Renormalization group for quantum walks

S Boettcher[1], **S Falkner**[1] **and R Portugal**[2]

[1]Department of Physics, Emory University, Atlanta, GA 30322; USA

[2]Laboratorio Nacional de Computacao Científica, Petropolis, RJ 25651-075; Brazil

E-mail: `http://www.physics.emory.edu/faculty/boettcher`

Abstract. We present a detailed introduction to the discrete-time quantum walk problem, in close analogy with the classical ordinary and persistent random walk. This approach facilitates a uniform application of the renormalization group that highlights similarities and differences between the classical and the quantum walk problem. Specifically, we discuss the renormalization group treatment for the mean-square displacement of a walker starting from a single site on the $1d$-line for ordinary and persistent random walks and the quantum walk. We outline the significance of universality for quantum walks and the control this might provide for quantum algorithms. We use our RG method to verify that all 2-state quantum walks on the $1d$-line are in the same universality class.

1. Introduction

A renormalization group (RG) treatment [1] of quantum walks (QW) holds significant promise for a better understanding of search algorithms for quantum computing [4, 5, 6, 7] or of quantum transport phenomena [3]. Much numerical and analytical work has been done to extract properties of QW in specific networks and lattice, typically with specific settings for the quantum coin, for a recent overview see [2]. Surprisingly little attention has been paid to categorizing asymptotic properties of QW in their relation with inherent symmetries of the quantum problem. For example, from the existing work it may stand to reason that a QW starting from a single site of a translational-invariant lattice may have a mean-square displacement (MSD) of

$$\left\langle r^2 \right\rangle \sim t^{\frac{2}{d_w}} \qquad (1)$$

with an exponent $d_w = 1$, describing ballistic spread in all directions, but a formal proof of such a statement has not yet been attempted. While it seems clear that quantum interference effects are the cause of this rapid spread, the strength and limits of such a causal relation have not been tested. Unlike for a stochastic classical random walk, the unitary description of a QW immediately entails additional internal degrees of freedom (the "coin" space) whose dimensionality depends on the neighborhood degrees in the lattice, which may vary for different lattices (square, triangular, or hexagonal, for instance) even within the same spatial dimension. This raises the question as to why there would be such a uniform result for d_w. More generally: What are the relevant parameters that determine universality classes [1] in QW and other quantum transport problems? Much of the work on QW has been driven by a computer science perspective that have lead to very strong results, for instance, regarding QW as a universal computing framework [7], similar to well-known results for random walks in

classical computation, or on optimizing search performance. We hope to contribute to the basic understanding of QW by connecting asymptotic scaling properties to symmetries and dynamics of the process, the ultimate tenants of control physicists have historically derived from RG.

In the following, we first review many of the technical details of the RG formalism as applied to RW. Then, we observe that the classical subject of persistent random walks (PRW) can be formulated (as a 2$^{\text{nd}}$-order Markov process) in a manner identical to a QW on the same geometry. We explore the RG formalism for these PRW at length so that our QW discussion significantly shortens. Finally, we demonstrate how this RG approach carries over to QW and highlight some of the important new aspects that inevitably arise when extending the RG into the complex plane. We then reproduce with the RG the familiar exponent $d_w = 1$ for QW on the 1d-line.

2. General Formulation of the Walk Problem

The generic master-equation for a discrete-time walk with a coin, whether classical or quantum, is

$$|\Psi(t+1)\rangle = \mathcal{U}|\Psi(t)\rangle, \qquad \mathcal{U} = \mathcal{S}(\mathcal{C} \otimes \mathcal{I}), \qquad (2)$$

where the time-evolution operator (or propagator) $\mathcal{U}$ is written in terms of the "shift" operator $\mathcal{S}$ and the "coin" $\mathcal{C}$ [2]. In the d-dimensional site-basis $|\vec{n}\rangle$, we can describe the state of the system in terms of the site amplitudes $\psi_{\vec{n},t} = \langle \vec{n}|\Psi(t)\rangle$, simply the probability density to be at that site for a classical walk, but representing in the quantum walk a vector in coin-space with each component holding the amplitude for transitioning out of site $\vec{n}$ along one of its edges. Application of the coin $\mathcal{C}$ entangles these components, with subsequent redistribution of the walk to neighboring sites by the shift operator $\mathcal{S}$, based on those amplitudes. The general shift operator for a homogeneous nearest-neighbor walk on the line is

$$\mathcal{S} = \sum_{\vec{n}} \sum_{\mu=1}^{d} \left\{ P \otimes |\vec{n} - \hat{e}_\mu\rangle \langle \vec{n}| + Q \otimes |\vec{n} + \hat{e}_\mu\rangle \langle \vec{n}| + R \otimes |\vec{n}\rangle \langle \vec{n}| \right\} \qquad (3)$$

with the shift matrices P, Q, and R for moving left or right, or not at all. From (2), we get

$$\mathcal{U} = \sum_{\vec{n}} \sum_{\mu=1}^{d} \left\{ A \otimes |\vec{n} - \hat{e}_\mu\rangle \langle \vec{n}| + B \otimes |\vec{n} + \hat{e}_\mu\rangle \langle \vec{n}| + M \otimes |\vec{n}\rangle \langle \vec{n}| \right\} \qquad (4)$$

with $A = P\mathcal{C}$, $B = Q\mathcal{C}$, and $M = R\mathcal{C}$.

In a quantum walk, the "hopping" operators A, B, and M are constrained by the requirement of unitary propagation, $\mathcal{I} = \mathcal{U}^\dagger \mathcal{U}$, which results in the conditions in coin-space,

$$\mathcal{I}_d = A^\dagger A + B^\dagger B + M^\dagger M, \quad 0 = A^\dagger M + M^\dagger B = A^\dagger B. \qquad (5)$$

As $\mathcal{C}$ is unitary, these conditions equally apply to P, Q, and R. They can not be satisfies by scalars (except for trivial cases). The algebra in (5) requires at least two-dimensional matrices, and the common interpretation is that the dimension c of the coin space has to match the degree of each site, i.e., each component of the site amplitudes $\psi_{\vec{n},t}$ refers to the transmission along a specific edge [2]. Thus, on a d-dimensional hypercubic lattice the coin-space dimension is $c = 2d$ (or $c = 2d + 1$ to allow for a component for remaining at the same site).

In a typical discussion of QW on the 1d-line, most authors simply choose trivial shift matrices that satisfy (5), such as $P = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $Q = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, and $R = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ and explore some version(s) of the most general unitary coin, in this case,

$$\mathcal{C} = \begin{pmatrix} \sqrt{\rho} & \sqrt{1-\rho}\,e^{i\phi} \\ \sqrt{1-\rho}\,e^{i\theta} & -\sqrt{\rho}\,e^{i(\phi+\theta)} \end{pmatrix}. \qquad (6)$$
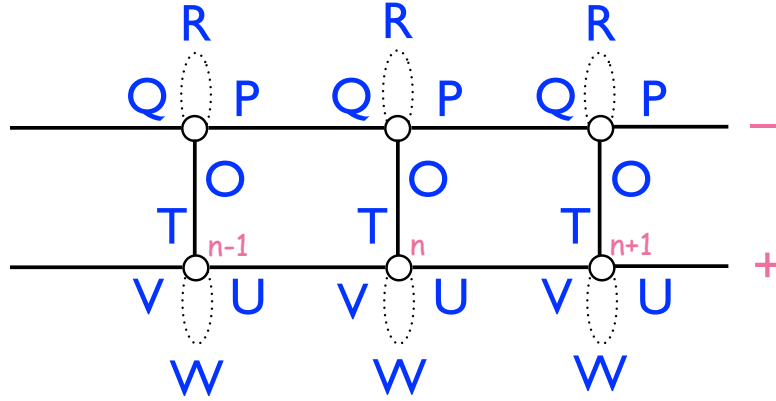
Figure 1. Depiction of the hopping operators for a quantum walk on a 1d-ladder. Each "rung" of the ladder is labeled by a consecutive site-index n while the upper and lower rail is labeled "-" and "+", respectively. Note that each vertex has degree 3, while the dashed loops are meant to indicate a hopping that remains at a site. Thus, the wavefunction on each site has either three or four components.

In general, the algebra in (5) is not limited to its lowest-ranked representatives (here, $c = 2$). It is not obvious that any higher-dimensional matrices obeying (5) would yield new universality classes, of course.

Classical RW are also represented by the propagator $\mathcal{U}$ in (4). In that case, the condition in (5) does not apply and we merely require $\mathcal{U}$ to be *stochastic*, i.e.,

$$A + B + M = \mathcal{C}, \tag{7}$$

where the coin $\mathcal{C}$ must be a stochastic matrix. In fact, (7) is satisfiable with scalars, say, $A = p\,(1-q)$, $B = (1-p)\,(1-q)$, and $M = q$, in which case $S = 1$. We note that the stochasticity constraint on $\mathcal{U}$ in (7) can be satisfied also by higher-dimensional matrices. For example, using the stochastic coin on the 1d-line,

$$\mathcal{C} = \begin{pmatrix} \rho & 1-\sigma \\ 1-\rho & \sigma \end{pmatrix}, \qquad 0 \le \rho, \sigma \le 1, \tag{8}$$

together with the basic shift matrices P, Q, and R (as above) reproduces the well-studied *persistent* RW [8, 9]. For this 2nd-order Markov process it is known that the freedom in the parameters ρ and σ does not provide a distinct universality from the ordinary RW, i.e., it behaves diffusive or ballistic for $\rho = \sigma$ or $\rho \ne \sigma$, respectively.

To illustrate the questions regarding universality in QW, consider the walk on a 1d-ladder, as shown in figure 1: We have in the site-basis $|n, \pm\rangle$ for the shift operator,

$$
\begin{aligned}
\mathcal{S} &= \sum_n \{ O\,|n,+\rangle\langle n,-| + P\,|n+1,-\rangle\langle n,-| + Q\,|n-1,-\rangle\langle n,-| + R\,|n,-\rangle\langle n,-| \\
&\quad + T\,|n,-\rangle\langle n,+| + U\,|n+1,+\rangle\langle n,+| + V\,|n-1,+\rangle\langle n,+| + W\,|n,+\rangle\langle n,+| \}, \\
&= \sum_n \left\{ \begin{bmatrix} R & O \\ T & W \end{bmatrix} |n\rangle\langle n| + \begin{bmatrix} P & 0 \\ 0 & U \end{bmatrix} |n+1\rangle\langle n| + \begin{bmatrix} Q & 0 \\ 0 & V \end{bmatrix} |n-1\rangle\langle n| \right\}, \tag{9}
\end{aligned}
$$

where the first row of the matrix correspond to "$|-\rangle$". Allowing even for two distinct coins for the upper and lower rail-sites,

$$\mathcal{C} = \left[\begin{array}{cc} C_- & 0 \\ 0 & C_+ \end{array} \right], \tag{10}$$

we obtain the propagator $\mathcal{U}$ in (4) with

$$M = \left[\begin{array}{cc} RC_- & OC_+ \\ TC_- & WC_+ \end{array} \right], \quad A = \left[\begin{array}{cc} PC_- & 0 \\ 0 & UC_+ \end{array} \right], \quad B = \left[\begin{array}{cc} QC_- & 0 \\ 0 & VC_+ \end{array} \right]. \tag{11}$$

For $\mathcal{U}$ to be unitary, A, B, and M again should satisfy (5). It appears from the degree-3 vertices of the 1d-ladder that the hopping operators in $\mathcal{S}$ should be at least 3×3 matrices (or 4×4, in case that $R \neq 0$ and $W \neq 0$ in figure 1 and an amplitude is needed for remaining at a site), which would make A, B, and M each 6×6 (or 8×8) matrices. Does a QW on this ladder always renormalize into the same universality class of the ordinary 2-state QW on a line? Can coins $C_\pm$ be devised such that the loops along the ladder alter the asymptotic scaling?

The fundamental quantity of interest for any walk is the amplitude $\psi_{\vec{n},t} = \langle \vec{n} | \Psi(t) \rangle$ to be at site $\vec{n}$ at time t [10]. Starting from some initial condition (IC) $\psi_{\vec{n},t=0}$, the time-evolution of the walk is governed by the master equation in (2), which now reads

$$\psi_{\vec{n},t+1} = \sum_{\vec{m}} U_{\vec{n},\vec{m}} \psi_{\vec{m},t} \tag{12}$$

with $U_{\vec{n},\vec{m}} = \langle \vec{n} | \mathcal{U} | \vec{m} \rangle$, for discrete "hops" from connected sites during a single time-step. To eliminate the aspect of time from this dynamic process in preparation for an RG treatment, we introduce generating functions (i.e., discrete Laplace transforms), here given by

$$\tilde{\psi}_{\vec{n}}(z) = \sum_{t=0}^{\infty} \psi_{\vec{n},t} z^t. \tag{13}$$

We typically suppress the argument z unless it appears explicitly. For simplicity, we merely consider IC localized at the origin, $\psi_{\vec{n},0} = \delta_{\vec{n},0}$, where $\delta_{\vec{n},\vec{m}}$ is the Kronecker symbol. Applying (13) to the master equation in (12) obtains

$$\tilde{\psi}_{\vec{n}} = \delta_{\vec{n},0} + z \sum_{\vec{m}} U_{\vec{n},\vec{m}} \tilde{\psi}_{\vec{m}}. \tag{14}$$

3. RG for a Classical Random Walk
For RW on a line, (12) with (4) reduces to

$$\tilde{\psi}_n = \delta_{n,0} + zq\tilde{\psi}_n + z(1-q)p\tilde{\psi}_{n-1} + z(1-q)(1-p)\tilde{\psi}_{n+1}, \tag{15}$$

where we have allowed for a probability q to remain at a given site and a bias p for jumps to the right, i.e., $1-p$ for jumps to the left. The RG start from (15) [11, 10], written for any even $n \neq 0$:

$$\begin{aligned} \tilde{\psi}_{n-1} &= a_k \tilde{\psi}_{n-2} + m_k \tilde{\psi}_{n-1} + b_k \tilde{\psi}_n = \frac{a_k}{1-m_k} \tilde{\psi}_{n-2} + \frac{b_k}{1-m_k} \tilde{\psi}_n, \\ \tilde{\psi}_n &= a_k \tilde{\psi}_{n-1} + m_k \tilde{\psi}_n + b_k \tilde{\psi}_{n+1}, \\ \tilde{\psi}_{n+1} &= a_k \tilde{\psi}_n + m_k \tilde{\psi}_{n+1} + b_k \tilde{\psi}_{n+2} = \frac{a_k}{1-m_k} \tilde{\psi}_n + \frac{b_k}{1-m_k} \tilde{\psi}_{n+2}, \end{aligned} \tag{16}$$

defining $a_0 = z(1-q)p$, $b_0 = z(1-q)(1-p)$, and $m_0 = zq$ as the "raw" hopping parameters. Eliminating algebraically all odd-site amplitudes (for $\ldots, n-3, n-1, n+1, n+3 \ldots$) and identifying in the remaining relations the renormalized hopping parameters,

$$\tilde{\psi}_n = a_{k+1}\tilde{\psi}_{n-2} + m_{k+1}\tilde{\psi}_n + b_{k+1}\tilde{\psi}_{n+2}, \qquad (17)$$

we obtain the RG-flow recursions

$$a_{k+1} = \frac{a_k^2}{1-m_k}, \qquad b_{k+1} = \frac{b_k^2}{1-m_k}, \qquad m_{k+1} = m_k + \frac{2a_k b_k}{1-m_k}. \qquad (18)$$

The fixed points (FP) arising from this RG-flow for $k \sim k+1 \to \infty$ are $(a^*, b^*, m^*) = (0, 0, m^*)$, $(1-m^*, 0, m^*)$, or $(0, 1-m^*, m^*)$ for any value of m^* (presumably on the unit interval). The 2nd and 3rd FP yield the ballistic solutions for $z \to 1$, with $d_w = 1$ easily obtained from the eigenvalues of the Jacobian

$$J = \left.\frac{\partial(a_{k+1}, b_{k+1}, m_{k+1})}{\partial(a_k, b_k, m_k)}\right|_{k \to \infty} = \begin{pmatrix} \frac{2a^*}{1-m^*}, & 0, & \frac{2b^*}{1-m^*} \\ 0, & \frac{2b^*}{1-m^*}, & \frac{2a^*}{1-m^*} \\ \frac{(a^*)^2}{(1-m^*)^2}, & \frac{(b^*)^2}{(1-m^*)^2}, & 1+\frac{2a^*b^*}{(1-m^*)^2} \end{pmatrix}, \qquad (19)$$

(From the exact solution one finds that in this case the FP value m^* depends in the initial values for the RG-flow, $1-m^* = |a_0 - b_0|_{z=1}$, which can not be obtained from the local analysis near FP.) The indeterminedness of m^* in the 1st FP is peculiar. In fact, only two possible values for m^* can be obtained. For any $|z| < 1$, only the trivial solution $m^* = 0$ can be reached. For $z = 1$, $a_k + b_k + m_k = 1$ for all k and starting from symmetric initial values $a_0 = b_0$, both remain identical and vanish together, $a_k \equiv b_k \to 0$, and $m_k \to m^* = 1$. Since both numerators and denominators in the Jacobian vanish, a correlated solution has to be constructed that "peals off" the leading behaviour to glance into the boundary layer. Using $a_k \equiv b_k \sim A_k \alpha^k$ and $m_k \sim 1 - M_k \alpha^k$ assuming large k and $|\alpha| < 1$, results in

$$A_{k+1} = \frac{A_k^2}{\alpha M_k}, \qquad M_{k+1} = \frac{1}{\alpha}M_k - \frac{2A_k^2}{\alpha M_k} \qquad (20)$$

with a single FP that self-consistently determines $\frac{A^*}{M^*} = \alpha = \frac{1}{2}$. The Jacobian of these recursions at its FP gives $\lambda = 4$ as the largest eigenvalue, i.e., $d_w = 2$ for the diffusive solution. In this formulation, even if we start with vanishing self-term, $q = 0$, initially, the self-term ultimately dominates, reflecting the fact that in diffusion the renormalized domain of size $L = 2^k$ outgrows the walk such that almost all hops remain within that domain.

We can conclude that the RG projects the salient, asymptotic properties of the walk into two universality classes (for any $0 \le q < 1$): ballistic motion, either to the left ($p < \frac{1}{2}$) or right ($p > \frac{1}{2}$), or diffusion ($p = \frac{1}{2}$), each characterized by a distinct exponent d_w. Each class reflects a fundamental (a)symmetry of the process.

4. Persistent Random Walks

The persistent random walk (PRW) [9, 8] on a $1d$-line is a well-studied process with many applications. On the line, persistence (or, anti-persistence) have a clear, intuitive meaning that can be described with a single parameter. PRW is a very useful precursor for the study of QW. In fact, we find that PRW is isomorph to QW up to the point when observables are considered. Then, of course, the difference between a dissipative stochastic process with direct probabilistic interpretation and one evolving a wave-function unitarily becomes very noticeable. But both are discrete processes in time and space described by a master equation that evolves a state variable

with internal degrees of freedom. In fact, as a 2nd-order Markov process [9], a PRW requires two separate pieces of initial conditions, for location and preferred direction, similar in structure to a Schroedinger wave-function in QW. The quantum coin-matrix in the QW has its equivalent here in a stochastic matrix driving the PRW. But the requirements for a stochastic matrix leads to very different physics than for a unitary coin. The results for the PRW on $1d$-lines provide hardly any new results compared to the ordinary RW studied above. On other lattices, such as the Hanoi networks [12], quite interesting and novel results can be achieved for PRW, though.

We will label the state variable describing a PRW also as $\psi_{n,t}$ with the understanding that it now represents a 2-component vector. Say, the upper component, $\psi_{n,t}^+$, refers to a walker with the preference to step to the right in the next time-step, and the lower component, $\psi_{n,t}^-$, indicates a left-hop preference. The value of each component describes the probability of finding a walker at that site n and time t in state "$\pm$", and the total probability of finding a walker there, irrespective of preference, is simply the sum of the two, $p_{n,t} = \psi_{n,t}^+ + \psi_{n,t}^-$. Otherwise, the formalism above leading to the master equation (14) for the generating function applies unchanged, providing us again with the basis of our RG-analysis. However, due to the internal structure of our state variable, the renormalizable quantities describing the effective hopping probabilities between domains now possess matrix form, and we will have to reconcile with the fact that our RG-flow equations will take on the form of non-linear matrix recursions.

Ignoring a possible left-right bias here, we consider a walker coming from the left (right) to have a probability ρ to continue to move right (left), and a probability $1 - \rho$ to reverse direction in the next step. For $\rho > \frac{1}{2}$ ($\rho < \frac{1}{2}$) the walker exhibits persistence (anti-persistence), and for $\rho = \frac{1}{2}$ becomes again an ordinary unbiased RW without memory. The master equations then read:

$$
\begin{array}{rcl}
\tilde{\psi}_n^+ & = & z\rho\tilde{\psi}_{n-1}^+ + z\left(1 - \rho\right)\tilde{\psi}_{n-1}^- + \delta_{n,0}\tilde{\psi}_{IC}^+, \\
\tilde{\psi}_n^- & = & z\left(1 - \rho\right)\tilde{\psi}_{n+1}^+ + z\rho\tilde{\psi}_{n+1}^- + \delta_{n,0}\tilde{\psi}_{IC}^-,
\end{array}
\tag{21}
$$

where $\tilde{\psi}_{IC}$ (with $\tilde{\psi}_{IC}^+ + \tilde{\psi}_{IC}^- = 1$) represents the IC of the PRW, which we place again at the origin. Initially, before RG, $\tilde{\psi}_n^+$ ($\tilde{\psi}_n^-$) only depends on hops from its left (right) neighbor; it is that inflow which induces the "+" ("−") state. (21) is conveniently rewritten in matrix notation as

$$
\tilde{\psi}_n = M\tilde{\psi}_n + A\tilde{\psi}_{n-1} + B\tilde{\psi}_{n+1} + \delta_{n,0}\tilde{\psi}_{IC}
\tag{22}
$$

with "raw" hopping matrices

$$
A_0 = z \left(\begin{array}{cc} \rho & 1 - \rho \\ 0 & 0 \end{array} \right), \qquad B_0 = z \left(\begin{array}{cc} 0 & 0 \\ 1 - \rho & \rho \end{array} \right),
\tag{23}
$$

and $M_0 = 0$ before renormalization, $k = 0$. Notice that A and B are different but do not express a bias as in (15). It is more revealing to decompose these matrices further and write

$$
A_k = P_k\mathcal{C}, \qquad B_k = Q_k\mathcal{C}, \qquad M_k = R_k\mathcal{C}
\tag{24}
$$

with initially

$$
P_0 = z \left(\begin{array}{cc} 1 & 0 \\ 0 & 0 \end{array} \right), \qquad Q_0 = z \left(\begin{array}{cc} 0 & 0 \\ 0 & 1 \end{array} \right), \qquad R_0 = z \left(\begin{array}{cc} 0 & 0 \\ 0 & 0 \end{array} \right)
\tag{25}
$$

and the *stochastic* "coin" matrix $\mathcal{C}$ in (8) (choosing $\sigma = \rho$), where each column sums to unity. Separating the generalized hopping matrices P_k, Q_k, R_k as the renormalizable quantities from the coin $\mathcal{C}$ will prove convenient to determine scalar recursion relations describing the RG-flow. The

same procedure will apply also for the coin operator in QW below. Clearly, different coins may lead to different RG-flows, which is exactly the purpose of the RG: discriminating universality classes amongst the coins. For PRW on the $1d$-line, the RG confirms that there are no new universality classes beyond those of an ordinary RW. Note that the most general stochastic coin in $1d$ has to be a 2×2 real matrix with two constraints on the rows, leaving room for at most two free parameters. By fixing $\sigma = \rho$, we already eliminated the directional asymmetry that leads to ballistic motion discussed for RW in section 3 for $p \neq \frac{1}{2}$.

As in section 3, to apply RG we write (22) as

$$
\begin{aligned}
\tilde{\psi}_{n-1} &= M_k \tilde{\psi}_{n-1} + A_k \tilde{\psi}_{n-2} + B_k \tilde{\psi}_n, \\
\tilde{\psi}_n &= M_k \tilde{\psi}_n + A_k \tilde{\psi}_{n-1} + B_k \tilde{\psi}_{n+1}, \\
\tilde{\psi}_{n+1} &= M_k \tilde{\psi}_{n+1} + A_k \tilde{\psi}_n + B_k \tilde{\psi}_{n+2}.
\end{aligned}
\tag{26}
$$

Again, solving for the central site yields

$$
\tilde{\psi}_n = M_{k+1} \tilde{\psi}_n + A_{k+1} \tilde{\psi}_{n-2} + B_{k+1} \tilde{\psi}_{n+2}
\tag{27}
$$

with the RG-flow

$$
\begin{aligned}
A_{k+1} &= A_k \left(I - M_k\right)^{-1} A_k, \\
B_{k+1} &= B_k \left(I - M_k\right)^{-1} B_k, \\
M_{k+1} &= M_k + A_k \left(I - M_k\right)^{-1} B_k + B_k \left(I - M_k\right)^{-1} A_k,
\end{aligned}
\tag{28}
$$

applying $M_0 = 0$ and (23). The key realization here is not to explore the FP of these matrix recursions directly. Instead, trial-and-error suggests that the RG-flow can be parameterized as

$$
P_k = \begin{pmatrix} a_k & 0 \\ 0 & 0 \end{pmatrix}, \qquad Q_k = \begin{pmatrix} 0 & 0 \\ 0 & a_k \end{pmatrix}, \qquad R_k = \begin{pmatrix} 0 & b_k \\ b_k & 0 \end{pmatrix}.
\tag{29}
$$

Then, one iteration of the RG-flow in (28) provides a closed set of exact recursions:

$$
\begin{aligned}
a_{k+1} &= \frac{\rho a_k^2}{(1 - b_k)\left[1 - (1 - 2\rho)b_k\right]}, \\
b_{k+1} &= b_k + \frac{a_k^2 \left[1 - \rho - (1 - 2\rho)\, b_k\right]}{(1 - b_k)\left[1 - (1 - 2\rho)b_k\right]}.
\end{aligned}
\tag{30}
$$

These recursions have only two FP, $(a^*, b^*) \in \left\{ (0, b^*), \left(\frac{\rho}{2\rho - 1}, \frac{1 - \rho}{1 - 2\rho} \right) \right\}$. The 2nd FP is physical only for $\rho = 0$ or $\rho = 1$ but not for any other value of ρ in between. It is important that we set $\rho = 0$ or $\rho = 1$ both, in the FP and the Jacobian, before we evaluate the Jacobian at the respective FP, since this FP is not physical on any open set of its analytic continuation. For $\rho = 1$ the Jacobian at this FP provides a degenerate eigenvalue, $\lambda = 2$, because the PRW starts with a given direction from the IC and can never reverse direction, i.e., it moves in a trivial, lock-step manner ballistically with $d_w = 1$. On the other hand, for $\rho = 0$, the totally anti-persistent walk should remain forever vacillating in a confined domain around its IC. Here, it is essential to evaluate the Jacobian for $\rho = 0$ first before we apply the FP; only then do we get the eigenvalues $\lambda = 1, 0$, suggesting that the walk maintains its initial position.

The first FP contains the trivial case $z < 1$ for $b^* = 0$. The general indeterminacy of b^* hints at the possibility of a scaling Ansatz again, as in section 3. With $a_k \sim \alpha^k a_k'$ and $b_k \sim 1 - \alpha^k b_k'$ for $\alpha < 1$ and $k \to \infty$ we get $a_{k+1}' = (a_k')^2 / (2\alpha b_k')$ and $b_{k+1}' = b_k' - a_{k+1}'$, which has an FP at $a'^* = b'^*$ and $\alpha = \frac{1}{2}$, *independent* of ρ, and its Jacobian $J' = \partial \left(a_{k+1}', b_{k+1}' \right) / \partial \left(a_k', b_k' \right)$ has the desired diffusive eigenvalues, $\lambda = 4, 1$. Thus, we arrive at the well-known result that a PRW has the same universality classes as the ordinary RW on a line.

5. Quantum Walks on a Line

Quantum walks (QW) are isomorph to PRW, but fundamentally differ in two respects: (1) Instead of a stochastic matrix, the evolution in a QW is driven by a unitary coin; (2) while asymptotic large-time behaviour of observables in ordinary walks are obtained in the limit of real $z \to 1$ from the generating functions, in QW observables typically result from integrals of the modulo-square of generating functions over the unit circle in the complex-z plane. While the first point provides conceptual challenges – and a much richer phenomenology – for QW, the integral in the second point is the main source of technical problems in the RG-analysis of QW: Instead of one FP, it seems that we have to study a wide-ranging set of asymptotic behaviours for the RG-flow anywhere near that circle. Those asymptotic behaviours may be grouped into a finite set of classes, yet, in some of those the RG-flow does not converge but remains forever oscillatory in a seemingly chaotic fashion. Interestingly, all of these features are already present for QW on a $1d$-line.

In close correspondence to the discussion of the PRW in section 4, we have the master equations for the QW in terms of generating functions

$$\tilde{\psi}_n = M\tilde{\psi}_n + A\tilde{\psi}_{n-1} + B\tilde{\psi}_{n+1} + \delta_{n,0}\psi_{IC} \tag{31}$$

with

$$A_k = P_k\mathcal{C}, \qquad B_k = Q_k\mathcal{C}, \qquad M_k = R_k\mathcal{C}, \tag{32}$$

where initially

$$P_0 = z \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad Q_0 = z \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \qquad R_0 = z \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \tag{33}$$

with the most general quantum "coin" given in (6).

The RG-analysis is formally *identical* to that already discussed in section 4. Starting from the same master equations in (26), we arrive at the same RG-flow equations in (28). Evolving those recursions from the initial conditions numerically for a few iterations (initially, for a conveniently chosen coin, such as Hadamard: $\rho = \frac{1}{2}$, $\phi = \theta = 0$), a recursive pattern emerges that suggest a similar Ansatz to that in (29):

$$P_k = \begin{pmatrix} a_k & 0 \\ 0 & 0 \end{pmatrix}, \qquad Q_k = \begin{pmatrix} 0 & 0 \\ 0 & -a_k \end{pmatrix}, \qquad R_k = \begin{pmatrix} 0 & b_k \\ b_k & 0 \end{pmatrix}, \tag{34}$$

the minute difference being due to having a unitary instead of a stochastic coin. We attain the initial values $a_1 = \sqrt{\rho}z^2, b_1 = \sqrt{1-\rho}z^2$ only after the first RG-step, the general behaviour ensuing thereafter. The resulting recursions are quite similar in appearance to the corresponding (30) for PRW and read

$$
\begin{aligned}
a_{k+1} &= \frac{\sqrt{\rho}a_k^2}{1 - 2\sqrt{1-\rho}b_k + b_k^2}, \\
b_{k+1} &= b_k + \frac{(b_k - \sqrt{1-\rho})a_k^2}{1 - 2\sqrt{1-\rho}b_k + b_k^2}
\end{aligned}
\tag{35}
$$

for general ρ and $\phi = \theta = 0$. In any case, however, a_k and b_k in general are complex variables that need to satisfy additional unitarity constraint which derive from (5).

In figure 2 we plot the modulo-square of the (1,1)-element of A_k (essentially, $|a_k|^2$), evolving the RG-flow recursions in (35) up to $k = 4$, starting with $z = e^{i\alpha}$. It demonstrates the complicated, oscillatory dependence on z of the renormalized quantities. In figure 3 we show the corresponding poles of that expression in the complex-z plane. A similar plot for PRW has
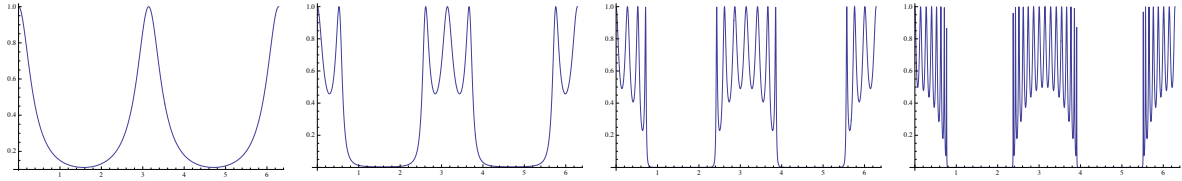
Figure 2. Plot of $\left| A_{1,1}^{(k)} \left(e^{i\alpha} \right) \right|^2$ as a function of $\alpha = \arg(z)$, $0 \leq \alpha \leq 2\pi$, for $k = 1, 2, 3$, and 4. Note that the function is periodic with period π and has significant support only between $-\frac{\pi}{4} < \alpha < \frac{\pi}{4}$ and its periodic iteration.
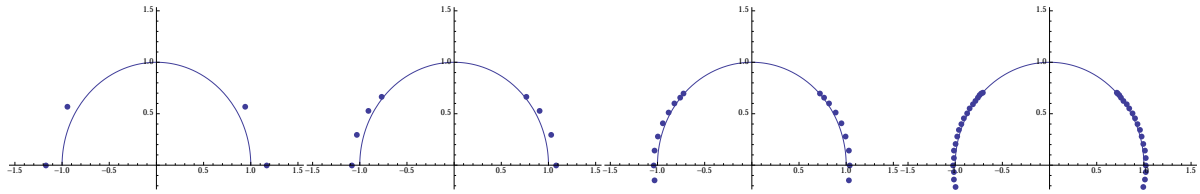

Figure 3. Plot of the poles of $A_{1,1}^{(k)}(z)$ in the complex-z plane for $k = 1, 2, 3$, and 4. Note that the poles cluster between $-\frac{\pi}{4} < \arg z < \frac{\pi}{4}$ and its periodic iteration, just outside and ever closer to the unit circle. Since $A^*(z) = A\left(\frac{1}{z}\right)$, the modulus has also a corresponding set of mirror-poles on the inside of the circle.

a set of entirely *real* poles that accumulate on the positive real axis just above, and ever closer to, $z = 1$ [10]. Thus, while in the classical case a fixed-point analysis near $z \to 1$ suffices, in QW we have to study the RG-flow on extensive parts of the unit circle in the complex-z plane.

We now focus on the analysis of the recursions in (35) for unimodular z and asymptotically for large k. As figures 2 and 3 reveal, a FP-analysis for some fixed limiting value, such as $z \to 1$ for the ordinary RW, appears to be insufficient. Rather, we have to confront the behaviour of the RG-flow over the entire circle of $\alpha = \arg(z)$. Since (35) do not explicitly contain z, it is straightforward to apply a conventional FP analysis, which result in the FP $(a^*, b^*) = \{(0, b^*), (\sqrt{\rho}, \sqrt{1-\rho})\}$. The 2$^\text{nd}$ FP can be reached only for one single value of z, namely that at $\alpha = 0$. As for ordinary RW above, the first FP requires a more extensive scaling Ansatz.

For $\alpha = 0$ (or $z \to 1$) we are apparently conducting the analysis typical for an ordinary RW. Yet, this turns out to be a "sweet-spot" at the center of the quantum domain, where the phases of the renormalized quantities evolves the least with the RG index k; in fact, the phase remains zero. Although the FP itself possesses an interesting explicit ρ-dependence, the Jacobian matrix and its eigenvalues at this FP are independent of ρ. The eigenvalue $\lambda = 2$ is degenerate but otherwise resembles the known result for QW on a 1d-line, $d_w = 1$. However, for $\alpha \neq 0$ such a traditional fixed-point analysis is useless, and we have developed a more sophisticated approach that confirms that $d_w = 1$ over the *entire* oscillatory regime.

Exactly at the edges of the oscillatory regime, i.e., at $\alpha = (2j + 1)\frac{\pi}{4}$ for $\rho = \frac{1}{2}$, we touch on the first FP in the same singular manner we also encountered for RW: The scaling Ansatz $a_k \sim i\epsilon^k a'_k$ and $b_k \sim \frac{1+i}{\sqrt{2}} - \epsilon^k b'_k$, near the pole of the denominator, leads to the new system, $\epsilon a'_{k+1} = (a'_k)^2 / (2b'_k)$ and $\epsilon b'_{k+1} = b'_k - a'_{k+1}$. Its FP provides $\epsilon = \frac{1}{2}$ and $a'^* = b'^*$, and a Jacobian with eigenvalues of $\lambda = 4, 1$. This isolated incidence of diffusive behaviour does not seem to affect the known true scaling. The entire discussion for this special case looks very much similar to the typical analysis for ordinary RW but here represents only a vanishingly small set of all

options for the QW.

For $\frac{\pi}{4} < \alpha < \frac{3\pi}{4}$ at $\rho = \frac{1}{2}$, and the other rapidly decaying regimes, no such scaling Ansatz will balance the vanishing numerator with a cancellation from the denominator of the a_k-recursion. Instead, the denominator simply approaches a complex constant and both a_k and b_k phase-lock with a phase difference of α, apparently. The Ansatz $a_k \sim \delta_k e^{i\mu}$ with vanishing δ_k for large k leads to $\delta_{k+1} \sim \delta_k^2$ while $b_k \sim -e^{i\mu}$ remains a constant (with μ undetermined, but see below). The exponentially decaying solution, $\delta_k \sim e^{-c2^k} / \left(2 + 2\sqrt{2}\cos\mu \right)$, provides no eigenvalue (or $\lambda = 0$) while the constant b_k-solution has $\lambda = 1$, i.e., there is no large-scale quantum transport in this entire sector of α. Any other domain for α simply reiterates these results due to the periodicity in $\alpha \to \alpha + \pi$.

Finally, it should be pointed out that the generic picture for the $1d$-behaviour remains unaltered for different choices of the quantum coin in (6). While figure 2 was generated with the Hadamard coin, $\rho = \frac{1}{2}$, $\phi = \theta = 0$, it is easy to show that any variation in either phase, ϕ or θ, merely rotates the unit circle (i.e., offsets α), whereas a variation in ρ alters the sizes of either type of domain without changing the behaviours within those. In all cases, our analysis reproduces the well-know fact [2] that $d_w = 1$ for the MSD in the QW is universal, unaffected by any variation in the coin parameters. Notice, that for either $\rho \to 0$ or $\rho \to 1$, the domains degenerate and either cover the entire circle or squeeze into a point, respectively, and special behaviour may arise.

Our presentation here has been largely focused on method and no new results have been produced, as we have only demonstrated our method to the well-known cases of RW, PRW, and QW on the $1d$-line. However, we are currently extending our analysis to more complicated cases, such a Sierpinski gaskets, where the exact RG-flow can be found and interesting new phenomena may arise due to the lack of translational invariance, which is bound to affect the universality class.

Acknowledgments

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