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Detection methods to rule out completely co-positive and bi-entangling operations

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Abstract. In this work we extend the quantum channel detection method developed in Refs. [1, 2] in order to detect other interesting convex sets of quantum channels. First we work out a procedure to detect non completely co-positive maps. Then we focus on the set of so-called bi-entangling operations and show how a map outside this set can be revealed. In both cases we provide explicit examples showing the theoretical technique and the corresponding experimental procedure.

1. Introduction

In quantum information it is of great importance to characterise quantum communication channels or quantum devices without necessarily performing quantum process tomography. Actually, quantum process tomography requires a large number of experimental resources, while one is usually interested in few properties of the quantum channel under consideration, as e.g. whether the channel has some entangling power. In many realistic implementations some a priori information on the form of the channel is available, hence, the quantum channel detection (QCD) method developed in [1, 2] can be applied. Besides being less informative than the full quantum process tomography, the QCD method allows us to test the property of interest with a much smaller experimental effort.

In this work we will discuss in detail how to detect two sets of quantum channels, namely quantum channels that are not completely co-positive (CCOP) and the set of operations that are not bi-entangling (BE). Both sets are of great interest as they are connected via the Choi-Jamiolkowski isomorphism to PPT states[1] [3] and to the problem of classical simulatability of quantum computation [4], respectively.

This work is organized as follows. In section 2 we will review the main idea of QCD following Refs. [1, 2]. In section 3 we will discuss a method to detect non CCOP maps. In section 4 we will study how to reveal quantum channels that are not BE operations, and we finally summarize the main results in section 5.

2. The general QCD method

The QCD method proposed in Refs. [1, 2] relies on the concept of witness operators [5] and the Choi-Jamiolkowski isomorphism [6]. We briefly remind both of them in the following.

[1] A state of a bipartite system is PPT if the partial transpose of its density matrix is positive semi-definite, otherwise it is NPT.

A state ρ is entangled if and only if there exists a hermitian operator W such that $\mathrm{Tr}[W\rho] < 0$ and $\mathrm{Tr}[W\rho_{sep}] \geq 0$ for all separable states; such an operator is called an entanglement witness. The Choi-Jamiolkowski isomorphism provides a one-to-one correspondence between completely positive (CP) maps $\mathscr{M}$ acting on $\mathcal{D}(\mathcal{H})$ (the set of density operators on $\mathcal{H}$, with finite dimension d) and positive operators $C_{\mathscr{M}}$ on $\mathcal{H}_{\mathcal{A}} \otimes \mathcal{H}_{\mathcal{B}}$ (named Choi states), where A and B here denote the two subsystems on which the Choi state is defined. The isomorphism can be stated as

$$\mathscr{M} \Longleftrightarrow C_{\mathscr{M}} = (\mathscr{M} \otimes \mathscr{I})[|\alpha\rangle\langle\alpha|], \tag{1}$$

where $\mathscr{I}$ is the identity map, and $|\alpha\rangle$ is the maximally entangled state with respect to the bipartite space $\mathcal{H} \otimes \mathcal{H}$, i.e. $|\alpha\rangle = \frac{1}{\sqrt{d}} \sum_{k=1}^{d} |k\rangle\,|k\rangle$. The above isomorphism can be exploited to link convex sets of quantum channels to particular convex sets of quantum states.

As a simple example consider the convex set of entanglement breaking (EB) channels. A channel $\mathscr{E}$ is EB if and only if its Choi state $C_{\mathscr{E}}$ is separable [7]. Therefore, the detection of entanglement of $C_{\mathscr{E}}$ by means of a suitable witness operator W_{EB} implies that the implemented quantum channel $\mathscr{E}$ is not EB [1].

Although the general QCD method applies to several convex sets of quantum channels, as e.g. EB and separable maps, in the following it will be explicitly studied for the convex sets of CCOP channels and BE operations.

3. Completely co-positive channels

In this section we will consider the set of CCOP channels. A CP map $\mathscr{C}$ acting on a qudit (d-dimensional system) is CCOP if and only if the composite map $\mathscr{C}_{\mathscr{T}} = \mathscr{T} \circ \mathscr{C}$, where $\mathscr{T}$ is the transposition map, is CP. Since a quantum map is CP if and only if the corresponding Choi operator is positive, we can restate the above definition as follows: a CP map $\mathscr{C}$ is CCOP if and only if the Choi operator $C_{\mathscr{C}_{\mathscr{T}}}$ related to the composite map $\mathscr{C}_{\mathscr{T}}$ is positive.

By the above correspondence we will develop a method to detect whether a map is non CCOP by adapting techniques developed for the detection of non positive partial transposed (NPT) entangled states [3]. Consider then a map $\mathscr{M}$ that does not belong to the set of CCOP channels. From the above definition it follows that the bipartite Choi state $C_{\mathscr{M}_{\mathscr{T}}} = (\mathscr{T}_A \otimes \mathscr{I})[C_{\mathscr{M}}]$ has at least one negative eigenvalue. Let λ_- be the most negative eigenvalue corresponding to the eigenvector $|\lambda_-\rangle$. The following operator, i.e.

$$W_{\mathrm{CCOP}} = |\lambda_-\rangle\langle\lambda_-|^{\mathscr{T}_A}, \tag{2}$$

is thus suitable to detect the NPT state $C_{\mathscr{M}_{\mathscr{T}}}$ corresponding to the non CCOP map $\mathscr{M}_{\mathscr{T}}$. Notice that the transposition map on the Choi state acts only on the first qudit, i.e. $\mathscr{T}_A$.

As an illustrative example we consider the case of the dephasing noise $\mathscr{D}$ acting on a single qubit, defined by the following trace preserving CP map

$$\mathscr{D}[\rho] = p\rho + (1-p)\sigma_z\rho\sigma_z\ , \tag{3}$$

where σ_z is a Pauli operator[2]. It is easy to see that the Choi state $C_{\mathscr{D}}$ corresponding to $\mathscr{D}$ takes the form

$$C_{\mathscr{D}} = p|\alpha\rangle\langle\alpha| + (1-p)|\phi^-\rangle\langle\phi^-|, \tag{4}$$

[2] Hence, dephasing noise consists in either leaving the input state unchanged (with probability p) or applying a phase flip σ_z (with $1-p$). Generally speaking, it represents a loss of quantum coherence in the off-diagonal terms of the regarded system.

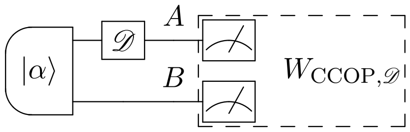

Figure 1. Experimental scheme showing the detection of the dephasing channel $\mathscr{D}$ as a non CCOP channel. Notice that the expectation value of the witness $W_{\text{CCOP},\mathscr{D}}$, namely $\text{Tr}[W_{\text{CCOP},\mathscr{D}} C_{\mathscr{D}}]$, can be measured locally.

with $|\phi^-\rangle = \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle)$. The above state can be shown to be NPT whenever $p \neq 1/2$. It is then possible to derive the following detection operator [8, 9] from Eq. 2:

$$W_{\text{CCOP},\mathscr{D}} = \begin{cases} \frac{1}{4}(\mathbb{1} \otimes \mathbb{1} + \sigma_x \otimes \sigma_x + \sigma_y \otimes \sigma_y - \sigma_z \otimes \sigma_z) & \text{for } p < \frac{1}{2}, \\ \frac{1}{4}(\mathbb{1} \otimes \mathbb{1} - \sigma_x \otimes \sigma_x - \sigma_y \otimes \sigma_y - \sigma_z \otimes \sigma_z) & \text{for } p > \frac{1}{2}. \end{cases} \tag{5}$$

This method can be experimentally implemented by preparing a two-qubit state in the maximally entangled state $|\alpha\rangle$, then operating with the quantum channel $\mathscr{D}$ to be detected on one of the two qubits and measuring the operator $W_{\text{CCOP},\mathscr{D}}$ acting on both qubits at the end (see Fig. 1). If the resulting average value $\text{Tr}[W_{\text{CCOP},\mathscr{D}} C_{\mathscr{D}}]$ is negative, we can then conclude that the Choi state $C_{\mathscr{D}_{\mathscr{T}}} = \mathscr{T} \circ \mathscr{D}$ is NPT and that the channel under consideration is not CCOP.

Finally, we would like to stress that, since every NPT state is entangled, the detection of a non CCOP channel $\mathscr{M}$ implies that $\mathscr{M}$ is not EB as well, but the opposite does not hold in general. Actually the set of EB channels is in general a subset of the CCOP channels. The two sets coincide only when the channels act on two-dimensional systems. From the perspective of QCD, this implies that for higher dimensional systems a quantum channel which is detected as non EB may nevertheless belong to the set of CCOP maps.

4. Bi-entangling operations

In this section we will focus on BE operations, a class of quantum channels that can generate at most bipartite entanglement. They were introduced in Ref. [4] in the context of quantum computation and were shown to be efficiently simulatable classically. BE operations are quantum channels acting on bipartite systems AB (of finite dimension d) in such a way that they can be expressed as convex combinations of (a) separable operations, (b) operations that swap the two qudits and then act as a separable operation, and (c) EB channels, that break any entanglement between the two qudits on which the channel acts and extra ancillae [4]. Via the Choi-Jamiolkowski isomorphism we can then characterize the set of BE operations in terms of the corresponding Choi states.

Consider a BE operation $\mathscr{M}_{BE}$ acting on the bipartite system AB. The Choi state $C_{\mathscr{M}_{BE}}$ associated to $\mathscr{M}_{BE}$ is then a four-partite state (composed of subsystems A, B, C and D). Separable channels have separable Choi states with respect to the bipartition $AC|BD$ [10]. As a consequence, channels of type (b), with a swap gate followed by separable channels, have separable Choi states in $AD|BC$. EB channels correspond to separable Choi states in the bipartition $AB|CD$. A general Choi state $C_{\mathscr{M}_{BE}}$ for a BE channel can then be written as a convex combination of four-partite states biseparable with respect to bipartitions $AC|BD$, $AD|BC$ and $AB|CD$, namely

$$C_{\mathscr{M}_{BE}} = p \sum_i p_i C_i^{(AC|BD)} + q \sum_j q_j C_j^{(AD|BC)} + r \sum_k r_k C_k^{(AB|CD)}, \tag{6}$$

where (p, q, r), $\{p_i\}$, $\{q_j\}$ and $\{r_k\}$ are probability distributions. Notice that the first term corresponds to the set (a), the second to (b) and the third to (c). In other words, the Choi states $C_{\mathscr{M}_{BE}}$ corresponding to BE operations lie in the convex hull of states biseparable with respect to the bipartitions $AC|BD$, $AD|BC$ and $AB|CD$ for the four-partite system $ABCD$. We name this convex set of four-partite Choi states corresponding to BE operations as S_{BE}. It is now possible to develop detection procedures for BE operations by employing suitable witness operators that detect the corresponding Choi state with respect to the biseparable states belonging to S_{BE}.

We will now focus on the case of a unitary transformation U acting on two d-dimensional systems. The corresponding Choi state is pure and given by $|U\rangle = (U \otimes \mathbb{1})\,|\alpha\rangle$. Therefore a suitable detection operator for U as a non BE operation can be constructed as [1, 2]

$$W_{BE,U} = \alpha_{BE}^2\, \mathbb{1} - C_U \ , \tag{7}$$

where $C_U = |U\rangle\langle U|$, and the coefficient α_{BE} is the overlap between the closest biseparable state in the set S_{BE} and the entangled state $|U\rangle$, namely

$$\alpha_{BE}^2 = \max_{\mathscr{M}_{BE}} \langle U|\, C_{\mathscr{M}_{BE}}\, |U\rangle \,. \tag{8}$$

Since the maximum of a linear function over a convex set is always achieved on the extremal points, the maximum involved in α_{BE} can be always calculated by maximising over the pure biseparable states in S_{BE}, i.e.

$$\alpha_{BE} = \max_{|\Xi\rangle \in S_{BE}} |\langle \Xi|\, U\rangle|. \tag{9}$$

By exploiting the Schmidt decomposition [11] of the state $|U\rangle$, the maximization above can be expressed analytically as

$$\alpha_{BE} = \max_i \max_\lambda \lambda_i(U), \tag{10}$$

where the index i labels the bipartitions $AC|BD$, $AD|BC$ and $AB|CD$, and $\lambda_i(U)$ are the Schmidt coefficients of $|U\rangle$ in the bipartition i. Therefore, in order to find the coefficient α_{BE} one has to find the maximal Schmidt coefficient of $|U\rangle$ for a fixed bipartite splitting and then maximize it among all the bipartitions involving only two versus two subsystems.

As an example of the above procedure consider the following unitary operation V acting on a two-qubit system

$$V = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{11}$$

The gate V is a modified swap gate such that it is no longer a BE operation. The coefficient α_{BE} for V can be computed following the steps outlined above. The Choi state $|V\rangle$ associated to the gate V is given by

$$|V\rangle = \frac{1}{\sqrt{2}}(|\alpha\rangle_{AD}\,|00\rangle_{BC} + |\phi^-\rangle_{AD}\,|11\rangle_{BC}), \tag{12}$$

and the Schmidt coefficients of V with respect to the bipartitions $AC|BD$, $AD|BC$ and $AB|CD$ can be easily computed as $\lambda_{AC|BD}(V) = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$, $\lambda_{AD|BC}(V) = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0, 0)$ and $\lambda_{AB|CD}(V) = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$. Therefore, the coefficient α_{BE} equals $1/\sqrt{2}$ and a suitable detection operator in order to detect V as a non BE operation takes the form

$$W_{BE,V} = \frac{1}{2}\,\mathbb{1} - C_V. \tag{13}$$

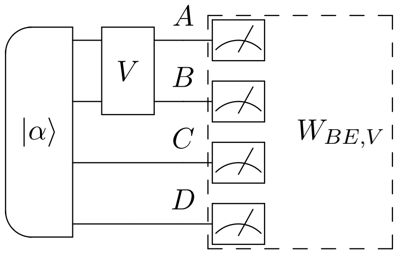

Figure 2. Experimental scheme implementing the detection of the gate V defined in Eq. 11 as a non BE operation.

From an experimental point of view, the detection procedure can be implemented as follows: prepare a four-partite qubit system in the state $|\alpha\rangle = |\alpha\rangle_{AC} |\alpha\rangle_{BD}$, apply the quantum gate V to qubits A and B, and finally perform a suitable set of local measurements in order to measure the operator (13). If the resulting average value $\mathrm{Tr}[W_{BE,V} C_V]$ is negative then the quantum channel is detected as a non BE operation. The experimental scheme is shown in Fig. 2.

We conclude this section by noticing that the method described above leads to different detection operators with respect to the detection method for non separable maps [1]. Indeed, already in the case of two qubits, the optimal witness operator that detects the gate V as a non separable channel is [1]

$$W_{Sep,V} = \frac{1}{4}\,\mathbb{1} - C_V. \tag{14}$$

As expected, the detection operator above is weaker than $W_{BE,V}$ in Eq. 13 in the sense that it leads to a negative expectation value for a smaller set of CP maps. This is due to the fact that BE maps are a strict subset of separable maps, and actually the set of separable Choi states in the bipartition $AC|BD$, corresponding to separable maps, is a strict subset of S_{BE}.

5. Conclusions

In summary, after a brief review of the general quantum channel detection method proposed in [1], based on the Choi-Jamiolkowski isomorphism and witness operators, we have developed a method to detect maps that do not belong to specific convex sets, i.e. the completely co-positive maps and the bi-entangling operations. Significant examples of non co-positive and non bi-entangling operations have been considered in detail, showing both the underlying theoretical techniques and the corresponding experimental schemes. We stress that the method works when some a priori knowledge on the quantum channel is available, it requires fewer measurements than quantum process tomography and it is achievable experimentally with present day technology [12]. In particular, the detection method for non entanglement breaking channels and for non separable maps has already been demonstrated in a quantum optical experiment [13].

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