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On $D = 4$ Stationary Black Holes

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Abstract. We review some recent results concerning non-extremal and extremal stationary, asymptotically flat black hole solutions in extended $D = 4$ supergravities, and their properties with respect to the global symmetries of the theory. More specifically we refer to the effective three-dimensional description of these solutions and their classification within orbits with respect to the action of the global symmetry group, illustrating, for single-center solutions, the general mathematical relation between the orbits of non-extremal and extremal black holes.

1. Introduction

The effective three-dimensional description [1] of (asymptotically flat) stationary solutions to $D = 4$ supergravity theories have provided a valuable tool for their classification [2, 3, 4, 5, 6, 7, 8, 9, 10]. This approach consists in describing this kind of solutions as solutions to an effective $D = 3$ Euclidean sigma-model which is formally obtained by reducing the $D = 4$ theory along the time direction and dualizing the vector fields into scalars. The main advantage of such a description, with respect to the $D = 4$ one, is that it makes a larger global symmetry group G manifest. The set of all stationary solutions to the original four-dimensional theory is indeed invariant with respect to the global symmetry group G of the Euclidean three-dimensional sigma-model, which is the isometry group of its target space. Being G larger that the analogous symmetry group of the parent $D = 4$ model, transformations can be used to generate new solutions from known ones which were not available in four dimensions. This *solution-generating technique* has been first used in order to construct non-extremal, rotating, electrically charged black hole solutions coupled to scalar fields [2, 11] and, more recently, found application in the context of *subtracted geometry* [12, 13, 14].

Stationary, asymptotically flat, black holes can therefore be conveniently classified in orbits of with respect to the action of G. We shall restrict ourselves here to the single-center case. General features of the solution like its rotation and extremality (related to the temperature) are in particular associated with invariants of G. As far as the rotational property of the black hole is concerned, this statement was proven in [15] by defining a matrix Q_ψ which, just like the Noether charge matrix Q, lies in the Lie algebra $\mathfrak{g}$ of G, and which vanishes if and only if the solution is static. In terms of Q and Q_ψ the regularity condition for the black hole solution was written in a G-invariant way. The matrix Q_ψ allows to easily infer how the angular momentum $\mathscr{J}$ transforms under G. These tools were then applied in [16] in order to define the general algebraic procedure for connecting the orbit of non-extremal solutions to those of extremal ones. In particular, as far as extremal under-rotating and static black holes are concerned, this

mechanism makes use of *singular Harrison transformations* and generalizes previous results in the literature, related to specific electric-magnetic frames. We shall review this analysis below.

2. The $D = 3$ description of stationary solutions

Our original setting is $D = 4$ extended (i.e. $\mathcal{N} > 1$), ungauged supergravity, whose bosonic sector consists in n_s scalar fields $\phi^r(x)$, n_v vector fields $A^\Lambda_\mu(x)$, $\Lambda = 0, \ldots, n_v - 1$, and the graviton $g_{\mu\nu}(x)$, which are described by the following Lagrangian [1]:

$$\mathcal{L}_4 = e \left(\frac{R}{2} - \frac{1}{2} G_{rs}(\phi^t)\, \partial_\mu \phi^r\, \partial^\mu \phi^s + \frac{1}{4} I_{\Lambda\Sigma}(\phi^r)\, F^\Lambda_{\mu\nu}\, F^{\Sigma\,\mu\nu} + \frac{1}{8\,e} R_{\Lambda\Sigma}(\phi^r)\, \epsilon^{\mu\nu\rho\sigma}\, F^\Lambda_{\mu\nu}\, F^\Sigma_{\rho\sigma} \right) . \quad (1)$$

A distinctive feature of supergravity models is that the scalar fields are described by a non-linear sigma-model, namely they are coordinates of a Riemannian target-space $\mathcal{M}^{(4)}_{scal}$, with positive-definite metric $G_{rs}(\phi)$. We shall restrict our analysis to scalar manifolds which are homogeneous symmetric, namely have the general form $\mathcal{M}^{(4)}_{scal} = G_4/H_4$, where G_4 is a semi-simple, non-compact Lie group and H_4 its maximal compact subgroup. Notice, moreover, that the scalar fields are non-minimally coupled to the vector fields through the (negative-definite) matrix $I_{\Lambda\Sigma}(\phi^r)$ (which generalizes the inverse squared-coupling constants) and the matrix $R_{\Lambda\Sigma}(\phi^r)$ (generalizing the theta-angle). The space-time metric of a stationary solution has the general form:

$$ds^2 = -e^{2U}\, (dt + \omega_i\, dx^i)^2 + e^{-2U}\, g_{ij}\, dx^i\, dx^j\,, \quad (2)$$

where $i, j = 1, 2, 3$ label the spatial coordinates and U, ω_i, g_{ij} are all functions of x^i. Since the seminal work by Breitenlohner, Gibbons and Maison [1], it is known that stationary solutions to the $D = 4$ theory are solutions to an effective sigma model defined on an Euclidean three-dimensional space, formally obtained by first reducing the four-dimensional model along the time direction, and then dualizing the $D = 3$ vector fields into scalars. The scalar fields of this effective model $\phi^I(x^i)$ are $n = 2 + n_s + 2n_v$ and comprise, besides the original four-dimensional scalars $\phi^r(x^i)$, the warp function U, the scalar a dual to the vector $\omega_i(x^j)$ and the $2n_v$ scalars $\mathcal{Z}^M = (\mathcal{Z}^\Lambda,\, \mathcal{Z}_\Lambda)$. The precise relation between the scalars a, $\mathcal{Z}^M$ and the four-dimensional fields is:

$$A^\Lambda = A^\Lambda_0 (dt + \omega) + A^\Lambda_{(3)}\,, \quad A^\Lambda_{(3)} \equiv A^\Lambda_i dx^i\,, \quad (3)$$

$$\mathbb{F}^M = \begin{pmatrix} F^\Lambda_{\mu\nu} \\ G_{\Lambda\,\mu\nu} \end{pmatrix} \frac{dx^\mu \wedge dx^\nu}{2} = d\mathcal{Z}^M \wedge (dt + \omega) + e^{-2U} \mathbb{C}^{MN} \mathcal{M}_{(4)NP}\, {}^{*_3} d\mathcal{Z}^P\,, \quad (4)$$

$$da = -e^{4U}\, {}^{*_3} d\omega - \mathcal{Z}^T \mathbb{C} d\mathcal{Z}\,, \quad (5)$$

where $\mathbb{C}_{MN}$ is the $(2n_v) \times (2n_v)$ symplectic invariant, antisymmetric matrix, $\omega = \omega_i\, dx^i$, $*_3$ is the Hodge operation in the $D = 3$ Euclidean space, $F^\Lambda_{\mu\nu}$ are the vector field strengths and $G_{\Lambda\,\mu\nu}$ their magnetic-duals. The symmetric, symplectic negative-definite matrix $\mathcal{M}_{(4)NP}$ is built out of $I = (I_{\Lambda\Sigma})$, $R = (R_{\Lambda\Sigma})$ as follows:

$$\mathcal{M}_{(4)MN} = \begin{pmatrix} (I + R\, I^{-1}\, R)_{\Lambda\Sigma} & -(R\, I^{-1})_\Lambda{}^\Gamma \\ -(I^{-1}\, R)^\Delta{}_\Sigma & I^{-1\,\Delta\Gamma} \end{pmatrix} . \quad (6)$$

The effective sigma-model Lagrangian has the general form:

$$\mathcal{L}_3 = e_3 \left(\frac{R_3}{2} - \frac{1}{2} G_{IJ}(\phi^K)\, \partial_i \phi^I\, \partial^i \phi^J \right) , \quad (7)$$

[1] Here we adopt the notations and conventions of [15, 16] (in particular we use the "mostly plus" convention and $8\pi G = c = \hbar = 1$).

where $R_3 = R[g_{ij}]$ and $e_3 = \sqrt{\det(g_{ij})}$. The scalars $\phi^I(x^i)$ span a new manifold $\mathcal{M}_{scal}$, which is homogeneous symmetric if and only if $\mathcal{M}^{(4)}_{scal}$ is. Therefore, in our analysis, it can then be expressed as $\mathcal{M}_{scal} = G/H$, where G is the (semisimple) isometry group and H is a maximal subgroup of G. This target manifold is pseudo-Riemannian and its metric $G_{IJ}(\phi^K)$ reads:

$$\frac{1}{2}\,G_{IJ}(\phi^K)\,\partial_i\phi^I\,\partial^i\phi^J = \partial_i U\partial^i U + \frac{1}{2}\,G_{rs}\,\partial_i\phi^r\,\partial^i\phi^s + \frac{1}{2}\mathrm{e}^{-2\,U}\,\partial_i\mathcal{Z}^T\,\mathcal{M}_{(4)}\,\partial^i\mathcal{Z} + \\ + \frac{1}{4}\mathrm{e}^{-4\,U}\,(\partial_i a + \mathcal{Z}^T\mathbb{C}\partial_i\mathcal{Z})(\partial^i a + \mathcal{Z}^T\mathbb{C}\partial^i\mathcal{Z})\,, \tag{8}$$

its negative-signature directions being related to the scalar fields $\mathcal{Z}^M$. This feature defines H as a suitable non-compact real form of the maximal compact subgroup of G.

Spherically-symmetric, asymptotically -flat black hole solutions are described in this setting by geodesics $\phi^I = \phi^I(\tau)$ on $\mathcal{M}_{scal}$, in which the affine parameter $\tau \leq 0$ is a harmonic function of the radial coordinate r. Let us first review the geodesic description of static solutions and then move to the description of more general ones. A geodesic is uniquely defined by its initial data: an initial point $\phi_0 = (\phi^I_0)$ at radial infinity, corresponding to the limit $\tau \to 0^-$, and an "initial velocity vector" Q in the tangent space $T_{\phi_0}(\mathcal{M}_{scal})$ to the manifold in ϕ_0, see Fig. 1. The global symmetry group of the effective three-dimensional theory is the isometry group G, whose action on a geodesic (ϕ_0, Q) can be described as follows: Using G/H we can freely move the initial point ϕ_0 over the manifold, to a new point ϕ'_0 with "velocity vector" Q'; then, for a fixed initial point ϕ'_0, we can still act on the initial velocity vector Q' using the stability group H of ϕ'_0, acting on $T_{\phi'_0}(\mathcal{M}_{scal})$, see Fig. 2. Since the action of G/H on ϕ_0 is transitive, we can always fix ϕ_0 to coincide with the origin O (defined by the vanishing values of all the scalars) and then classify the orbits of the geodesics under the action of G (i.e. in maximal sets of solutions connected through the action of G) in terms of the orbits of the velocity vector $Q \in T_O(\mathcal{M}_{scal})$ under the action of H.

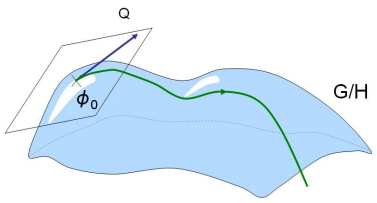

Figure 1. A geodesic on $\mathcal{M}_{scal}$ defined by its initial data.

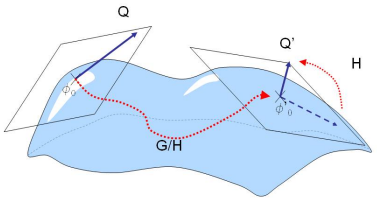

Figure 2. Action of G on a geodesic.

As we consider more general stationary solutions, they are clearly no longer described by geodesics on the scalar manifold, since the scalars ϕ^I will also depend on the angular coordinates. We shall focus our attention on axisymmetric, single-center black holes, for which the scalar fields will be functions of r and θ only: $\phi^I = \phi^I(r,\theta)$. Nevertheless, the solution will still be characterized by a unique point ϕ_0 at radial infinity ("initial point"):

$$\phi^I_0 = \lim_{r\to\infty}\phi^I(r,\theta)\,, \tag{9}$$

and by an "initial velocity" vector Q. By the same token, we can fix $\phi_0 \equiv O$ and classify these solutions according to the action of H on $Q \in T_O(\mathcal{M}_{scal})$.

This tangent space at the origin is isomorphic to subspace $\mathfrak{K}$ (the *coset space*) of the Lie algebra $\mathfrak{g}$ of G, complement to the Lie algebra $\mathfrak{H}$ of H: $T_O(\mathcal{M}_{scal}) \sim \mathfrak{K}$. The space $\mathfrak{K}$, in turn, is the carrier of a representation of H with respect to its adjoint action: $H^{-1}\,\mathfrak{K}\,H \subset \mathfrak{K}$, which realizes the action of the isotropy group H on the tangent space. Therefore the "initial velocity" vector Q should be viewed as a matrix in $\mathfrak{K}$. In fact, Q is the Noether charge matrix of the solution:

$$Q = \frac{1}{4\pi}\int_{S_2} {}^{*_3}J\,, \tag{10}$$

$J = J_i\,dx^i$ being the Noether current. The explicit form of J is given by the standard theory of sigma models on coset manifolds:

$$J_i \equiv \frac{1}{2}\partial_i\phi^I\,\mathcal{M}^{-1}\partial_I\mathcal{M}\,, \tag{11}$$

where $\mathcal{M}(\phi^I) = \mathbb{L}(\phi^I)\eta\mathbb{L}(\phi^I)^\dagger$ is an H-invariant symmetric matrix built out of the representative $\mathbb{L}(\phi^I)$ of G/H at the point ϕ^I. The involution defining the subalgebra $\mathfrak{H}$ of H in $\mathfrak{g}$ is defined in terms of an H-invariant real metric η as follows: $h \in \mathfrak{H} \;\Leftrightarrow\; \eta\,h^\dagger\,\eta = -h$ (here we work in some matrix representation of G). The components of Q along a suitable basis $\{K_A\} = \{K_0,\, K_\bullet,\, K_r,\, K_\Lambda,\, K^\Lambda\}$ of $\mathfrak{K} \sim T_O(\mathcal{M}_{scal})$ are the physical quantities characterizing the solution at spatial infinity, namely the ADM mass M_{ADM}, the NUT-charge n_{NUT}, the $D = 4$ scalar charges Σ_r and the electric and magnetic charges q_Λ, p^Λ:

$$Q \propto M_{ADM}\,K_0 + \Sigma^r\,K_r + n_{NUT}\,K_\bullet + p^\Lambda K_\Lambda + q_\Lambda K^\Lambda \in \mathfrak{K}\,.$$

Notice that the angular momentum $\mathscr{J}$ does not appear in this expansion.

2.1. Angular Momentum and Duality

In [15] we posed the question: Can we describe, just as all the other physical quantities, the angular momentum as a suitable component of some characteristic vector Q_ψ in the tangent space? In other words: Can we describe the angular momentum in terms of quantities which are intrinsic to the $D = 3$ sigma-model? The answer was given by defining Q_ψ as the following $\mathfrak{K}$-valued matrix:

$$Q_\psi = -\frac{3}{4\pi}\int_{S^\infty_2} \psi_{[i}J_{j]}\;dx^i\wedge dx^j = \frac{3}{8\pi}\int_{S^\infty_2} g_{\varphi\varphi}\,J_\theta d\theta d\varphi \propto \mathscr{J}\,K_\bullet + \cdots \in \mathfrak{K}\,, \tag{12}$$

$\psi = \partial_\varphi$ being the angular Killing vector of the axisymmetric solution. Q_ψ is a matrix in the coset space which describes the rotation of the solution. This quantity, as we review below, is important since it allows to easily derive the action of G on $\mathscr{J}$.

2.2. Action of G on a Solution

Being G the global symmetry of the $D = 3$ model, if $\phi^I(x^i)$ is a solution, the configuration $\phi'^I(x^i)$, obtained by acting on it by means of an element g of G, is still a solution. Let us denote the g-transformed $\phi' = (\phi'^I)$ of the point $\phi = (\phi^I)$ by $\phi' = g\star\phi$. The (non-linear) relation between ϕ'^I and ϕ^I follows from basic coset-space geometry:

$$\forall g \in G\;:\;\; \mathcal{M}(g\star\phi) = D[g]\,\mathcal{M}(\phi)\,D[g]^\dagger\,, \tag{13}$$

where $D[g]$ is the matrix associated with g in the chosen representation of G. Equation (13) can be used to devise a *solution generating technique* [2, 11] in order to construct new stationary

solutions from known ones. In particular, from eq.s (11), (10) and (12), it follows that, under a global symmetry transformation g, J, Q, Q_ψ transform through the adjoint action:

$$J \;\to\; (D[g]^{-1})^\dagger\, J\, D[g]^\dagger \;;\; Q \;\to\; (D[g]^{-1})^\dagger\, Q\, D[g]^\dagger \;;\; Q_\psi \;\to\; (D[g]^{-1})^\dagger\, Q_\psi\, D[g]^\dagger \,. \tag{14}$$

Recall now that, having fixed $\phi_0 = O$, G was broken to H, so that the action of G on a solution amounts to an action of H on the tangent space quantities: Q, Q_ψ. The above properties allow us to infer the physical features of the transformed solution (including the angular momentum, by virtue of (12)), without having to directly solve the matrix equation (13).

Moreover, in light of eq.s (14), we can characterize the rotational property of a black hole in the effective $D = 3$ description by the following G-invariant statement:

$$\text{Static solution} \;\;\Leftrightarrow\;\; Q_\psi = 0 \,. \tag{15}$$

The presence of a non-vanishing Q_ψ is a characteristic of the G-orbits of rotating solutions and therefore one cannot *generate* rotation on a static $D = 4$ solution using G !

Using Q and Q_ψ one can recast the regularity condition of a rotating solution in a G-invariant form. Consider the Kerr-Newman solution to Einstein-Maxwell theory:

$$ds^2 = \frac{\tilde\Delta}{\rho^2}(dt+\omega)^2 - \frac{\rho^2}{\tilde\Delta}\left(\frac{\tilde\Delta}{\Delta}dr^2 + \tilde\Delta d\theta^2 + \Delta\sin^2\theta d\varphi^2\right) \tag{16}$$

where

$$\begin{aligned}
\Delta &= (r-m)^2 - \mathbf{c}^2 \,, \\
\mathbf{c}^2 &= m^2 - \frac{1}{2}(q^2+p^2) - \alpha^2 \,, \\
\tilde\Delta &= \Delta - \alpha^2\sin^2\theta \,, \\
\rho^2 &= r^2 + \alpha^2\cos^2\theta \,, \\
\omega &= \alpha\sin^2\theta\,\frac{\rho^2-\tilde\Delta}{\tilde\Delta}\,d\varphi \,,
\end{aligned} \tag{17}$$

where m, q, p are the mass and the electric and magnetic charges while the angular momentum is given by $\mathscr{J} = m\,\alpha$. The vector potential A^0 is given by

$$A^0 \;=\; (-q\,r + p\,\alpha\,\cos\theta)\,\frac{dt}{\rho^2} + +[-p\,(\alpha^2+r^2)\cos\theta + q\,\alpha\,r\,\sin^2\theta]\,\frac{d\varphi}{\rho^2} \,. \tag{18}$$

This solution is regular provided the following condition is satisfied:

$$m^2 - \frac{1}{2}(q^2+p^2) \geq \alpha^2 \,. \tag{19}$$

In this case Q and Q_ψ are diagonalizable matrices and one finds that:

$$\frac{k}{2}\,\mathrm{Tr}(Q^2) = m^2 - \frac{p^2+q^2}{2} \quad ; \quad \mathrm{Tr}(Q_\psi^2) = \frac{\mathscr{J}^2}{m^2}\,\mathrm{Tr}(Q^2) \,,$$

where k is a representation-dependent constant. Using the above relations we can rewrite the regularity condition (19) in the following form:

$$\frac{k}{2}\,\mathrm{Tr}(Q^2) \geq \frac{\mathrm{Tr}(Q_\psi^2)}{\mathrm{Tr}(Q^2)} \,, \tag{20}$$

which, in light of the transformation properties (14), is manifestly G-invariant. This means that it is satisfied by all the solutions in the same G orbit as the regular KN one. Equality in eq. (19), or (20), holds for the *extremal* solutions (i.e, with vanishing Hawking temperature).

3. Extremal Limits

In [16] we addressed the problem of defining a general mechanism for connecting the orbit of non-extremal Kerr (or Kerr-Newman) solutions to the orbits of the known extremal ones. The regularity bound (20) can be saturated while keeping both sides non-vanishing (e.g. Q and Q_ψ diagonalizable just as in the Kerr-Newman case). The resulting solution exhibits an ergo-sphere and is dubbed *extremal over-rotating*. The bound can alternatively be saturated in a non-trivial way, by letting both sides of the inequality vanish separately. The resulting solutions can either be *extremal static* (see for instance [17] and references therein) or *extremal under-rotating* (i.e. rotating with no ergo-sphere) [18, 19, 20, 21, 22].

These limits to extremal (under-rotating or static) solutions have been considered in the literature in specific contexts: Heterotic theory [11, 23]; Kaluza-Klein supergravity [18, 19]. In [16] we defined a general geometric prescription for connecting the non-extremal Kerr-orbit to the extremal static or under-rotating ones, in a way which is frame-independent (i.e. does not depend on the particular string theory and compactification yielding the four-dimensional supergravity). This procedure makes use of *singular Harrison transformations* by means of which an Inönü–Wigner contraction on the matrices Q and Q_ψ is implemented, resulting in the nilpotent matrices $Q^{(0)}$ and $Q_\psi^{(0)}$ associated with extremal static or under-rotating black holes.

Harrison transformations [1] are H-transformations which play a special role in the solution generating techniques: They are not present among the global symmetries of the $D = 4$ theory and have the distinctive property of switching on electric or magnetic charges when acting on neutral solutions (like the Kerr or Schwarzshild ones). Their generators $(\mathbb{J}_M) = (\mathbb{J}_\Lambda, \mathbb{J}^\Lambda)$ in $\mathfrak{H}$ are in one-to-one correspondence with the electric and magnetic charges $(\mathcal{P}^M) = (p^\Lambda, q_\Lambda)$ and are non-compact (i.e. are represented, in a suitable basis, by hermitian matrices). The space $\mathrm{Span}(\mathbb{J}_M)$ generated by $\{\mathbb{J}_M\}$ is the coset space of the symmetric manifold H/H_c, H_c being the maximal compact subgroup of H, and thus it is the carrier of a representation of H_c (the same representation in which the charges $\mathcal{P}^M$ transform with respect to H_c). More specifically $H_c = \mathrm{U}(1)_E \times H_4$, where $\mathrm{U}(1)_E$ is an Ehlers transformation, and H_4 is the maximal compact subgroup of G_4.

In [16] we considered the *maximal abelian subalgebra* (MASA) of the space $\mathrm{Span}(\mathbb{J}_M)$. This is a subspace whose generators $\mathbb{J}^{(N)} = \{\mathcal{J}_\ell\}_{\ell=1,\ldots,p}$ are defined by the *normal form* of the electric and magnetic charges, i.e. the minimal subset of charges into which the charges of the most general solution can be rotated by means of H_c. Its dimension p is therefore just the *rank of the coset* H/H_c. In the maximal supergravity, for example, $p = \mathrm{rank}\left(\frac{\mathrm{SO}^*(16)}{\mathrm{U}(8)}\right) = 4$, the same being true for the half-maximal theory, $p = \mathrm{rank}\left(\frac{\mathrm{SO}(6,2)\times\mathrm{SO}(2,6+n)}{\mathrm{SO}(2)^2\times\mathrm{SO}(6)\times\mathrm{SO}(6+n)}\right) = 4$, and for the $\mathcal{N} = 2$ symmetric models with rank-3 scalar manifold in $D = 4$ (for this class of theories, $p =$ rank $+1$). The simplest representative of the latter class of models is the STU one, which is a consistent truncation of all the others, besides being a truncation of the maximal and half-maximal theories. Therefore its space $\mathbb{J}^{(N)}$ is contained in the spaces of Harrison generators of all the above mentioned symmetric models. As a consequence of this, for the sake of simplicity, we can restrict ourselves to the simplest STU model since the G-orbits of non-extremal and extremal regular solutions to the broad class of symmetric models mentioned above have a representative in the common STU truncation. As for the restricted number of $\mathcal{N} = 2$ symmetric models for which the rank of $\mathcal{M}^{(4)}_{scal}$ is less than 3, the following discussion has a straightforward generalization. Depending on the symplectic frame, i.e. on the higher-dimensional origin of the four-dimensional theory, this normal form can consist of different kinds of charges. If the $D = 4$ supergravity originates from a dimensional reduction of a $D = 5$ theory on a circle, one normal form is $\{p^0, q_i\}$ and another is $\{q_0, p^i\}$, $i = 1, \ldots, p-1$.

The procedure devised in [16] consists in first acting on the Kerr solution, given by eq.s (16),(17),(18) with $p = q = 0$, by means of a Harrison transformation generated by $\mathbb{J}^{(N)}$, of the

form:

$$\mathcal{O} = \exp\left(\sum_{\ell=0}^{p-1} \log(\beta_\ell)\mathcal{J}_\ell\right) . \tag{21}$$

The resulting solution is a non-extremal, rotating one, coupled to scalar fields, with charges in the normal form. In the Heterotic model where the normal form consists of two electric and two magnetic charges, this solution was first constructed in [2]. This construction reviewed here applies to any symplectic frame. If we denote by $Q^{(K)}$, $Q_\psi^{(K)}$ the matrices Q, Q_ψ pertaining to the Kerr solution

$$Q^{(K)} = 2\, m_K\, H_0 \ , \ \ Q_\psi^{(K)} = 2\, \mathscr{J}_K\, K_\bullet \, , \tag{22}$$

m_K, $\mathscr{J}_K$ being the mass and the angular momentum of the solution, the corresponding matrices for the transformed one are readily computed from (14):

$$Q' = \mathcal{O}^{-1\,\dagger}\, Q^{(K)}\, \mathcal{O}^\dagger \ ; \ \ Q'_\psi = \mathcal{O}^{-1\,\dagger}\, Q_\psi^{(K)}\, \mathcal{O}^\dagger \, . \tag{23}$$

We refer to [16] for the precise dependence of the mass, angular momentum and charges of the new solution on m_K, $\mathscr{J}_K$ and β_ℓ. The next step is to rescale the Harrison parameters $\beta_\ell > 0$ and the original angular momentum $\mathscr{J}_K$ as follows:

$$\beta_\ell = (m_K)^{\sigma_\ell}\, \alpha_\ell \ \ ; \ \ \mathscr{J}_K = m_K^2\, \Omega \, , \tag{24}$$

where $\sigma_\ell = \pm 1$, and send $m_K \to 0$ while keeping α_ℓ and Ω fixed. Although this is clearly a singular limit for the Harrison transformation, the resulting limiting solution is well defined. The normal-form charges in the limit will only depend on α_ℓ and on the signs σ_ℓ and the effect on Q' and Q'_ψ is, as anticipated above, an Inönü–Wigner contraction yielding nilpotent matrices:

$$\{Q',\, Q'_\psi\} \ \stackrel{m_K \to 0}{\longrightarrow} \ \ \{Q^{(0)},\, Q_\psi^{(0)}\} \ \text{nilpotent} \, . \tag{25}$$

As a consequence of this $\mathrm{Tr}(Q'^2)$ as well as $\mathrm{Tr}(Q'^2_\psi)$ will both vanish in this limit. However the latter quantity vanishes "faster" than the former and this guarantees that both sides of the regularity bound (20) vanish separately. The resulting solution is therefore extremal. Let us illustrate this in some more detail.

Four-dimensional black hole solutions can be classified in orbits with respect to the global symmetry group G_4 in $D = 4$ (which are clearly smaller than the G-orbits in the $D = 3$ description, being G larger than G_4). This group acts on the $D = 4$ fields as a generalized electric-magnetic duality, which respect to which the electric and magnetic charges transform in some characteristic symplectic representation. Among the G-invariant quantities labeling these orbits, the most important is the *quartic invariant* $I_4(p^\Lambda, q_\Lambda)$ which only depends on the electric and magnetic charges and totally characterizes the geometry near the horizon of the extremal static solutions, by virtue of the *attractor mechanism* [24, 25][2]. Being the horizon area proportional to $\sqrt{|I_4|}$, regular extremal solutions can either have $I_4 > 0$ or $I_4 < 0$. In the whole class of supergravities mentioned above, which have the STU one as a consistent truncation, the former solutions can either be supersymmetric (BPS) or non-supersymmetric, while the latter can only be non-supersymmetric. The sign of I_4 in the limiting solution depends on the signs σ_ℓ. We distinguish between two relevant cases.

[2] For a classification of extremal solutions with respect to G_4 see [26].

Case 1. *Only* for the choices of σ_ℓ yielding $I_4 < 0$, we have a residual angular momentum given by:

$$\mathscr{J} = \frac{\Omega}{2}\sqrt{|I_4(p,q)|} \neq 0\,. \tag{26}$$

It is apparent, from the above expression, that the angular momentum, though not G-invariant, is invariant, as expected, under the action of the global symmetry group G_4 in four-dimensions. Indeed $I_4(p,q)$ is G_4-invariant and Ω, pertaining to a Kerr solution, is G_4-invariant as well. The resulting solution is a non-BPS extremal under-rotating black hole. It is important to emphasize here that, in this limit, we find the most general solution of this class *modulo* action of G (generating solution with respect to G).

We also find that the *degree of nilpotency* of $Q^{(0)}_\psi$ is one unit less than that of $Q^{(0)}$. This explains why $\mathrm{Tr}(Q'^2_\psi)$ vanishes faster than $\mathrm{Tr}(Q'^2)$, as mentioned above.

Case 2. For choices of σ_ℓ yielding $I_4 > 0$, we have no residual rotation, i.e. $Q^{(0)}_\psi = 0$, and the limiting solution is either BPS or extremal non-BPS *static*. $Q^{(0)}$ is nilpotent and non-vanishing. Also in this case we find the generating solutions of these classes of black holes with respect to G.

Singular non-extremal limit. There is an other kind of limit we can consider, which consists in keeping $\mathscr{J}_K/m_K$, instead of $\mathscr{J}_K/m_K^2$, fixed. In this case we only have finite limits for $I_4 > 0$. The matrices $Q^{(0)}$ and $Q^{(0)}_\psi$ of the limiting solution are still both nipotent, though they have now the same degree. As a consequence of this, while in the limit the left hand side of (20) vanishes, the right hand side stays finite. Therefore the resulting solution, violating the regularity bound (20), is *singular*. This class of black holes comprises the rotating BPS ones studied, for instance, in [27].

4. Conclusions

We have reviewed a general mechanism for retrieving, in the effective $D = 3$ description, all the extremal limits, including the (nilpotent) G-orbits of regular single-center extremal static and under-rotating solutions from that of non-extremal regular solutions (the Kerr solution). The procedure does not depend on the particular symplectic frame in which the electric and magnetic charges of the four-dimensional supergravity are defined. A valuable tool in this analysis was the matrix Q_ψ, first introduced in [15], which encodes the rotation-properties of the black hole and allows to infer the transformation properties of the angular momentum under G without having to explicitly derive the new solution.

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