(E_{ij}, E_{kl})c.$$

The non-central generators have generating series

$$E^A(z, w) = \sum_{i,j \in \mathbb{Z}} E_{ij} z^{i-1} w^{-j}, \tag{3.12}$$

and relations

$$\begin{aligned} [E^A(z_1, w_1), E^A(z_2, w_2)] &= E^A(z_1, w_2)\delta(z_2 - w_1) - E^A(z_2, w_1)\delta(z_1 - w_2) \\ &+ \iota_{z_1,w_2} \frac{1}{z_1 - w_2} \iota_{w_1,z_2} \frac{1}{w_1 - z_2} c - \iota_{w_2,z_1} \frac{1}{z_1 - w_2} \iota_{z_2,w_1} \frac{1}{w_1 - z_2} c. \end{aligned} \tag{3.13}$$

Here we used the formal delta function notation $\delta(z - w) := \sum_{n \in \mathbb{Z}} z^n w^{-n-1} = \delta(w - z)$ (see e.g. [Kac98], [FBZ04], [ACJ13]).

Further, a_∞ has a triangular decomposition

$$a_\infty = a^-_\infty \oplus a^0_\infty \oplus a^+_\infty. \tag{3.14}$$

Here $a^\pm_\infty$ consists of correspondingly the strictly upper (strictly lower) triangular infinite matrices; $a^0_\infty = \mathfrak{gl}_0 \oplus \mathbb{C}c$ where $\mathfrak{gl}_0$ denotes the diagonal matrices.

The root system of a_∞ is $\Delta = \{\epsilon_i - \epsilon_j \,|\, i,j \in \mathbb{Z}, i \neq j\}$ where $\epsilon_i \in (\mathfrak{gl}_0)^*$ is defined by $\epsilon_i(E_{jj}) = \delta_{ij}$ $(i,j \in \mathbb{Z})$. There is a conjugate linear, involutive anti-automorphism $\omega \in \mathrm{End}(a_\infty)$ defined by $\omega(E_{ij}) = E_{ji}$ and this is called "the compact anti-involution".

For $¢ \in \mathbb{C}$ and $\Lambda \in \bigoplus_{i\in\mathbb{Z}, i\neq 0}(\mathbb{C}E_{ii})^*$, set

$$\begin{aligned} {}^a\lambda_i &:= \Lambda(E_{ii}) \\ {}^aH_i &:= E_{ii} - E_{i+1,i+1} + \delta_{i,0}c \\ {}^ah_i &:= \Lambda({}^aH_i) = {}^a\lambda_i - {}^a\lambda_{i+1} + \delta_{i,0}¢ \end{aligned}$$

Define ${}^a\Lambda_j \in (a_\infty^0)^*$ by

$$\begin{aligned} {}^a\Lambda_j(E_{ii}) &= \begin{cases} 1, & \text{for } 0 < i \le j, \\ -1, & \text{for } j < i \le 0, \\ 0, & \text{otherwise}, \end{cases} \\ {}^a\Lambda_j(c) &= 0. \end{aligned}$$

Define also ${}^a\hat{\Lambda}_0 \in (a_\infty^0)^*$ by ${}^a\hat{\Lambda}_0(c) = 1$, ${}^a\hat{\Lambda}_0(E_{ii}) = 0$ for $i \in \mathbb{Z}$. Then the i-th fundamental weight is

$${}^a\hat{\Lambda}_j = {}^a\Lambda_j + {}^a\hat{\Lambda}_0, \quad i \in \mathbb{Z}.$$

Let $L(a_\infty; {}^a\Lambda, ¢) = L(\widehat{\mathfrak{gl}}_\infty; {}^a\Lambda, ¢)$ denote the highest weight a_∞-module with highest weight Λ and central charge $¢$.

The algebra $\bar{d}_\infty$ is defined as the **subalgebra** of $\bar{a}_\infty$, consisting of the infinite matrices preserving the bilinear form $D(v_i, v_j) = \delta_{i,1-j}$, i.e.,

$$\bar{d}_\infty = \{(a_{ij}) \in \bar{a}_\infty |\ a_{ij} = -a_{1-j,1-i}\}. \tag{3.15}$$

Denote by d_∞ the central extension of $\bar{d}_\infty$ by a central element c, $d_\infty = \bar{d}_\infty \oplus \mathbb{C}c$, with the same cocycle as for a_∞, (3.11). The commutation relations for the elementary matrices in d_∞ are obtained using the relations in a_∞: [1]

$$[E_{ij}, E_{kl}] = \delta_{jk}E_{il} - \delta_{li}E_{kj} + C(E_{ij}, E_{kl})c.$$

The generators for the algebra d_∞ can be written in terms of these elementary matrices as:

$$\{E_{i,j} - E_{1-j,1-i},\ i,j \in \mathbb{Z}; \text{and}\ \ c\}.$$

We can arrange the non-central generators in a generating series

$$E^D(z,w) = \sum_{i,j\in\mathbb{Z}} (E_{ij} - E_{1-j,1-i})z^{i-1}w^{-j}. \tag{3.16}$$

The generating series $E^D(z,w)$ obeys the following relations:

$$E^D(z,w) = -E^D(w,z)$$

and

$$\begin{aligned} [E^D(z_1,w_1), E^D(z_2,w_2)] &= E^D(z_1,w_2)\delta(z_2-w_1) - E^D(z_2,w_1)\delta(z_1-w_2) \\ &\quad + E^D(w_2,w_1)\delta(z_1-z_2) - E^D(z_1,z_2)\delta(w_1-w_2) \\ &+ 2\iota_{z_1,w_2}\frac{1}{z_1-w_2}\iota_{w_1,z_2}\frac{1}{w_1-z_2}c - 2\iota_{z_2,w_1}\frac{1}{z_2-w_1}\iota_{w_2,z_1}\frac{1}{w_2-z_1}c \\ &- 2\iota_{z_1,z_2}\frac{1}{z_1-z_2}\iota_{w_1,w_2}\frac{1}{w_1-w_2}c + 2\iota_{z_2,z_1}\frac{1}{z_1-z_2}\iota_{w_2,w_1}\frac{1}{w_2-w_1}c. \end{aligned}$$

[1] Note that in [Kac90] the commutation relation $[E_{ij}, E_{kl}] = \delta_{jk}E_{il} - \delta_{li}E_{kj} + \frac{1}{2}C(E_{ij}, E_{kl})c$ is used instead.

The assignment $E(z,w) \mapsto :\phi^D(z)\phi^D(w):$, $c \mapsto \frac{1}{2} Id_{F^{\otimes \frac{1}{2}}}$ gives a representation of the Lie algebra d_∞ on $F^{\otimes \frac{1}{2}}$ (see e.g. [DJKM81b], [KW94], [KWY98]), which we denote by $r_{\frac{1}{2}}$. Further, it is known (see e.g. [KW94], [KWY98], [Wan99a]) that as d_∞ modules

$$F_{\bar{0}}^{\otimes \frac{1}{2}} \cong L(d_\infty; {}^d\hat{\Lambda}_0, \frac{1}{2}); \quad F_{\bar{1}}^{\otimes \frac{1}{2}} \cong L(d_\infty; {}^d\hat{\Lambda}_1, \frac{1}{2}). \tag{3.17}$$

where $L(d_\infty; {}^d\Lambda, \cent)$ denotes the highest weight d_∞-module with highest weight ${}^d\Lambda$ and central charge $\cent$. The highest weights are defined by the following, using the symmetry in d_∞:

$${}^d\hat{\Lambda}_0(E_{i,i} - E_{1-i,1-i}) = 0 \quad \text{for any} \quad i \in \mathbb{Z};$$
$${}^d\hat{\Lambda}_1(E_{i,i} - E_{1-i,1-i}) = 1; \quad \text{for } i = 1; \quad {}^d\hat{\Lambda}_1(E_{i,i} - E_{1-i,1-i}) = 0 \quad \text{for } i \neq 0, 1;$$
$${}^d\hat{\Lambda}_0(c) = {}^d\hat{\Lambda}_1(c) = \frac{1}{2}.$$

As a d_∞-module with central charge $c = \frac{1}{2}$ $F^{\otimes \frac{1}{2}}$ then decomposes as

$$F^{\otimes \frac{1}{2}} = F_{\bar{0}}^{\otimes \frac{1}{2}} \oplus F_{\bar{1}}^{\otimes \frac{1}{2}} \cong L(d_\infty; {}^d\hat{\Lambda}_0, \frac{1}{2}) \oplus L(d_\infty; {}^d\hat{\Lambda}_1, \frac{1}{2}),$$

Next we will show that $F^{\otimes \frac{1}{2}}$ is also a module for a_∞ (and thus d_∞) with central charge $c = 1$.

Remark 3.2 It is well known (in the context of representation theory it was introduced by I. Frenkel in [Fre81] and extensively used afterwards) that

$$F^{\otimes \frac{1}{2}} \otimes F^{\otimes \frac{1}{2}} \cong F^{\otimes 1};$$

where $F^{\otimes 1}$ is the Fock space of **1 pair** of two charged fermions. The two charged fermions are the fields $\psi^+(z)$ and $\psi^-(z)$ with operator product expansions (OPEs):

$$\psi^+(z)\psi^-(w) \sim \frac{1}{z-w} \sim \psi^-(z)\psi^+(w), \quad \psi^+(z)\psi^+(w) \sim 0 \sim \psi^-(z)\psi^-(w)$$

where the 1 above denotes the identity map $Id_{F^{\otimes 1}}$. The modes ψ_n^+ and ψ_n^-, $n \in \mathbb{Z}$ of the fields $\psi^+(z)$ and $\psi^-(z)$, which we index as follows:

$$\psi^+(z) = \sum_{n\in\mathbf{Z}} \psi_n^+ z^{-n-1}, \quad \psi^-(z) = \sum_{n\in\mathbf{Z}} \psi_n^- z^{-n-1}, \tag{3.18}$$

form a Clifford algebra Cl_A with relations

$$\{\psi_m^+, \psi_n^-\} = \delta_{m+n,-1}1, \quad \{\psi_m^+, \psi_n^+\} = \{\psi_m^-, \psi_n^-\} = 0. \tag{3.19}$$

The Fock space $F^{\otimes 1}$ is the highest weight representation of Cl_A generated by the vacuum vector $|0\rangle$, so that $\psi_n^+|0\rangle = \psi_n^-|0\rangle = 0$ for $n \geq 0$ (see e.g. [Fre81], [KR87], [KW94], [Kac98] for more details on $F^{\otimes 1}$). It is well known (see e.g., [Kac98], [FBZ04], [LL04]) that $F^{\otimes 1}$ has a structure of a super vertex algebra (i.e., with a single point of locality at $z = w$ in the OPEs); this vertex algebra is often called "charged free fermion vertex algebra". It is also well known (introduced by I. Frenkel, [Fre81]; and [DJKM81a]) and extensively used (e.g., [KR87], [FFR91], [Kac90], [Wan99b], [Wan99a] among many others) that $F^{\otimes 1}$ is a module for the a_∞ algebra, moreover

$$F^{\otimes 1} \cong \oplus_{n\in\mathbb{Z}} L(a_\infty; {}^a\hat{\Lambda}_n, 1).$$

This isomorphism has often been referred to as equivalent to the well known charged free boson-fermion correspondence (an isomorphism between the super vertex algebra on $F^{\otimes 1}$ and the super vertex algebra of the rank one odd lattice). Here we are concerned with a more subtle point: a boson-fermion correspondence is not just an equivalence of Lie algebra representations, but an isomorphism of appropriate field theories (or vertex algebra structures). As shown and used by I. Frenkel ([Fre81]) one has

$$F^{\otimes 1} \cong F^{\otimes \frac{1}{2}} \otimes F^{\otimes \frac{1}{2}},$$

which is implemented by

$$\phi_1(z) = \frac{1}{\sqrt{2}}\left(\psi^+(z) + \psi^-(z)\right), \ \phi_2(z) = \frac{i}{\sqrt{2}}\left(\psi^+(z) - \psi^-(z)\right).$$

In today's language we would say that this fields map generates an isomorphism of super vertex algebras, a fact which is extensively used by many authors (e.g., [FFR91], [MTZ08], [Bar11] among many). But as we will show below, as a_∞ modules

$$F^{\otimes 1} \cong F^{\otimes \frac{1}{2}} \cong \oplus_{n\in\mathbb{Z}} L(a_\infty; {}^a\hat{\Lambda}_n, 1).$$

In other words, as vector spaces and as a_∞ modules $F^{\otimes \frac{1}{2}}$ and $F^{\otimes 1}$ are indistinguishable. On the other hand, in the case of $F^{\otimes 1}$ the boson-fermion correspondence (of type A) is an isomorphism of **super vertex algebras**, requiring locality only at $z = w$ (see e.g., [Kac98]). But, in the case of $F^{\otimes \frac{1}{2}}$ the boson-fermion correspondence (of type D-A) is an isomorphism of **twisted vertex algebras**, requiring locality at both $z = w$ and $z = -w$ (see [Ang12], [ACJ13]).

We obtain a representation of a_∞ on $F^{\otimes \frac{1}{2}}$ by introducing fields arising from a 2-point local twisted vertex algebra:

Proposition 3.3 *Let*

$$\phi^{+DA}(z) = \frac{\phi^D(z) - \phi^D(-z)}{2}, \quad \phi^{-DA}(z) = \frac{\phi^D(z) + \phi^D(-z)}{2}. \tag{3.20}$$

The assignment $zE(z^2, w^2) \mapsto :\phi^{+DA}(z)\phi^{-DA}(w):,\ \ c \mapsto Id_{F^{\otimes \frac{1}{2}}}$ *gives a representation of the Lie algebra* a_∞ *on* $F^{\otimes \frac{1}{2}}$ *with central charge* $c = 1$.

Proof: We will use Wick's Theorem. We have $\phi^{+DA}(z) = \sum_{n\in\mathbb{Z}} \phi^D_{-2n+\frac{1}{2}} z^{2n-1}$ and $\phi^{-DA}(z) = \sum_{n\in\mathbb{Z}} \phi^D_{-2n-\frac{1}{2}} z^{2n}$, thus the modes $\phi^{+DA}(z)$ (resp. $\phi^{-DA}(z)$) are the operator coefficients of $\phi^D(z)$ in front of odd (resp. even) powers of the formal variable z. Hence, since $\phi^D(z)$ obeys the second condition of Wick's theorem, $\phi^{+DA}(z)$ and $\phi^{-DA}(z)$ obey it too. We also have

$$\phi^{+DA}(z)\phi^{+DA}(w) \sim 0, \quad \phi^{-DA}(z)\phi^{-DA}(w) \sim 0; \tag{3.21}$$

$$\phi^{+DA}(z)\phi^{-DA}(w) \sim \frac{1}{2}\left(\frac{1}{z-w} + \frac{1}{z+w}\right) \sim \frac{z}{z^2 - w^2}; \tag{3.22}$$

$$\phi^{-DA}(z)\phi^{+DA}(w) \sim \frac{1}{2}\left(\frac{1}{z-w} - \frac{1}{z+w}\right) \sim \frac{w}{z^2 - w^2}; \tag{3.23}$$

hence the first condition of Wick's theorem is also satisfied. Thus from Wick's theorem we have

$$\begin{aligned}
&:\phi^{+DA}(z_1)\phi^{-DA}(w_1)::\phi^{+DA}(z_2)\phi^{-DA}(w_2): \\
&\quad \sim \frac{z_1}{z_1^2 - w_2^2} :\phi^{-DA}(w_1)\phi^{+DA}(z_2): + \frac{z_2}{w_1^2 - z_2^2} :\phi^{+DA}(z_1)\phi^{-DA}(w_2): \\
&\quad + \frac{z_1}{z_1^2 - w_2^2}\frac{z_2}{w_1^2 - z_2^2}.
\end{aligned}$$

Hence

$$\begin{aligned}&[:\phi^{+DA}(z_1)\phi^{-DA}(w_1):,:\phi^{+DA}(z_2)\phi^{-DA}(w_2):]\\&=z_1\delta(z_1^2-w_2^2):\phi^{-DA}(w_1)\phi^{+DA}(z_2):+z_2\delta(z_2^2-w_1^2):\phi^{+DA}(z_1)\phi^{-DA}(w_2):\\&\quad+\iota_{z_1,w_2}\frac{z_1}{z_1^2-w_2^2}\iota_{w_1,z_2}\frac{z_2}{w_1^2-z_2^2}-\iota_{w_2,z_1}\frac{z_1}{z_1^2-w_2^2}\iota_{z_2,w_1}\frac{z_2}{w_1^2-z_2^2};\end{aligned}$$

and we use the fact that $:\phi^{-DA}(w)\phi^{+DA}(z):=-:\phi^{+DA}(z)\phi^{-DA}(w):$. □

We will denote this new representation on $F^{\otimes\frac{1}{2}}$ by r_1. Since d_∞ is a subalgebra of a_∞, we have the following

Corollary 3.4 *$F^{\otimes\frac{1}{2}}$ is a module for d_∞ with central charge $c=1$ via the restriction of the representation r_1.*

If we introduce a normal ordered product $:\phi^D_m\phi^D_n:$ on the modes ϕ^D_m of the field $\phi^D(z)$, compatible with the normal ordered product of fields (Definition 2.3), we have to have

$$:\phi^D(z)\phi^D(w):=\sum_{m,n\in\mathbf{Z}+\frac{1}{2}}:\phi^D_{-m-\frac{1}{2}}\phi^D_{-n-\frac{1}{2}}:z^mw^n,$$

and thus

$$:\phi^D_{-m-\frac{1}{2}}\phi^D_{-n-\frac{1}{2}}:=\begin{cases}\phi^D_{-m-\frac{1}{2}}\phi^D_{-n-\frac{1}{2}} & \text{for } m+n\neq 1\\ \phi^D_{-m-\frac{1}{2}}\phi^D_{-n-\frac{1}{2}}-1=-\phi^D_{-n-\frac{1}{2}}\phi^D_{-m-\frac{1}{2}} & \text{for } m+n=-1, n\geq 0,\\ \phi^D_{-m-\frac{1}{2}}\phi^D_{-n-\frac{1}{2}} & \text{for } m+n=-1, m\geq 0.\end{cases} \tag{3.24}$$

Hence the well known representation $r_{\frac{1}{2}}$ of d_∞ on $F^{\otimes\frac{1}{2}}$ with central charge $c=\frac{1}{2}$ is defined by

$$r_{\frac{1}{2}}(E_{m,n}-E_{1-n,1-m})=:\phi^D_{-m+1/2}\phi^D_{n-\frac{1}{2}}:$$

for all $m,n\in\mathbb{Z}$. Now Proposition 3.3 gives us a new representation r_1 of a_∞ on $F^{\otimes\frac{1}{2}}$ given by

$$r_1(E_{m,n})=:\phi^D_{-2m+\frac{1}{2}}\phi^D_{2n-\frac{1}{2}}:. \tag{3.25}$$

for all $m,n\in\mathbb{Z}$. Hence by restriction the representation of d_∞ on $F^{\otimes\frac{1}{2}}$ with central charge $c=1$ is

$$r_1(E_{m,n}-E_{1-n,1-m}):=:\phi^D_{-2m+\frac{1}{2}}\phi^D_{2n-\frac{1}{2}}:-:\phi^D_{2n-3/2}\phi^D_{-2m+3/2}:.$$

for all $m,n\in\mathbb{Z}$.

Proposition 3.5 *(The upper triangular elements annihilate v_n) For all $n\in\mathbb{Z}$, $a^+_\infty v_n=0$.*

Proof: We need to prove that for any $k\geq 1$ and any $i,n\in\mathbb{Z}$, $r_1(E_{i,i+k})v_n=0$. We have $r_1(E_{i,i+k})=:\phi^D_{-2i+\frac{1}{2}}\phi^D_{2i+2k-\frac{1}{2}}:$. We start with $n=0$, $v_0=|0\rangle$. There are two cases: The case of $2i+2k>0$ is trivial. In the case $2i+2k\leq 0$, then $:\phi^D_{-2i+\frac{1}{2}}\phi^D_{2i+2k-\frac{1}{2}}:=-\phi^D_{2i+2k-\frac{1}{2}}\phi^D_{-2i+\frac{1}{2}}$. But $-2i\geq 2k\geq 2$, hence

$$r_1(E_{i,i+k})v_0=r_1(E_{i,i+k})|0\rangle=-\phi^D_{2i+2k-\frac{1}{2}}\phi^D_{-2i+\frac{1}{2}}|0\rangle=0.$$

We continue with $n > 0$, where $v_n = \phi^D_{-2n+1-\frac{1}{2}} \cdots \phi^D_{-3-\frac{1}{2}} \phi^D_{-1-\frac{1}{2}}|0\rangle$. Again, there are two cases: first we will consider the case when $2i + 2k \le 0$:

$$r_1(E_{i,i+k})\phi^D_{-2n+1-\frac{1}{2}} \cdots \phi^D_{-3-\frac{1}{2}} \phi^D_{-1-\frac{1}{2}}|0\rangle = -\phi^D_{2i+2k-\frac{1}{2}} \phi^D_{-2i+\frac{1}{2}} \phi^D_{-2n+1-\frac{1}{2}} \cdots \phi^D_{-3-\frac{1}{2}} \phi^D_{-1-\frac{1}{2}}|0\rangle.$$

Now since it is impossible to have $-2i + \frac{1}{2} = -(-2l + 1 - \frac{1}{2})$ for any $l \in \mathbb{Z}$, $\phi^D_{-2i+\frac{1}{2}}$ will anticommute with any $\phi^D_{-2l+1-\frac{1}{2}}$, and thus $\phi^D_{-2i+\frac{1}{2}} \phi^D_{-2n+1-\frac{1}{2}} \cdots \phi^D_{-3-\frac{1}{2}} \phi^D_{-1-\frac{1}{2}}|0\rangle = 0$.

Next let $2i + 2k > 0$, then $: \phi^D_{-2i+\frac{1}{2}} \phi^D_{2i+2k-\frac{1}{2}} := \phi^D_{-2i+\frac{1}{2}} \phi^D_{2i+2k-\frac{1}{2}}$, and

$$r_1(E_{i,i+k})\phi^D_{-2n+1-\frac{1}{2}} \cdots \phi^D_{-3-\frac{1}{2}} \phi^D_{-1-\frac{1}{2}}|0\rangle = \phi^D_{-2i+\frac{1}{2}} \phi^D_{2i+2k-\frac{1}{2}} \phi^D_{-2n+1-\frac{1}{2}} \cdots \phi^D_{-3-\frac{1}{2}} \phi^D_{-1-\frac{1}{2}}|0\rangle.$$

Now $\phi^D_{2i+2k-\frac{1}{2}}$ is an operator annihilating the vacuum, and unless we have $2i + 2k - \frac{1}{2} = -(-2l + 1 - \frac{1}{2})$ for some $1 \le l \le n$, $l \in \mathbb{Z}$, then it will anticommute with any of the $\phi^D_{-2l+1-\frac{1}{2}}$, $1 \le l \le n$, and thus $\phi^D_{2i+2k-\frac{1}{2}} \phi^D_{-2n+1-\frac{1}{2}} \cdots \phi^D_{-3-\frac{1}{2}} \phi^D_{-1-\frac{1}{2}}|0\rangle = 0$. If, on the other hand $2i + 2k - \frac{1}{2} = -(-2l + 1 - \frac{1}{2})$ for some $1 \le l \le n$, $l \in \mathbb{Z}$, then we have

$$\begin{aligned} r_1(E_{i,i+k})&\phi^D_{-2n+1-\frac{1}{2}} \cdots \phi^D_{-3-\frac{1}{2}} \phi^D_{-1-\frac{1}{2}}|0\rangle \\ &= \phi^D_{-2i+\frac{1}{2}} \phi^D_{2i+2k-\frac{1}{2}} \phi^D_{-2n+1-\frac{1}{2}} \cdots \phi^D_{-2l+1-\frac{1}{2}} \cdots \phi^D_{-1-\frac{1}{2}}|0\rangle \\ &= \pm\phi^D_{-2i+\frac{1}{2}} \phi^D_{-2n+1-\frac{1}{2}} \cdots \widehat{\phi^D_{-2l+1-\frac{1}{2}}} \phi^D_{-2l+3-\frac{1}{2}} \cdots \phi^D_{-1-\frac{1}{2}}|0\rangle; \end{aligned}$$

here $\widehat{\phi^D_{-2l+1-\frac{1}{2}}}$ denotes the fact that $\phi^D_{-2l+1-\frac{1}{2}}$ is absent. But then from $2i + 2k - 1/2 = -(-2l+1-\frac{1}{2})$, we have $i+k = l$ and $-2i+\frac{1}{2} = -2l+2k+1-\frac{1}{2}$, and we know from $k \ge 1$ that either $-2l + 3 - \frac{1}{2} \le -2i + \frac{1}{2} \le -1 - \frac{1}{2}$, or $-2i + \frac{1}{2} \ge \frac{1}{2}$. If $-2i + \frac{1}{2} \ge \frac{1}{2}$ (i.e., $k \ge l$), then since $\phi^D_{-2i+\frac{1}{2}}$ anticommutes with all $\phi^D_{-2l+1-\frac{1}{2}}$, we have $\phi^D_{-2i+\frac{1}{2}} \phi^D_{-2n+1-\frac{1}{2}} \cdots \widehat{\phi^D_{-2l+1-\frac{1}{2}}} \phi^D_{-2l+3-\frac{1}{2}} \cdots \phi^D_{-1-\frac{1}{2}}|0\rangle$. If on the other hand $-2l + 3 - \frac{1}{2} \le -2i + \frac{1}{2} \le -1 - \frac{1}{2}$, then that means $\phi^D_{-2i+\frac{1}{2}} = \phi^D_{-2l_1+1-\frac{1}{2}}$ for $1 \le l_1 \le l - 1$ and

$$\begin{aligned} \phi^D_{-2i+\frac{1}{2}}&\phi^D_{-2n+1-\frac{1}{2}} \cdots \widehat{\phi^D_{-2l+1-\frac{1}{2}}} \phi^D_{-2l+3-\frac{1}{2}} \cdots \phi^D_{-1-\frac{1}{2}}|0\rangle \\ &= \phi^D_{-2i+\frac{1}{2}} \phi^D_{-2n+1-\frac{1}{2}} \cdots \widehat{\phi^D_{-2l+1-\frac{1}{2}}} \phi^D_{-2l+3-\frac{1}{2}} \cdots \phi^D_{-2l_1+1-\frac{1}{2}} \cdots \phi^D_{-1-\frac{1}{2}}|0\rangle \\ &= \phi^D_{-2l_1+1-\frac{1}{2}} \phi^D_{-2n+1-\frac{1}{2}} \cdots \widehat{\phi^D_{-2l+1-\frac{1}{2}}} \phi^D_{-2l+1-\frac{1}{2}} \cdots \phi^D_{-2l_1+1-\frac{1}{2}} \cdots \phi^D_{-1-\frac{1}{2}}|0\rangle \\ &= \pm\phi^D_{-2n+1-\frac{1}{2}} \cdots \widehat{\phi^D_{-2l+1-\frac{1}{2}}} \phi^D_{-2l+1-\frac{1}{2}} \cdots \phi^D_{-2l_1+1-\frac{1}{2}} \phi^D_{-2l_1+1-\frac{1}{2}} \cdots \phi^D_{-1-\frac{1}{2}}|0\rangle = 0; \end{aligned}$$

since $\phi^D_{-2l_1+1-\frac{1}{2}} \phi^D_{-2l_1+1-\frac{1}{2}} = 0$.

Consider now $v_{-n} = \phi^D_{-2n+2-\frac{1}{2}} \cdots \phi^D_{-2-\frac{1}{2}} \phi^D_{-\frac{1}{2}}|0\rangle$, $n > 0$. Again, first we consider the case $2i + 2k > 0$, when $: \phi^D_{-2i+\frac{1}{2}} \phi^D_{2i+2k-\frac{1}{2}} := \phi^D_{-2i+\frac{1}{2}} \phi^D_{2i+2k-\frac{1}{2}}$.

$$r_1(E_{i,i+k})\phi^D_{-2n+2-\frac{1}{2}} \cdots \phi^D_{-2-\frac{1}{2}} \phi^D_{-\frac{1}{2}}|0\rangle = \phi^D_{-2i+\frac{1}{2}} \phi^D_{2i+2k-\frac{1}{2}} \phi^D_{-2n+2-\frac{1}{2}} \cdots \phi^D_{-2-\frac{1}{2}} \phi^D_{-\frac{1}{2}}|0\rangle.$$

Now since it is impossible to have $2i + 2k - \frac{1}{2} = -(-2l + 2 - \frac{1}{2})$ for any $l \in \mathbb{Z}$, $\phi^D_{2i+2k-\frac{1}{2}}$ will anticommute with any $\phi^D_{-2l-\frac{1}{2}}$, and thus $\phi^D_{2i+2k-\frac{1}{2}} \phi^D_{-2n+2-\frac{1}{2}} \cdots \phi^D_{-2-\frac{1}{2}} \phi^D_{-\frac{1}{2}}|0\rangle = 0$.

Next, let $2i+2k \leq 0$, then : $\phi^D_{-2i+\frac{1}{2}}\phi^D_{2i+2k-\frac{1}{2}} := -\phi^D_{2i+2k-\frac{1}{2}}\phi^D_{-2i+\frac{1}{2}}$, and

$$r_1(E_{i,i+k})\phi^D_{-2n+2-\frac{1}{2}}\cdots\phi^D_{-2-\frac{1}{2}}\phi^D_{-\frac{1}{2}}|0\rangle = -\phi^D_{2i+2k-\frac{1}{2}}\phi^D_{-2i+\frac{1}{2}}\phi^D_{-2n+2-\frac{1}{2}}\cdots\phi^D_{-2-\frac{1}{2}}\phi^D_{-\frac{1}{2}}|0\rangle.$$

Now $\phi^D_{-2i+\frac{1}{2}}$ is an operator annihilating the vacuum, and unless we have $-2i+\frac{1}{2} = -(-2l-\frac{1}{2})$ for some $1 \leq l \leq n-1,\ l \in \mathbb{Z}$, then it will anticommute with any of the $\phi^D_{-2l-\frac{1}{2}}$, $1 \leq l \leq n-1$, and thus $\phi^D_{-2i+\frac{1}{2}}\phi^D_{-2n+2-\frac{1}{2}}\cdots\phi^D_{-2-\frac{1}{2}}\phi^D_{-\frac{1}{2}}|0\rangle = 0$. If, on the other hand $-2i+\frac{1}{2} = -(-2l-\frac{1}{2})$ for some $1 \leq l \leq n-1,\ l \in \mathbb{Z}$, then we have

$$\begin{aligned}
r_1(E_{i,i+k})&\phi^D_{-2n+2-\frac{1}{2}}\cdots\phi^D_{-2-\frac{1}{2}}\phi^D_{-\frac{1}{2}}|0\rangle \\
&= -\phi^D_{2i+2k-\frac{1}{2}}\phi^D_{-2i+\frac{1}{2}}\phi^D_{-2n+2-\frac{1}{2}}\cdots\phi^D_{-2l+2-\frac{1}{2}}\cdots\phi^D_{-\frac{1}{2}}|0\rangle \\
&= \pm\phi^D_{2i+2k-\frac{1}{2}}\phi^D_{-2n+2-\frac{1}{2}}\cdots\widehat{\phi^D_{-2l-\frac{1}{2}}}\phi^D_{-2l+2-\frac{1}{2}}\cdots\phi^D_{-\frac{1}{2}}|0\rangle.
\end{aligned}$$

But then from $-2i+1/2 = -(-2l-\frac{1}{2})$, we have $i=-l$ and $2i+2k-\frac{1}{2} = 2k-2l-\frac{1}{2}$, and we know from $i+k \leq 0$ that $-2l+2-\frac{1}{2} \leq 2i+2k-\frac{1}{2} \leq -\frac{1}{2}$. That means $\phi^D_{2i+2k-\frac{1}{2}} = \phi^D_{-2l_1-\frac{1}{2}}$ for $0 \leq l_1 \leq l-1$ and

$$\begin{aligned}
\phi^D_{2i+2k-\frac{1}{2}}&\phi^D_{-2n+2-\frac{1}{2}}\cdots\widehat{\phi^D_{-2l-\frac{1}{2}}}\phi^D_{-2l+2-\frac{1}{2}}\cdots\phi^D_{-\frac{1}{2}}|0\rangle \\
&= \phi^D_{2i+2k-\frac{1}{2}}\phi^D_{-2n+2-\frac{1}{2}}\cdots\widehat{\phi^D_{-2l-\frac{1}{2}}}\phi^D_{-2l+2-\frac{1}{2}}\cdots\phi^D_{-2l_1-\frac{1}{2}}\cdots\phi^D_{-\frac{1}{2}}|0\rangle \\
&= \phi^D_{-2l_1-\frac{1}{2}}\phi^D_{-2n+2-\frac{1}{2}}\cdots\widehat{\phi^D_{-2l-\frac{1}{2}}}\phi^D_{-2l+2-\frac{1}{2}}\cdots\phi^D_{-2l_1-\frac{1}{2}}\cdots\phi^D_{-\frac{1}{2}}|0\rangle \\
&= \pm\phi^D_{-2n+2-\frac{1}{2}}\cdots\widehat{\phi^D_{-2l-\frac{1}{2}}}\phi^D_{-2l+2-\frac{1}{2}}\cdots\phi^D_{-2l_1-\frac{1}{2}}\phi^D_{-2l_1-\frac{1}{2}}\cdots\phi^D_{-\frac{1}{2}}|0\rangle = 0;
\end{aligned}$$

since $\phi^D_{-2l_1-\frac{1}{2}}\phi^D_{-2l_1-\frac{1}{2}} = 0$. $\square$

Lemma 3.6 *(Calculating the weights) For all $i,n \in \mathbb{Z}$, $r_1(E_{ii})v_n = {}^a\hat{\Lambda}_n(E_{ii})v_n$.*

Proof: We want to prove

$$\begin{aligned}
&r_1(E_{i,i})v_0 = 0\cdot v_0;\\
&r_1(E_{i,i})v_n = 1\cdot v_n \quad \text{for } 0<i\leq n; \quad r_1(E_{i,i})v_n = 1\cdot v_n, \quad \text{for } i>n\geq 1;\\
&r_1(E_{i,i})v_n = 0\cdot v_n \quad \text{for } i \leq 0,\ n>0;\\
&r_1(E_{i,i})v_{-n} = -1\cdot v_{-n} \quad \text{for } -n+1 \leq i \leq 0; \quad r_1(E_{i,i})v_{-n} = 0\cdot v_{-n} \quad \text{for } -n+1 > i > 0;\\
&r_1(E_{i,i})v_{-n} = 0\cdot v_{-n} \quad \text{for } i>0,\ n>0.
\end{aligned}$$

We have

$$r_1(E_{i,i}) =: \phi^D_{-2i+\frac{1}{2}}\phi^D_{2i-\frac{1}{2}} := \begin{cases} \phi^D_{-2i+\frac{1}{2}}\phi^D_{2i-\frac{1}{2}}, & \text{for } i>0,\\ -\phi^D_{2i-\frac{1}{2}}\phi^D_{-2i+\frac{1}{2}}, & \text{for } i\leq 0 \end{cases}$$

For $n=0$, $v_0 = |0\rangle$, and $r_1(E_{i,i}) =: \phi^D_{-2i+\frac{1}{2}}\phi^D_{2i-\frac{1}{2}}$:, and it is clear that $r_1(E_{i,i})|0\rangle = 0$.
Let $n>0$ and $v_n = \phi^D_{-2n+1-\frac{1}{2}}\cdots\phi^D_{-3-\frac{1}{2}}\phi^D_{-1-\frac{1}{2}}|0\rangle$. If $i>0$, $\phi^D_{2i-\frac{1}{2}}$ from $r_1(E_{i,i})$ will anticommute

with any $\phi^D_{-2l+1-\frac{1}{2}}$ in v_n, **except** for l such that $-2l+1-\frac{1}{2} = -(2i-\frac{1}{2})$, i.e., $l=i$. In the case there exist an l such that $l=i$, then

$$\begin{aligned} r_1(E_{i,i})v_n &= r_1(E_{i,i})\phi^D_{-2n+1-\frac{1}{2}}\cdots\phi^D_{-3-\frac{1}{2}}\phi^D_{-1-\frac{1}{2}}|0\rangle \\ &= \phi^D_{-2i+\frac{1}{2}}\phi^D_{2i-\frac{1}{2}}\phi^D_{-2n+1-\frac{1}{2}}\cdots\phi^D_{-2l+1-\frac{1}{2}}\cdots\phi^D_{-1-\frac{1}{2}}|0\rangle \\ &= (-1)^{n-l}\phi^D_{-2i+\frac{1}{2}}\phi^D_{-2n+1-\frac{1}{2}}\cdots\phi^D_{2i-\frac{1}{2}}\phi^D_{-2l+1-\frac{1}{2}}\cdots\phi^D_{-1-\frac{1}{2}}|0\rangle \\ &= (-1)^{n-l}\phi^D_{-2l+1-\frac{1}{2}}\phi^D_{-2n+1-\frac{1}{2}}\cdots\widehat{\phi^D_{-2l+1-\frac{1}{2}}}\cdots\phi^D_{-1-\frac{1}{2}}|0\rangle \\ &= \phi^D_{-2n+1-\frac{1}{2}}\cdots\phi^D_{-2l+1-\frac{1}{2}}\cdots\phi^D_{-1-\frac{1}{2}}|0\rangle = v_n \end{aligned}$$

Hence $r_1(E_{i,i})v_n = v_n$ for $i \le 0$ when $n \ge i$ (i.e., exist an l such that $l=i$), and $r_1(E_{i,i})v_n = 0$ for $i \le 0$ when $n < i$.

If $i \le 0$, then $r_1(E_{i,i}) = -\phi^D_{2i-\frac{1}{2}}\phi^D_{-2i+\frac{1}{2}}$ and $\phi^D_{-2i+\frac{1}{2}}$ will anticommute with **any** $\phi^D_{-2l+1-\frac{1}{2}}$ in v_n, as it is impossible to have $-2l+1-\frac{1}{2} = -(-2i+\frac{1}{2})$. Hence $r_1(E_{i,i})v_n = 0$ for $i \le 0$.

Let $n > 0$ and $v_{-n} = \phi^D_{-2n+2-\frac{1}{2}}\cdots\phi^D_{-2-\frac{1}{2}}\phi^D_{-\frac{1}{2}}|0\rangle$. If $i \le 0$, $\phi^D_{-2i+\frac{1}{2}}$ from $r_1(E_{i,i})$ will anticommute with any $\phi^D_{-2l+2-\frac{1}{2}}$ in v_{-n}, **except** for l such that $-2l+2-\frac{1}{2} = -(2i+\frac{1}{2})$, i.e., $l = -i-1$. In the case there exist an l such that $l=-i-1$, then

$$\begin{aligned} r_1(E_{i,i})v_n &= r_1(E_{i,i})\phi^D_{-2n+2-\frac{1}{2}}\cdots\phi^D_{-2-\frac{1}{2}}\phi^D_{-\frac{1}{2}}|0\rangle \\ &= -\phi^D_{2i-\frac{1}{2}}\phi^D_{-2i+\frac{1}{2}}\phi^D_{-2n+2-\frac{1}{2}}\cdots\phi^D_{-2l+2-\frac{1}{2}}\cdots\phi^D_{-\frac{1}{2}}|0\rangle \\ &= -(-1)^{n-l}\phi^D_{2i-\frac{1}{2}}\phi^D_{-2n+2-\frac{1}{2}}\cdots\phi^D_{-2i+\frac{1}{2}}\phi^D_{-2l+2-\frac{1}{2}}\cdots\phi^D_{-\frac{1}{2}}|0\rangle \\ &= -(-1)^{n-l}\phi^D_{-2l+2-\frac{1}{2}}\phi^D_{-2n+2-\frac{1}{2}}\cdots\widehat{\phi^D_{-2l+2-\frac{1}{2}}}\cdots\phi^D_{-\frac{1}{2}}|0\rangle \\ &= -\phi^D_{-2n+2-\frac{1}{2}}\cdots\phi^D_{-2l+2-\frac{1}{2}}\cdots\phi^D_{-\frac{1}{2}}|0\rangle = v_n \end{aligned}$$

Hence $r_1(E_{i,i})v_{-n} = -v_{-n}$ for $i \le 0$ when $n \ge -i-1$ (i.e., exist an l such that $l=-i-1$), and $r_1(E_{i,i})v_{-n} = 0$ for $i \le 0$ when $n < -i-1$.

If $i > 0$, then $r_1(E_{i,i}) = \phi^D_{-2i+\frac{1}{2}}\phi^D_{2i-\frac{1}{2}}$ and $\phi^D_{2i-\frac{1}{2}}$ will anticommute with **any** $\phi^D_{-2l+2-\frac{1}{2}}$, as it is impossible to have $-2l+2-\frac{1}{2} = -(2i-\frac{1}{2})$. Hence $r_1(E_{i,i})v_{-n} = 0$ for $i > 0$. □

Proposition 3.7 *For any $n \in \mathbb{Z}$, $F^{\otimes\frac{1}{2}}_{(n)}$ is an a_∞-submodule of $F^{\otimes\frac{1}{2}}$ and $F^{\otimes\frac{1}{2}}_{(n)} = U(a^-_\infty)v_n$.*

Proof: First, $F^{\otimes\frac{1}{2}}_{(n)}$ is an a_∞-submodule of $F^{\otimes\frac{1}{2}}$ as $r_1(E_{ij})$ is a homogenous operator acting on $F^{\otimes\frac{1}{2}}$ of degree 0 with respect to the grading given in Lemma 3.1 and in (3.5). Then certainly that gives us $U(a^-_\infty)v_n \subseteq F^{\otimes\frac{1}{2}}_{(n)}$. The proof that $F^{\otimes\frac{1}{2}}_{(n)} = U(a^-_\infty)v_n$ is similar for each $n \in \mathbb{Z}$, thus we will show it only for $n=0$. Let $v \in F^{\otimes\frac{1}{2}}_{(0)}$, without loss of generality here we can assume v is homogeneous, i.e. $v = \phi^D_{-n_k-\frac{1}{2}}\cdots\phi^D_{-n_2-\frac{1}{2}}\phi^D_{-n_1-\frac{1}{2}}|0\rangle$. Since $v \in F^{\otimes\frac{1}{2}}_{(0)}$, we have $k=2l$ and precisely half of the indexes $n_1, n_2, \ldots, n_k$ are even, the other half are odd. Thus we can write after eventual use of the anticommutation relations in Cl_D

$$v = \pm\phi^D_{-n^o_l-\frac{1}{2}}\phi^D_{-n^e_l-\frac{1}{2}}\cdots\phi^D_{-n^o_1-\frac{1}{2}}\phi^D_{-n^e_1-\frac{1}{2}}|0\rangle,$$

i.e., we have rearranged the factors in pairs, so that the indexes in the pairs are n_s^e and n_s^o, and we have $n_l^o > \cdots > n_1^o$, $n_l^e > \cdots > n_1^e$. Hence we have $n_s^e = 2q_s$ for some $q_s \in \mathbb{Z}$ and $n_s^o = 2p_s - 1$ for some $p_s \in \mathbb{Z}$. Thus

$$v = \pm\phi^D_{-n_l^o-\frac{1}{2}}\phi^D_{-n_l^e-\frac{1}{2}}\cdots\phi^D_{-n_1^o-\frac{1}{2}}\phi^D_{-n_1^e-\frac{1}{2}}|0\rangle = \pm r_1(E_{p_l,-q_l})\cdots r_1(E_{p_1,-q_1}|0\rangle;$$

which proves that $F^{\otimes\frac{1}{2}}_{(0)} = U(a_\infty^-)v_0$. The proof that $F^{\otimes\frac{1}{2}}_{(n)} = U(a_\infty^-)v_n$ is very similar for the other $n \in \mathbb{Z}$ and we will omit it. $\square$

Proposition 3.8 *For any $n \in \mathbb{Z}$, $F^{\otimes\frac{1}{2}}_{(n)}$ is an irreducible submodule for the representation r_1 of a_∞ inside $F^{\otimes\frac{1}{2}}$. Moreover for any $n \in \mathbb{Z}$, $F^{\otimes\frac{1}{2}}_{(n)} \cong L(a_\infty;{}^a\hat{\Lambda}_n, 1)$.*

Proof: This proof uses the uniqueness property of the contragradient Hermitian symmetric form on the Verma module $V(a_\infty;{}^a\hat{\Lambda}_n, 1)$ (see [Jan79], [Kac90] or [MP95]). For a more direct, but calculational proof the reader can see the Appendix. It is well known that one can define ω the conjugate linear involutive anti-automorphism on Cl_D by $\omega(\phi^D_m) = \phi^D_{-m}$ for all $m \in \mathbb{Z} + \frac{1}{2}$ and $\omega(1) = 1$. Recall the module $F^{\otimes\frac{1}{2}}$ is defined to be the induced module

$$F^{\otimes\frac{1}{2}} = Cl_D \otimes_{\mathbb{C}1\otimes Cl_{D+}} \mathbb{C}|0\rangle$$

whereby $1|0\rangle = |0\rangle$ and $Cl_{D+}|0\rangle = 0$. The conjugate linear involutive anti-automorphism (or antilinear antiautomorphism) $\omega : Cl_D \to Cl_D$ defined by $\omega(\phi_n) = \phi_{-n}$ gives rise to a non-degenerate positive definite form $\langle\ |\ \rangle$ defined on $F^{\otimes\frac{1}{2}}$ whereby

$$\langle Xv\,|\,w\rangle = \langle v\,|\,\omega(X)w\rangle$$

for all $v, w \in F^{\otimes\frac{1}{2}}$ and $X \in Cl_D$. Observe that $F^{\otimes\frac{1}{2}}{}_{(n)} \perp F^{\otimes\frac{1}{2}}{}_{(m)}$ for $m \neq n$.

It is straightforward to check that

$$\omega(r_1(E_{m,n})) = r_1(E_{n,m}) = r_1(\omega(E_{m,n})). \tag{3.26}$$

Thus ω defined on Cl_D agrees with the compact anti-involution defined on a_∞ given earlier.

We have from Proposition 3.5, Lemma 3.6, and Proposition 3.7 that $F^{\otimes\frac{1}{2}}_{(n)}$ is a highest weight a_∞-module. By the universal mapping property of Verma modules there exists an a_∞-module homomorphism $\pi : V(a_\infty;{}^a\hat{\Lambda}_n, 1) \to F^{\otimes\frac{1}{2}}_{(n)}$ sending the highest weight vector $v_{{}^a\hat{\Lambda}_n}$ of the Verma module $V(a_\infty;{}^a\hat{\Lambda}_n, 1)$ to v_n in $F^{\otimes\frac{1}{2}}_{(n)}$. Moreover the Hermitian symmetric ω-contragradient form $\langle\ ,\ \rangle$ on $F^{\otimes\frac{1}{2}}_{(n)}$ pulls back to a Hermitian symmetric form $(\ ,\ )$ on the Verma module $V(a_\infty;{}^a\hat{\Lambda}_n, 1)$. In other words

$$(uv_{{}^a\hat{\Lambda}_n}, u'v_{{}^a\hat{\Lambda}_n}) := \langle\pi(uv_{{}^a\hat{\Lambda}_n}), \pi(u'v_{{}^a\hat{\Lambda}_n})\rangle = \langle r_1(u)v_n, r_1(u')v_n\rangle. \tag{3.27}$$

for all $u, u' \in a_\infty$. By (3.26), we have

$$\begin{aligned}(Xuv_{{}^a\hat{\Lambda}_n}, u'v_{{}^a\hat{\Lambda}_n}) &= \langle r_1(X)r_1(u)v_n, r_1(u')v_n\rangle \\ &= \langle r_1(u)v_n, \omega(r_1(X))r_1(u')v_n\rangle \\ &= \langle r_1(u)v_n, r_1(\omega(X)u')v_n\rangle \\ &= (uv_{{}^a\hat{\Lambda}_n}, \omega(X)u'v_{{}^a\hat{\Lambda}_n})\end{aligned}$$

Hence (,) is contragradient with respect to ω.

Now it is known that there is a unique contragradient Hermitian symmetric form on the Verma module $V(a_\infty;{}^a\hat{\Lambda}_n,1)$ with $(v_{^a\hat{\Lambda}_n}, v_{^a\hat{\Lambda}_n}) = \langle v_n, v_n\rangle$ and its radical is precisely the unique maximal submodule $\overline{V(a_\infty;{}^a\hat{\Lambda}_n,1)}$ of $V(a_\infty;{}^a\hat{\Lambda}_n,1)$ (see [Jan79], [Kac90] or [MP95]). So

$$0 = (\overline{V(a_\infty;{}^a\hat{\Lambda}_n,1)}, u'v_{^a\hat{\Lambda}_n}) := \langle \pi(\overline{V(a_\infty;{}^a\hat{\Lambda}_n,1)}), \pi(u'v_{^a\hat{\Lambda}_n})\rangle \tag{3.28}$$

for all $u' \in U(a_\infty)$. Since $\langle\ ,\ \rangle$ is nondegenerate and π is surjective one must have $\overline{V(a_\infty;{}^a\hat{\Lambda}_n,1)} \subseteq \ker\pi$. Thus $L(a_\infty;{}^a\hat{\Lambda}_n,1) = V(a_\infty;{}^a\hat{\Lambda}_n,1)/\overline{V(a_\infty;{}^a\hat{\Lambda}_n,1)} \cong F^{\otimes\frac{1}{2}}{}_{(n)}$. $\square$

Remark 3.9 An alternative proof of the irreducibility of $F^{\otimes\frac{1}{2}}_{(n)}$ can be given as follows. In [Ang12] we considered the Heisenberg algebra $\mathcal{H}_\mathbb{Z}$, which is a subalgebra of a_∞ represented by

$$h_n \mapsto \sum_{i\in\mathbb{Z}} E_{i,i+n}$$

In [Ang12] we prove that each $F^{\otimes\frac{1}{2}}_{(n)}$ is irreducible under $\mathcal{H}_\mathbb{Z}$. Since $\mathcal{H}_\mathbb{Z}$ is a subalgebra of a_∞, then $F^{\otimes\frac{1}{2}}_{(n)}$ is irreducible under a_∞.

The previous three propositions can now be combined in the following

Theorem 3.10 *As a_∞ modules with central charge $c = 1$ $F^{\otimes\frac{1}{2}}_{(n)} \cong L(a_\infty;{}^a\hat{\Lambda}_n,1)$ and $F^{\otimes\frac{1}{2}}$ decomposes into irreducible submodules as follows:*

$$F^{\otimes\frac{1}{2}} \cong \oplus_{n\in\mathbb{Z}} L(a_\infty;{}^a\hat{\Lambda}_n,1).$$

Hence $F^{\otimes\frac{1}{2}} \cong F^{\otimes 1}$ as a_∞ modules with central charge $c = 1$.

Here $F^{\otimes 1}$ denotes the fermionic Fock space of the charged free fermions (see Remark 3.2 above, after [Fre81], [KR87], [KW94], [Kac98], [KWY98], [Wan99a]).

Since $F^{\otimes\frac{1}{2}} \cong F^{\otimes 1}$ as a_∞ modules with central charge $c = 1$, we can use Theorem 3.2 of [Wan99b] to get the decomposition of the irreducible central charge $c = \frac{1}{2}$ d_∞ modules $F^{\otimes\frac{1}{2}}_{\bar{0}}$ and $F^{\otimes\frac{1}{2}}_{\bar{1}}$ in terms of the new d_∞ central charge $c = 1$ action:

Corollary 3.11 *As d_∞ modules with central charge $c = 1$ we have*

$$F^{\otimes\frac{1}{2}}_{(n)} \cong L(d_\infty;{}^{\mathfrak{d}}\hat{\Lambda}_n,1) \quad \text{for } n \neq 0 \qquad F^{\otimes\frac{1}{2}}_{(0)} \cong L(d_\infty;{}^{\mathfrak{d}}\hat{\Lambda}_0,1) \oplus L(d_\infty;{}^{\mathfrak{d}}\hat{\Lambda}_{det},1); \tag{3.29}$$

where for $n \neq 0$ $F^{\otimes\frac{1}{2}}_{(n)}$ has highest weight vector v_n, and $F^{\otimes\frac{1}{2}}_{(0)}$ decomposes into two irreducible highest weight modules, with highest weight vectors v_0 and $\tilde{v}_0 = \phi^D_{-\frac{3}{2}}\phi^D_{-\frac{1}{2}}|0\rangle \in F^{\otimes\frac{1}{2}}_{(0)}$ (note $\tilde{v}_0$ is not a highest weight vector for the a_∞ action, only the d_∞ action). Thus as d_∞ modules with central charge $c = 1$

$$F^{\otimes\frac{1}{2}} = \left(\oplus_{\substack{n\in\mathbb{Z}\\ n\neq 0}} L(d_\infty;{}^{\mathfrak{d}}\hat{\Lambda}_n,1)\right) \bigoplus \left(L(d_\infty;{}^{\mathfrak{d}}\hat{\Lambda}_0,1) \oplus L(d_\infty;{}^{\mathfrak{d}}\hat{\Lambda}_{det},1)\right),$$

and

$$\begin{aligned} F^{\otimes\frac{1}{2}}_{\bar{1}} &= \oplus_{n\in\mathbb{Z}} L(d_\infty;{}^{\mathfrak{d}}\hat{\Lambda}_{2n-1},1); \\ F^{\otimes\frac{1}{2}}_{\bar{0}} &= \left(\oplus_{\substack{n\in\mathbb{Z}\\ n\neq 0}} L(d_\infty;{}^{\mathfrak{d}}\hat{\Lambda}_{2n},1)\right) \bigoplus \left(L(d_\infty;{}^{\mathfrak{d}}\hat{\Lambda}_0,1) \oplus L(d_\infty;{}^{\mathfrak{d}}\hat{\Lambda}_{det},1)\right). \end{aligned}$$

Here, as in [Wan99b], the highest weights ${}^{\mathfrak{d}\mathfrak{d}}\hat{\Lambda}_n$ are obtained from the restrictions of ${}^{a}\hat{\Lambda}_n$, except for ${}^{\mathfrak{d}\mathfrak{d}}\hat{\Lambda}_{det}$ which is defined by

$$ {}^{\mathfrak{d}\mathfrak{d}}\hat{\Lambda}_{det}(E_{i,i} - E_{1-i,1-i}) = 2; \quad \text{for } i = 1, \quad {}^{\mathfrak{d}\mathfrak{d}}\hat{\Lambda}_{det}(E_{i,i} - E_{1-i,1-i}) = 0 \quad \text{for } i \neq 0, 1. $$

Proof: Follows directly from Theorem 3.2 of [Wan99b] and Theorem 3.10. □

4. Appendix

For readers who would like to see a more computational proof of Proposition 3.8 we present one below.

Alternate proof of Proposition 3.8: First we will prove that the action of a_∞ on $F^{\otimes \frac{1}{2}}$ will preserve the $\mathbb{Z}$ grading gd; that will show that each $F^{\otimes \frac{1}{2}}_{(n)}$ is a submodule for the r_1 action of a_∞ on $F^{\otimes \frac{1}{2}}$. Then we will show that the submodule $F^{\otimes \frac{1}{2}}_{(n)}$ is irreducible.

Let $v = \phi^D_{-n_k-\frac{1}{2}} \dots \phi^D_{-n_2-\frac{1}{2}} \phi^D_{-n_1-\frac{1}{2}} |0\rangle$ be any homogeneous vector in $F^{\otimes \frac{1}{2}}_{(n)}$. We have

$$ r_1(E_{p,q}) =: \phi^D_{-2p+\frac{1}{2}} \phi^D_{2q-\frac{1}{2}} := \begin{cases} \phi^D_{-2p+\frac{1}{2}} \phi^D_{2q-\frac{1}{2}}, & \text{unless } q \leq 0 \text{ and } 2q - \frac{1}{2} = -(-2p + \frac{1}{2}) \\ -\phi^D_{2q-\frac{1}{2}} \phi^D_{-2p+\frac{1}{2}}, & \text{otherwise.} \end{cases} $$

Consider first the case when $-(2q - \frac{1}{2}) = -2p + \frac{1}{2}$ and $q \leq 0$. Then $-2p + \frac{1}{2} > 0$, and in

$$ r_1(E_{p,q})v = -\phi^D_{2q-\frac{1}{2}} \phi^D_{-2p+\frac{1}{2}} \phi^D_{-n_k-\frac{1}{2}} \dots \phi^D_{-n_2-\frac{1}{2}} \phi^D_{-n_1-\frac{1}{2}} |0\rangle $$

$\phi^D_{-2p+\frac{1}{2}}$ will either anticommute with all $\phi^D_{-n_s-\frac{1}{2}}$, $s = 1, \dots, k$, in which case $r_1(E_{p,q})v = 0$ as $\phi^D_{-2p+\frac{1}{2}} |0\rangle = 0$; or otherwise we will have $-2p + \frac{1}{2} = -(-n_s - \frac{1}{2})$, for some $s = 1, \dots, k$. In that case we also have $-n_s - \frac{1}{2} = 2q - \frac{1}{2}$ and

$$ \begin{aligned} r_1(E_{p,q})v &= -\phi^D_{2q-\frac{1}{2}} \phi^D_{-2p+\frac{1}{2}} \phi^D_{-n_k-\frac{1}{2}} \dots \phi^D_{-n_s-\frac{1}{2}} \dots \phi^D_{-n_1-\frac{1}{2}} |0\rangle \\ &= \pm \phi^D_{2q-\frac{1}{2}} \phi^D_{-n_k-\frac{1}{2}} \dots \widehat{\phi^D_{-n_s-\frac{1}{2}}} \dots \phi^D_{-n_1-\frac{1}{2}} |0\rangle \\ &= \pm \phi^D_{-n_k-\frac{1}{2}} \dots \phi^D_{-n_s-\frac{1}{2}} \dots \phi^D_{-n_1-\frac{1}{2}} |0\rangle. \end{aligned} $$

This shows that in both cases when $-(2q - \frac{1}{2}) = -2p + \frac{1}{2}$ we have if $v \in F^{\otimes \frac{1}{2}}_{(n)}$ then $r_1(E_{p,q})v \in F^{\otimes \frac{1}{2}}_{(n)}$.

Next, consider the case when $-(2q - \frac{1}{2}) \neq -2p + \frac{1}{2}$, but still $q \leq 0$, which implies again that $r_1(E_{p,q}) = -\phi^D_{2q-\frac{1}{2}} \phi^D_{-2p+\frac{1}{2}}$ (anticommutation) and again we consider the possible cases for

$$ r_1(E_{p,q})v = -\phi^D_{2q-\frac{1}{2}} \phi^D_{-2p+\frac{1}{2}} \phi^D_{-n_k-\frac{1}{2}} \dots \phi^D_{-n_2-\frac{1}{2}} \phi^D_{-n_1-\frac{1}{2}} |0\rangle. $$

The first case is when $p < 0$ and $\phi^D_{-2p+\frac{1}{2}}$ anticommutes with all $\phi^D_{-n_s-\frac{1}{2}}$, $s = 1, \dots, k$, in which case $r_1(E_{p,q})v = 0$. The second case is again as above when $-2p + \frac{1}{2} = -(-n_s - \frac{1}{2})$, for some $s = 1, \dots, k$. We again have

$$ \begin{aligned} r_1(E_{p,q})v &= -\phi^D_{2q-\frac{1}{2}} \phi^D_{-2p+\frac{1}{2}} \phi^D_{-n_k-\frac{1}{2}} \dots \phi^D_{-n_s-\frac{1}{2}} \dots \phi^D_{-n_1-\frac{1}{2}} |0\rangle \\ &= \pm \phi^D_{2q-\frac{1}{2}} \phi^D_{-n_k-\frac{1}{2}} \dots \widehat{\phi^D_{-n_s-\frac{1}{2}}} \dots \phi^D_{-n_1-\frac{1}{2}} |0\rangle. \end{aligned} $$

In other words, we have "removed" $\phi^D_{-n_s-\frac{1}{2}}$ with **even** index $n_s = 2p$. Now since $q < 0$, we either have $2q - \frac{1}{2} = -n_t - \frac{1}{2}$ for some $t = 1, \ldots, k, t \neq s$ in which case again we have $r_1(E_{p,q})v = 0$ as $\phi^D_{-2q-\frac{1}{2}}\phi^D_{-2q-\frac{1}{2}} = 0$. Or otherwise we have "added" $\phi^D_{-n_t-\frac{1}{2}}$ with **even** index $n_t = -2q$, which doesn't change the degree dg, as we have "removed" an even index $n_s = 2p$ and "added" an even index $n_t = -2q$. Thus in all cases when $q < 0$ we have if $v \in F^{\otimes \frac{1}{2}}_{(n)}$ then $r_1(E_{p,q})v \in F^{\otimes \frac{1}{2}}_{(n)}$.

Lastly, let $q \geq 1$. Similar argument as above show that either $r_1(E_{p,q})v = 0$, or

$$\begin{aligned} r_1(E_{p,q})v &= \phi^D_{-2p+\frac{1}{2}}\phi^D_{2q-\frac{1}{2}}\phi^D_{-n_k-\frac{1}{2}}\cdots\phi^D_{-n_t-\frac{1}{2}}\cdots\phi^D_{-n_1-\frac{1}{2}}|0\rangle \\ &= \pm\phi^D_{-2p+1-\frac{1}{2}}\phi^D_{-n_k-\frac{1}{2}}\cdots\widehat{\phi^D_{-n_t-\frac{1}{2}}}\cdots\phi^D_{-n_1-\frac{1}{2}}|0\rangle; \end{aligned}$$

where $2q - \frac{1}{2} = -(-n_t - \frac{1}{2})$ for some $t = 1, \ldots, k, t \neq s$. Thus $n_t = 2q - 1$ and we have "removed" an **odd** index n_t. Now if $-2p + 1 - \frac{1}{2} < 0$, we are "adding" back an **odd** index $-2p + 1$. If, on the other hand $-2p + 1 - \frac{1}{2} > 0$, then either we get 0, or we remove also an **even** index. Thus as a summary, in all cases we either remove an odd (even) index and add an odd (even) index back; we remove both an odd and an even index; or we get 0. Hence in all cases the action of $r_1(E_{p,q})$ will preserve the grading. Hence each $F^{\otimes \frac{1}{2}}_{(n)}$ is a submodule for the representation r_1 of a_∞ on $F^{\otimes \frac{1}{2}}$.

Further, from the observations above we can summarize the action r_1 on $F^{\otimes \frac{1}{2}}$ as follows. We have 3 nontrivial cases: in case 1 $r_1(E_{p,q})$ acting on a homogeneous vector v "**adds**" two factors $\phi^D_{-n_{s_1}-\frac{1}{2}}$ and $\phi^D_{-n_{s_2}-\frac{1}{2}}$, so that one of the indexes n_{s_1}, n_{s_2} is **even** , the other is **odd** (we will call it for short "adding an even and an odd index"). In case 2, we **replace** a factor $\phi^D_{-n_{s_1}-\frac{1}{2}}$ with another factor $\phi^D_{-n_{s_2}-\frac{1}{2}}$, where either both factors n_{s_1}, n_{s_2} are **even**, or both factors n_{s_1}, n_{s_2} are **odd** (we will call it for short "replacing even with even index" and "replacing odd with odd index"). And case 3 is when we "**remove**" two factors $\phi^D_{-n_{s_1}-\frac{1}{2}}$ and $\phi^D_{-n_{s_2}-\frac{1}{2}}$, so that one of the indexes n_{s_1}, n_{s_2} is **even** , the other is **odd** ("removing an even and an odd index"). Not that "adding an index" that is already present, or "removing an index" that was absent, will of course produce the 0 vector.

Now we want to prove that for each $n \in \mathbb{Z}$ $F^{\otimes \frac{1}{2}}_{(n)}$ is an irreducible module for a_∞. This will be done in two steps. The first step is to prove that each vector in $F^{\otimes \frac{1}{2}}_{(n)}$ can be generated from the "n-th vacuum vector" v_n, i.e., $F^{\otimes \frac{1}{2}}_{(n)} = U(a^-_\infty)v_n$. The second step is to prove that for any vector $v \in F^{\otimes \frac{1}{2}}_{(n)}$, we have $v_n \in U(a_\infty)v$.

The proof that $F^{\otimes \frac{1}{2}}_{(n)} = U(a^-_\infty)v_n$ is done in Proposition 3.7. Finally, we prove step 2, i.e., for any vector $v \in F^{\otimes \frac{1}{2}}_{(n)}$, we have $v_n \in U(a_\infty)v$. Let $v \in F^{\otimes \frac{1}{2}}_{(n)}$, not necessary homogeneous: $v = \sum_{hk} c_{hk}v^{hk}$, where v^{hk} are homogeneous vectors,

$$v^{hk} = \phi^D_{-n_k-\frac{1}{2}}\cdots\phi^D_{-n_2-\frac{1}{2}}\phi^D_{-n_1-\frac{1}{2}}|0\rangle, \quad c_k \in \mathbb{C}.$$

It is clear that by the operation "adding an even and an odd index" we can reduce v only to a linear combination of homogeneous elements with minimal possible length $\tilde{L}$ among the v^{hk} by the following two steps: we would "add an even and an odd index" starting from the already existing indexes in the vector v^{hk} with the largest length $\tilde{L}$ (which will annihilate it), and then we would remove the same combination back (which will bring the elements with lower length back to their original length). This is always possible, as any two lengths within $F^{\otimes\frac{1}{2}}_{(n)}$ differ always by an even number; and we always have at least two differing elements $\phi^D_{-n_k-\frac{1}{2}}$ between the homogeneous elements with the two consecutive lengths, as well as two differing elements $\phi^D_{-n_k-\frac{1}{2}}$ between homogeneous elements with same lengths. Now among the remaining different vectors v^{hk} with minimal possible length $\tilde{L}$ we can use the operations of "replacing even index with even" and "replacing odd index with odd" until only a single v^{hk} with largest index n_k being the minimal possible remains. We will show how this algorithm on an example: consider a vector $v \in F^{\otimes\frac{1}{2}}_{(0)}$ that will illustrate these 3 steps,

$$\begin{aligned} v = {} & \phi^D_{-7-\frac{1}{2}}\phi^D_{-4-\frac{1}{2}}\phi^D_{-3-\frac{1}{2}}\phi^D_{-2-\frac{1}{2}}|0\rangle + \phi^D_{-5-\frac{1}{2}}\phi^D_{-3-\frac{1}{2}}\phi^D_{-2-\frac{1}{2}}\phi^D_{-0-\frac{1}{2}}|0\rangle \\ & + \phi^D_{-8-\frac{1}{2}}\phi^D_{-5-\frac{1}{2}}|0\rangle + \phi^D_{-6-\frac{1}{2}}\phi^D_{-1-\frac{1}{2}}|0\rangle + \phi^D_{-5-\frac{1}{2}}\phi^D_{-4-\frac{1}{2}}|0\rangle + \phi^D_{-9-\frac{1}{2}}\phi^D_{-8-\frac{1}{2}}|0\rangle. \end{aligned}$$

We have Step 1:

$$\begin{aligned} E_{4,-2}v = {} & \phi^D_{-7-\frac{1}{2}}\phi^D_{-4-\frac{1}{2}}v = \phi^D_{-7-\frac{1}{2}}\phi^D_{-4-\frac{1}{2}}\phi^D_{-7-\frac{1}{2}}\phi^D_{-4-\frac{1}{2}}\phi^D_{-3-\frac{1}{2}}\phi^D_{-2-\frac{1}{2}}|0\rangle \\ & + \phi^D_{-7-\frac{1}{2}}\phi^D_{-4-\frac{1}{2}}\phi^D_{-5-\frac{1}{2}}\phi^D_{-3-\frac{1}{2}}\phi^D_{-2-\frac{1}{2}}\phi^D_{-0-\frac{1}{2}}|0\rangle \\ & + \phi^D_{-7-\frac{1}{2}}\phi^D_{-4-\frac{1}{2}}\phi^D_{-8-\frac{1}{2}}\phi^D_{-5-\frac{1}{2}}|0\rangle + \phi^D_{-7-\frac{1}{2}}\phi^D_{-4-\frac{1}{2}}\phi^D_{-6-\frac{1}{2}}\phi^D_{-1-\frac{1}{2}}|0\rangle \\ & + \phi^D_{-7-\frac{1}{2}}\phi^D_{-4-\frac{1}{2}}\phi^D_{-5-\frac{1}{2}}\phi^D_{-4-\frac{1}{2}}|0\rangle + \phi^D_{-7-\frac{1}{2}}\phi^D_{-4-\frac{1}{2}}\phi^D_{-9-\frac{1}{2}}\phi^D_{-8-\frac{1}{2}}|0\rangle \\ = {} & \phi^D_{-7-\frac{1}{2}}\phi^D_{-4-\frac{1}{2}}\phi^D_{-5-\frac{1}{2}}\phi^D_{-3-\frac{1}{2}}\phi^D_{-2-\frac{1}{2}}\phi^D_{-0-\frac{1}{2}}|0\rangle \\ & + \phi^D_{-7-\frac{1}{2}}\phi^D_{-4-\frac{1}{2}}\phi^D_{-8-\frac{1}{2}}\phi^D_{-5-\frac{1}{2}}|0\rangle + \phi^D_{-7-\frac{1}{2}}\phi^D_{-4-\frac{1}{2}}\phi^D_{-6-\frac{1}{2}}\phi^D_{-1-\frac{1}{2}}|0\rangle \\ & + \phi^D_{-7-\frac{1}{2}}\phi^D_{-4-\frac{1}{2}}\phi^D_{-9-\frac{1}{2}}\phi^D_{-8-\frac{1}{2}}|0\rangle; \end{aligned}$$

Step 2:

$$\begin{aligned} E_{-2,4}E_{4,-2}v = {} & \phi^D_{4+\frac{1}{2}}\phi^D_{7+\frac{1}{2}}\phi^D_{-7-\frac{1}{2}}\phi^D_{-4-\frac{1}{2}}v \\ = {} & \phi^D_{4+\frac{1}{2}}\phi^D_{7+\frac{1}{2}}\phi^D_{-7-\frac{1}{2}}\phi^D_{-4-\frac{1}{2}}\phi^D_{-5-\frac{1}{2}}\phi^D_{-3-\frac{1}{2}}\phi^D_{-2-\frac{1}{2}}\phi^D_{-0-\frac{1}{2}}|0\rangle \\ & + \phi^D_{4+\frac{1}{2}}\phi^D_{7+\frac{1}{2}}\phi^D_{-7-\frac{1}{2}}\phi^D_{-4-\frac{1}{2}}\phi^D_{-8-\frac{1}{2}}\phi^D_{-5-\frac{1}{2}}|0\rangle \\ & + \phi^D_{4+\frac{1}{2}}\phi^D_{7+\frac{1}{2}}\phi^D_{-7-\frac{1}{2}}\phi^D_{-4-\frac{1}{2}}\phi^D_{-6-\frac{1}{2}}\phi^D_{-1-\frac{1}{2}}|0\rangle \\ & + \phi^D_{4+\frac{1}{2}}\phi^D_{7+\frac{1}{2}}\phi^D_{-7-\frac{1}{2}}\phi^D_{-4-\frac{1}{2}}\phi^D_{-9-\frac{1}{2}}\phi^D_{-8-\frac{1}{2}}|0\rangle \\ = {} & \phi^D_{-5-\frac{1}{2}}\phi^D_{-3-\frac{1}{2}}\phi^D_{-2-\frac{1}{2}}\phi^D_{-0-\frac{1}{2}}|0\rangle \\ & + \phi^D_{-8-\frac{1}{2}}\phi^D_{-5-\frac{1}{2}}|0\rangle + \phi^D_{-6-\frac{1}{2}}\phi^D_{-1-\frac{1}{2}}|0\rangle + \phi^D_{-9-\frac{1}{2}}\phi^D_{-8-\frac{1}{2}}|0\rangle; \end{aligned}$$

We repeat Step 1:

$$\begin{aligned} E_{3,-1}E_{-2,4}E_{4,-2}v = {} & \phi^D_{-5-\frac{1}{2}}\phi^D_{-2-\frac{1}{2}}E_{-2,4}E_{4,-2}v \\ = {} & \phi^D_{-5-\frac{1}{2}}\phi^D_{-2-\frac{1}{2}}\phi^D_{-5-\frac{1}{2}}\phi^D_{-3-\frac{1}{2}}\phi^D_{-2-\frac{1}{2}}\phi^D_{-0-\frac{1}{2}}|0\rangle \\ & + \phi^D_{-5-\frac{1}{2}}\phi^D_{-2-\frac{1}{2}}\phi^D_{-8-\frac{1}{2}}\phi^D_{-5-\frac{1}{2}}|0\rangle + \phi^D_{-5-\frac{1}{2}}\phi^D_{-2-\frac{1}{2}}\phi^D_{-6-\frac{1}{2}}\phi^D_{-1-\frac{1}{2}}|0\rangle \\ & + \phi^D_{-5-\frac{1}{2}}\phi^D_{-2-\frac{1}{2}}\phi^D_{-9-\frac{1}{2}}\phi^D_{-8-\frac{1}{2}}|0\rangle \\ = {} & \phi^D_{-5-\frac{1}{2}}\phi^D_{-2-\frac{1}{2}}\phi^D_{-6-\frac{1}{2}}\phi^D_{-1-\frac{1}{2}}|0\rangle + \phi^D_{-5-\frac{1}{2}}\phi^D_{-2-\frac{1}{2}}\phi^D_{-9-\frac{1}{2}}\phi^D_{-8-\frac{1}{2}}|0\rangle. \end{aligned}$$

We repeat Step 2:

$$E_{-1,3}E_{3,-1}E_{-2,4}E_{4,-2}v = \phi^D_{-6-\frac{1}{2}}\phi^D_{-1-\frac{1}{2}}|0\rangle + \phi^D_{-9-\frac{1}{2}}\phi^D_{-8-\frac{1}{2}}|0\rangle.$$

Finally Step 3:

$$\begin{aligned} E_{1,5}E_{-1,3}E_{3,-1}E_{-2,4}E_{4,-2}v &= \phi^D_{-1-\frac{1}{2}}\phi^D_{9+\frac{1}{2}}\phi^D_{-6-\frac{1}{2}}\phi^D_{-1-\frac{1}{2}}|0\rangle + \phi^D_{-1-\frac{1}{2}}\phi^D_{9+\frac{1}{2}}\phi^D_{-9-\frac{1}{2}}\phi^D_{-8-\frac{1}{2}}|0\rangle \\ &= \phi^D_{-8-\frac{1}{2}}\phi^D_{-1-\frac{1}{2}}|0\rangle. \end{aligned}$$

Hence similarly we can reduce any potentially nonhomogeneous vector v we started with to a homogeneous vector v^{hk} by successive action of $r_1(E_{pq})$. Thus we consider the homogeneous vector $v^{hk} = \phi^D_{-n_k-\frac{1}{2}}\cdots\phi^D_{-n_2-\frac{1}{2}}\phi^D_{-n_1-\frac{1}{2}}|0\rangle$. If v^{hk} has length $\tilde{L}(v^{hk}) > |n|$, then $\tilde{L}(v^{hk}) = |n|+2l$ and we can use the the operation of "removing an even and an odd index" l times in succession until we get a vector of minimal length $\tilde{L} = |n|$. After that we just have to eventually "replace some even indexes with even" and "replace some odd indexes with odd" to produce the highest weight vector v_n. Hence, we have proved that $v_n \in U(a_\infty)v$ for any $v \in F^{\otimes\frac{1}{2}}_{(n)}$, which since $F^{\otimes\frac{1}{2}}_{(n)} = U(a^-_\infty)v_n$ proves that $F^{\otimes\frac{1}{2}}_{(n)}$ is an irreducible highest weight module for a_∞. The highest weights are calculated in Lemma 3.6, and that proves Proposition 3.8. □

References

[ACJ13] Iana I Anguelova, Ben Cox, and Elizabeth Jurisich. *N-point locality for vertex operators: normal ordered products, operator product expansions, twisted vertex algebras.* arXiv:1307.4830, submitted, 2013.

[Ang12] Iana I Anguelova. Twisted vertex algebras, bicharacter construction and boson-fermion correspondences. arXiv:1206.4026, submitted, 2012.

[Bar11] Katrina Barron. Twisted modules for N=2 supersymmetric vertex operator superalgebras arising from finite automorphisms of the N=2 Neveu-Schwarz algebra. arXiv:1110.0229, submitted, 2011.

[BS83] N N Bogoliubov and D V Shirkov. *Quantum fields.* Benjamin/Cummings Publishing Co. Inc. Advanced Book Program, Reading, MA, 1983. Translated from the Russian by D. B. Pontecorvo.

[DJKM81a] Etsurō Date, Michio Jimbo, Masaki Kashiwara, and Tetsuji Miwa. Transformation groups for soliton equations. III. Operator approach to the Kadomtsev-Petviashvili equation. *J. Phys. Soc. Japan*, 50(11):3806–3812, 1981.

[DJKM81b] Etsurō Date, Michio Jimbo, Masaki Kashiwara, and Tetsuji Miwa. Transformation groups for soliton equations. VI. KP hierarchies of orthogonal and symplectic type. *J. Phys. Soc. Japan*, 50(11):3813–3818, 1981.

[DKM81] Etsurō Date, Masaki Kashiwara, and Tetsuji Miwa. Transformation groups for soliton equations. II. Vertex operators and τ functions. *Proc. Japan Acad. Ser. A Math. Sci.*, 57(8):387–392, 1981.

[FBZ04] Edward Frenkel and David Ben-Zvi. *Vertex algebras and algebraic curves*, volume 88 of *Mathematical Surveys and Monographs.* American Mathematical Society, Providence, RI, second edition, 2004.

[FFR91] Alex J Feingold, Igor B Frenkel, and John F X Ries. *Spinor construction of vertex operator algebras, triality, and* $E_8^{(1)}$, volume 121 of *Contemporary Mathematics.* American Mathematical Society, Providence, RI, 1991.

[Fre81] Igor B Frenkel. Two constructions of affine Lie algebra representations and boson-fermion correspondence in quantum field theory. *J. Funct. Anal.*, 44(3):259–327, 1981.

[FHL93] Igor B Frenkel, Yi-Zhi Huang, and James Lepowsky. On axiomatic approaches to vertex operator algebras and modules. *Mem. Amer. Math. Soc.*, 104(494):viii+64, 1993.

[FLM88] Igor B Frenkel, James Lepowsky, and Arne Meurman. *Vertex operator algebras and the Monster*, volume 134 of *Pure and Applied Mathematics.* Academic Press Inc., Boston, MA, 1988.

[Hua98] Kerson Huang. *Quantum field theory.* John Wiley & Sons Inc., New York, 1998. From operators to path integrals.

[Jan79] Jens Carsten Jantzen. *Moduln mit einem höchsten Gewicht*, volume 750 of *Lecture Notes in Mathematics.* Springer, Berlin, 1979.

[Kac90] Victor Kac. *Infinite-dimensional Lie algebras.* Cambridge University Press, Cambridge, third edition, 1990.

[Kac98] Victor Kac. *Vertex algebras for beginners*, volume 10 of *University Lecture Series.* American Mathematical Society, Providence, RI, second edition, 1998.

[KR87] Victor Kac and A K Raina. *Bombay lectures on highest weight representations of infinite-dimensional Lie algebras*, volume 2 of *Advanced Series in Mathematical Physics.* World Scientific Publishing Co. Inc., Teaneck, NJ, 1987.

[KW94] Victor Kac and Weiqiang Wang. Vertex operator superalgebras and their representations. In *Mathematical aspects of conformal and topological field theories and quantum groups (South Hadley, MA, 1992)*, volume 175 of *Contemp. Math.*, pages 161–191. Amer. Math. Soc., Providence, RI, 1994.

[KWY98] Victor Kac, Weiqiang Wang, and Catherine H Yan. Quasifinite representations of classical Lie subalgebras of $W_{1+\infty}$. *Adv. Math.*, 139(1):56–140, 1998.

[LL04] James Lepowsky and Haisheng Li. *Introduction to vertex operator algebras and their representations*, volume 227 of *Progress in Mathematics.* Birkhäuser Boston Inc., Boston, MA, 2004.

[MP95] Robert V Moody and Arturo Pianzola. *Lie algebras with triangular decompositions.* Canadian Mathematical Society Series of Monographs and Advanced Texts. John Wiley & Sons Inc., New York, 1995. A Wiley-Interscience Publication.

[MTZ08] Geoffrey Mason, Michael P Tuite, and Alexander Zuevsky. Torus n-point functions for $\mathbb{R}$-graded vertex operator superalgebras and continuous fermion orbifolds. *Comm. Math. Phys.*, 283(2):305–342, 2008.

[Wan99a] Weiqiang Wang. Dual pairs and infinite dimensional Lie algebras. In *Recent developments in quantum affine algebras and related topics (Raleigh, NC, 1998)*, volume 248 of *Contemp. Math.*, pages 453–469. Amer. Math. Soc., Providence, RI, 1999.

[Wan99b] Weiqiang Wang. Duality in infinite-dimensional Fock representations. *Commun. Contemp. Math.*, 1(2):155–199, 1999.