

# QED effective action for an $O(2) \times O(3)$ symmetric field in the full mass range

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**Abstract.** An interesting class of background field configurations in QED are the  $O(2) \times O(3)$  symmetric fields. Those backgrounds have some instanton-like properties and yield a one-loop effective action that is highly nontrivial but amenable to numerical calculation, for both scalar and spinor QED. Here we use the recently developed “partial-wave-cutoff method” for a numerical analysis of both effective actions in the full mass range. In particular, at large mass we are able to match the asymptotic behavior of the physically renormalized effective action against the leading two mass levels of the inverse mass (or heat kernel) expansion. At small mass we obtain good numerical results even in the massless case for the appropriately (unphysically) renormalized effective action after the removal of the chiral anomaly term through a small radial cutoff factor. In particular, we show that the effective action after this removal remains finite in the massless limit, which also provides indirect support for M. Fry’s hypothesis that the QED effective action in this limit is dominated by the chiral anomaly term.

## 1. The $O(2) \times O(3)$ symmetric background

G. Dunne *et al* [1, 2, 3, 4] initiated the application of the “partial wave-cutoff method”, to be explained below, to the important class of  $O(2) \times O(3)$  symmetric fields first introduced by S. L. Adler [5, 6]. These backgrounds can be defined, in Euclidean metric, as

$$A_\mu(x) = \eta_{\mu\nu}^3 x_\nu g(r), \quad g(r) \equiv \nu \frac{e^{-\alpha r^2}}{\rho^2 + r^2}, \quad (1)$$

where  $\eta_{\mu\nu}^3$  is a ’t Hooft symbol,  $r^2 = x_\mu x^\mu$  and  $\alpha \geq 0$ . Note that  $g(r)$  is spherically symmetric, and has the further properties

- (i)  $\alpha > 0 \rightarrow \int d^4x F^2$  is finite.
- (ii)  $\alpha = 0 \rightarrow g(r) \propto \frac{1}{r^2}$ , which is what we need to see the chiral anomaly term  $\int d^4x F_{\mu\nu} \tilde{F}_{\mu\nu}$ .

According to M. Fry [7, 8], the following general remarks hold for the spinor QED effective action in the background (1) with  $\alpha = 0$ : let  $\mathcal{R}$  denote the (scheme independent) effective action obtained after subtraction of the two-point contribution. It behaves for small  $m$  as

$$\mathcal{R} \sim \frac{\nu^2}{4} \ln m^2 + \text{less singular in } m^2. \quad (2)$$



The logarithmic term is determined entirely by the chiral anomaly,

$$-\frac{1}{4\pi^2} \int d^4x F_{\mu\nu} \tilde{F}_{\mu\nu} = \frac{\nu^2}{2}. \quad (3)$$

## 2. The partial wave-cutoff method

After decomposing the negative chirality part of the Dirac operator into partial-wave radial operators with quantum numbers  $l$  and  $l_3$ , the corresponding effective action is :

$$\Gamma_L^{(-)} = - \sum_{s=\pm\frac{1}{2}} \sum_{l=0,\frac{1}{2},1,\dots}^L \Omega(l) \sum_{l_3=-l}^l \ln \left( \frac{\det(m^2 + \mathcal{H}_{(l,l_3,s)})}{\det(m^2 + \mathcal{H}_{(l,l_3,s)}^{\text{free}})} \right). \quad (4)$$

We concentrate on the negative chirality sector of the spinor effective action where  $\Omega(l) = (2l+1)$  is the degeneracy factor, and the  $s$  sum comes from adding the contributions of each spinor component. The partial-wave cutoff method separates the sum over the quantum number  $l$  into a low partial-wave contribution, each term of which is computed using the (numerical) Gel'fand-Yaglom method, and a high partial-wave contribution, whose sum is computed analytically using WKB. Then we apply a regularization and renormalization procedure and combine these two contributions to yield the finite and renormalized effective action. The Gel'fand-Yaglom method [1, 2, 3], can be summarized as follows: Let  $\mathcal{M}_1$  and  $\mathcal{M}_2$  denote two second-order radial differential operators on the interval  $r \in [0, \infty)$  and let  $\Phi_1(r)$  and  $\Phi_2(r)$  be solutions to the initial value problem

$$\mathcal{M}_i \Phi_i(r) = 0; \quad \Phi_i(r) \sim r^{2l} \quad \text{as } r \rightarrow 0. \quad (5)$$

Then the ratio of the determinants is given by

$$\frac{\det \mathcal{M}_1}{\det \mathcal{M}_2} = \lim_{R \rightarrow \infty} \left( \frac{\Phi_1(R)}{\Phi_2(R)} \right).$$

In our case

$$\Phi_-''(r) + \frac{4l+3}{r} \Phi_-' - \left( m^2 + 4l_3 g(r) + r^2 g(r)^2 + [4g(r) + r g'(r)] \right) \Phi_-(r) = 0.$$

The high-mode contribution, which remains to be calculated using WKB, is

$$\Gamma_H^{(-)} = - \sum_{s=\pm\frac{1}{2}} \sum_{l=L+\frac{1}{2}}^{\infty} \Omega(l) \sum_{l_3=-l}^l \ln \left( \frac{\det(m^2 + \mathcal{H}_{(l,l_3,s)})}{\det(m^2 + \mathcal{H}_{(l,l_3,s)}^{\text{free}})} \right). \quad (6)$$

## 3. Two versions of the effective action

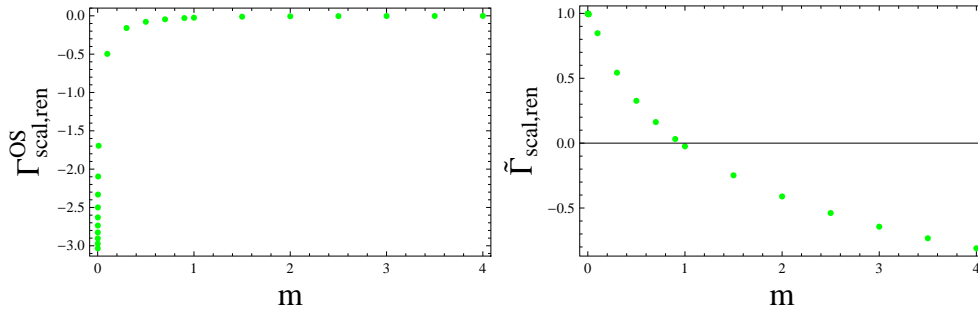
For the class of backgrounds considered here, the partial-wave-cutoff method works well for any value of the mass up to numerical accuracy. The effective action calculated as above is finite for any non-zero value of the mass. When we use on-shell ('OS') renormalization ( $\mu = m$ ), its leading small-mass behavior contains the logarithmically divergent term [4]

$$\Gamma_{\text{ren}}^{\text{OS}}(m) \sim \left( - \int_0^\infty dr Q_{\log}(r) \right) \ln m, \quad m \rightarrow 0. \quad (7)$$

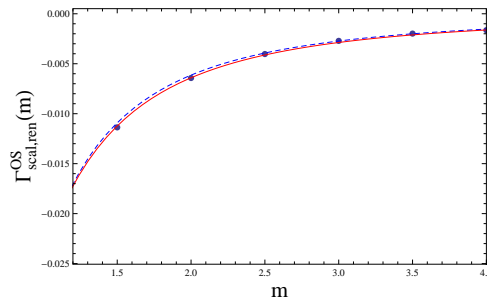
Thus for the study of this small  $m$  regime we introduce a modified effective action,

$$\tilde{\Gamma}_{\text{ren}}(m) \equiv \Gamma_{\text{ren}}(m, \mu) + \left( \int_0^\infty dr Q_{\log}(r) \right) \ln \mu \quad (= \Gamma_{\text{ren}}(m, \mu = 1)). \quad (8)$$

It turns out that  $\tilde{\Gamma}$  is finite for  $m = 0$ , which supports Fry's conjecture, mentioned above, for the case of the backgrounds with  $\alpha > 0$  (where the chiral anomaly term is absent). Both variants of the effective action are contrasted in Fig. 1.



**Figure 1.** Effective action. *Left panel*, On shell, eq. (7). *Right panel*, modified, eq. (8).



**Figure 2.** Large mass limit of the scalar effective action for  $\alpha = 1/120$ . Dots represent the exact effective action, the dashed curve considers only the leading term  $c_2/m^2$ , and the solid curve, both the leading and subleading terms  $c_2/m^2 + c_4/m^4$ .

#### 4. Large mass asymptotic behavior

In this section we exhibit the leading and subleading terms in the inverse mass (= heat kernel) expansion of the one-loop scalar QED effective action. The first two terms are (we calculated them using the worldline formalism along the lines of [9, 10])

$$\Gamma_{\text{scal}}^{\text{OS}}(m) = \frac{c_{\text{scal},2}(\alpha)}{m^2} + \frac{c_{\text{scal},4}(\alpha)}{m^4} + O\left(\frac{1}{m^6}\right), \quad (9)$$

where the coefficients in the limit  $\alpha \rightarrow 0$  are, up to cubic order in  $\alpha$ ,

$$\begin{aligned} c_{\text{scal},2}(\alpha) &= -\frac{1}{460}\alpha^3 \left( 256 \log(2\alpha) + 256\gamma_E + \frac{304}{5} \right) - \frac{31\alpha^2}{300} + \frac{23\alpha}{600} - \frac{2}{75}, \\ c_{\text{scal},4}(\alpha) &= -\frac{37\gamma_E\alpha^3}{135} + \frac{187349\alpha^3}{396900} - \frac{2\gamma_E\alpha^2}{15} - \frac{36853\alpha^2}{529200} - \frac{1}{135}(37\alpha + 18)\alpha^2 \log(2\alpha) \\ &\quad - \frac{571\alpha}{22050} + \frac{107}{52920}. \end{aligned} \quad (10)$$

The large-mass behavior of the effective action is shown in Fig. 2 for the scalar QED case.

#### 5. Finiteness of the massless four-point contribution

In this section we show that the four-point contribution to the effective action in the “standard”  $O(2) \times O(3)$  symmetric background, (1) with  $\alpha = 0$  and  $\nu = \rho = 1$ , is finite in the massless limit. This is a detail of some importance for Fry’s investigation that had been missing in the

analysis of [7], although it has been anticipated in [8]. In the worldline formalism, we can write this quartic contribution to the effective action as (in either scalar or spinor QED)

$$\Gamma^{(4)}[A] = - \prod_{i=1}^4 \int \frac{d^4 k_i}{(2\pi)^4} \bar{a}(k_i^2) (2\pi)^4 \delta^4(\sum k_i) \Gamma[k_1, \varepsilon_1; \dots; k_4, \varepsilon_4], \quad (11)$$

where  $\Gamma$  is the worldline path integral representation of the off-shell Euclidean four-photon amplitude and  $\bar{a}(k^2) = 4c\pi^2 K_2(\rho\sqrt{k^2})/k^2$ , where  $K_2(x)$  is the modified Bessel function of the second kind. After performing the path integral, suitable integrations by parts, a rescaling  $\tau_i = Tu_i, i = 1, \dots, 4$  and the elimination of the global  $T$  integral, we obtain (see [10] for details)

$$\Gamma[k_1, \varepsilon_1; \dots; k_4, \varepsilon_4] = - \frac{e^4}{(4\pi)^2} \int_0^1 du_1 du_2 du_3 du_4 \frac{Q_4(\dot{G}_{B12}, \dots, \dot{G}_{B34})}{\left(m^2 - \frac{1}{2} \sum_{i,j=1}^4 G_{Bij} k_i \cdot k_j\right)^2}. \quad (12)$$

Here  $G_{Bij} \equiv G_B(u_i, u_j) = |u_i - u_j| - (u_i - u_j)^2$  is the worldline Green's function and  $\dot{G}_{Bij}$  its derivative.  $Q_4$  is a polynomial in the various  $\dot{G}_{Bij}$ 's, as well as in the momenta and polarizations. Now, the QED Ward identity implies that the rhs of (12) is  $O(k_i)$  in each of the four momenta, which can also be easily verified using properties of the numerator polynomial  $Q_4$ . Using this fact and (11) we see that there is no singularity at  $k_i = 0$ , and convergence at large  $k_i$ .

## 6. Conclusions

We have continued and extended here the full mass range analysis of the scalar and spinor QED effective actions for the  $O(2) \times O(3)$  symmetric backgrounds, started in [4], by a more detailed numerical study of both the small and large mass behaviors. In [4] only the unphysically renormalized versions  $\tilde{\Gamma}_{\text{ren}}(m)$  of these effective actions were considered (corresponding to  $\mu = 1$ ), which are appropriate for the small mass limit, but have a logarithmic divergence in  $m$  in the large  $m$  limit. Here we have instead used the physically renormalized effective actions  $\Gamma_{\text{ren}}^{\text{OS}}(m)$  for the study of the large mass expansions, which made it possible to achieve a numerical matching of both this leading and even the subleading term in the inverse mass expansions of the effective actions. In our study of the small mass limit, we have improved on [4] by obtaining good numerical results for  $\tilde{\Gamma}_{\text{ren}}(m)$  even at  $m = 0$ , and showing continuity for  $m \rightarrow 0$  for various values of  $\alpha$ . Moreover, we have presented numerical evidence that  $\tilde{\Gamma}_{\text{ren}}(m = 0)$  stays finite even in the limit  $\alpha \rightarrow 0$ . This fact is important in the spinor case, where it supports indirectly Fry's conjecture [7] that, for the case at hand, the only source of a divergence of  $\tilde{\Gamma}_{\text{ren}}(m)$  for  $\alpha = 0$  at  $m \rightarrow 0$  should be the chiral anomaly term. As a side result, we have proved the finiteness of the massless limit four-point contribution to the effective action in scalar and spinor QED for the standard  $O(2) \times O(3)$  symmetric background ( $\alpha = 0, \nu = \rho = 1$ ).

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