

Studying the behaviour of averaged models in the third body perturbation problem

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Abstract. The main goal of the present paper is to perform a study of the behaviour of the single and double averaged methods used to study the third-body perturbation problem in celestial mechanics. Both models are compared with the full elliptic restricted three-body problem. The study shows the differences between the values of the eccentricity and inclination of the satellite predicted by those models, showing the differences at every instant of time, as well as the integral over the time of the differences between the models. This integral is a new way to look at those differences and can give us a more complete view of the quality of the approximations, instead of just instantaneous views.

1. Introduction

There are many researches in the literature studying the third-body perturbation in a satellite in orbit around the Earth. Among the first ones are Kozai [1] that developed long-period and secular terms due to the perturbing potential of the Sun and the Moon. Cook [2] used the Lagrange's planetary equations of motion to study the long-period terms of the Sun and the Moon. Later, Kozai [3] considered the secular terms for asteroids with high inclination and eccentricity, considering the perturbation from Jupiter. Kaula [4] also studied this problem, obtaining new equations to calculate those effects. After that, Giacaglia [5] studied this problem with ecliptic elements for the Moon and equatorial elements for the satellite. Kozai [6] worked again in this problem and obtained the short period terms in closed forms and the secular and long period terms based in numerical integrations. Collins and Cefola [7] applied double averaged techniques to study satellites in high altitudes around the Earth, in particular for a long time span. Some other papers using the double average technique can be seen in Broucke [8], for expansions of second order, and Prado [9], for expansions up to the fourth order. Some other papers studied other planetary systems, like Carvalho ([10], [11]), Lara [12], Paskowitz and Scheers ([13], [14]). Domingos et al. [15] and Xiaodong et al. [16] included an inclination between the perturbing body and the orbital plane of the satellite. The goal of the single average is to eliminate only short periodic terms of the satellite to give an understanding of the long-term stability of the orbits, instead of considering also an average over the motion of the disturbing body, as done by the double-averaged model. The present paper makes a comparison of the three models, assuming that the perturbing body is in an elliptical orbit.



2. Dynamical model

It is assumed the existence of a main body, with mass m_o , that is fixed in the center of the reference system x - y - z . The perturbing body, having mass m' , is in an elliptic orbit with keplerian elements: semi-major axis a' , eccentricity e' , inclination i' , argument of periapsis ω' , longitude of the ascending node Ω' , and mean motion n' . The spacecraft is in an orbit given by the orbital elements: semi-major axis a , eccentricity e , inclination i , argument of periapsis ω , longitude of the ascending node Ω , and mean motion n .

Using an expansion in Legendre's polynomials (considering $r' \gg r$), the second order part of the disturbing potential, considering an average over the eccentric anomaly of the spacecraft, is [15]:

$$\langle R_2 \rangle = \frac{\mu' a'^2 n'^2}{2} \left(\frac{a'}{r'} \right)^3 \left\{ \left(1 + \frac{3}{2} e^2 \right) \left[\frac{3}{2} (\alpha^2 + \beta^2) - 1 \right] + \frac{15}{4} (\alpha^2 - \beta^2) e^2 \right\} \quad (1)$$

$$\text{where } r' = \frac{a'(1-e'^2)}{1+e' \cos f'}, \quad \cos f' = \cos M' + e'(\cos 2M' - 1) + \frac{9}{8} e'^2 (\cos 3M' - \cos M') + \dots$$

$$\text{and } \sin f' = \sin M' + e' \sin 2M' + e'^2 \left(\frac{9}{8} \sin 3M' - \frac{7}{8} \sin M' \right) + \dots$$

For the situation of considering elliptic orbits for the perturbing body, the values of α and β are given by $\alpha = \cos \omega \cos D - \cos i \sin \omega \sin D$, $\beta = -\sin \omega \cos D - \cos i \cos \omega \sin D$ and $D = \Omega - f' - \omega'$ [15]. The mean anomaly of the perturbing body M' is given by $M' = M'_o + n't$ and the Lagrange's planetary equations of motion are [17]:

$$\frac{de}{dt} = K \frac{15}{4} \mu' n'^2 \frac{e \sqrt{1-e^2}}{n} [\sin 2\omega (\cos^2 D - \cos^2 i \sin^2 D) - \cos i \cos 2\omega \sin 2D] \quad (2)$$

$$\begin{aligned} \frac{di}{dt} = K \frac{1}{\sin i [\mu a (1-e^2)]^{\frac{1}{2}}} \frac{3}{4} \mu' n'^2 a^2 \left\{ e^2 \left[-5 \cos 2\omega \sin 2D \cos^2 i + \frac{3}{2} (-\sin 2D + \sin 2D \cos^2 i) + \right. \right. \\ \left. \left. - \frac{5}{2} \sin D \cos 2\omega (1 + \cos^2 i) \right] + (\sin D - \sin 2D \cos^2 i) - 5 \cos 2D \sin 2\omega \cos i \right\} \end{aligned} \quad (3)$$

$$\begin{aligned} \frac{d\Omega}{dt} = K \frac{3}{4} \mu' n'^2 \frac{a^2}{\sin i [\mu a (1-e^2)]^{\frac{1}{2}}} \left\{ \left(1 + \frac{3}{2} e^2 \right) (-\sin^2 D \sin 2i) + [\cos 2i \cos 2\omega \sin^2 D + \right. \\ \left. + \sin i \sin 2D \sin 2\omega] \left(\frac{5}{2} e^2 \right) \right\} \end{aligned} \quad (4)$$

$$\begin{aligned} \frac{d\omega}{dt} = K \frac{3}{4} \mu' n'^2 \left\{ -\frac{a^2}{\sin i [\mu a (1-e^2)]^{\frac{1}{2}}} \left[\left(1 + \frac{3}{2} e^2 \right) (-\sin^2 D \sin 2i) + \right. \right. \\ \left. \left. + \left(\frac{5}{2} e^2 \right) (\cos 2i \sin^2 D \cos 2\omega) + \frac{1}{2} \sin 2i \sin 2D \sin 2\omega \right] + \right. \\ \left. + \frac{\sqrt{1-e^2}}{n} [5(\cos^2 D - \cos^2 i \sin^2 D) \cos 2\omega - 5 \cos i \sin 2D \sin 2\omega + \frac{6}{2} (\cos^2 D + \cos^2 i \sin^2 D)] \right\} \end{aligned} \quad (5)$$

where
$$K = \frac{1 + 3e' \cos f' + 3e'^2 \cos^2 f'}{1 - 6e' + 15e'^2}$$

3. Numerical results

The high-altitude circular orbits, assumed to be an orbit with semi-major axis of 42000 km in the present paper, near the geostationary orbits, are the ones more affected by the third body perturbations, so a detailed study is made regarding the comparisons among the different models. This task is done by choosing a specific value for the initial inclination and then analyzing the evolution of the magnitude of the differences between the values obtained by the two averaged models mentioned here and the full restricted elliptic three-body problem. The instantaneous errors between each of the averaged models and the full model are plotted as a function of the eccentricity and inclination of the orbit of the spacecraft. Then, the integral of the magnitude of these differences over the time is also obtained. This quantity shows better how far the curves are from each other taking into account the whole interval of time and not focusing only in individual points. All the comparisons are made for a short time period of two days (which correspond to ~ 0.46 canonical units), because the idea is to see the difference of the models for shorter times.

Figure 1 shows the results for an initial inclination of 20 degrees, that is a value below the critical angle. The dashed line represents the results obtained by the double average model and the continuous line represents the single averaged model. Eccentricities of the primaries of 0.1 are shown in red, 0.2 in green and 0.3 in blue. The circular case is not shown here because the results are too close among the models and eccentricities higher than 0.3 are also omitted because the approximations are not good. Note the very small errors, in particular for the inclination, for both averaged models.

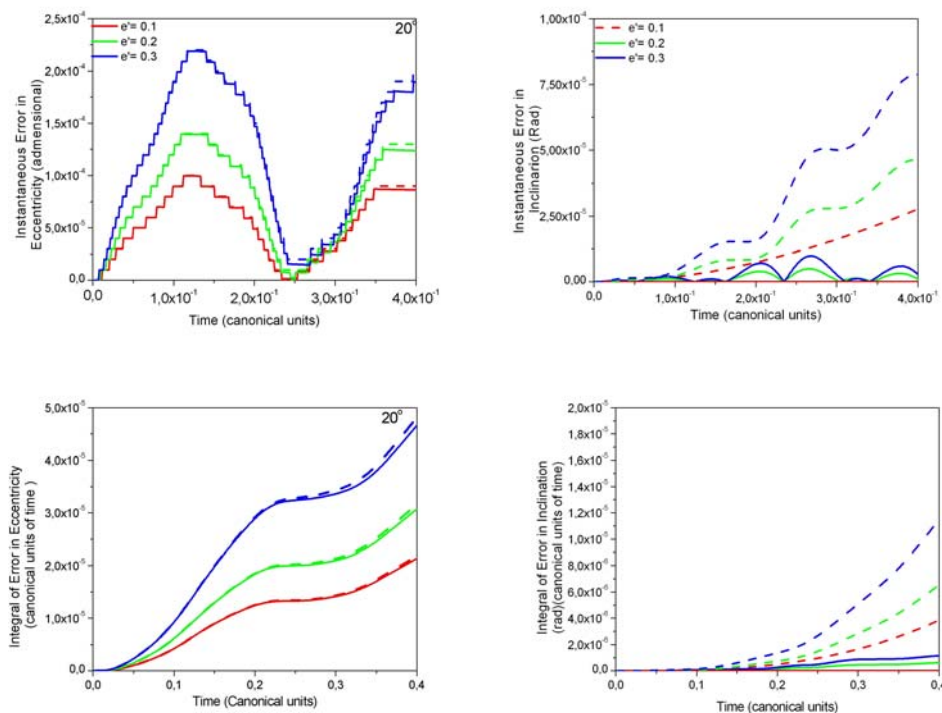


Figure 1. Instantaneous and integral errors in eccentricity and inclination as a function of time for High-Earth orbits with initial inclination of 20°. The eccentricity of the perturbing body (e') is shown in colours codes: 0.1 (red), 0.2 (green) and 0.3 (blue). The lines are: single-averaged model (solid line) and double-averaged model (dashed line).

It is visible that the quality of the results decreases when the orbit of the primaries become more eccentric. The magnitude of the error has shorter period for the inclination, when compared to the eccentricity. It is visible that the single-averaged model is better than the double-averaged model for shorter times, as expected, since it takes into account the short period motion of the spacecraft.

Figure 2 shows the results for an initial inclination of 40 degrees, which is very near the critical angle. This type of analyses confirms the ones made before. Note the decrease of the quality of the second order averaged approximations with the increase of the eccentricity of the primaries. It is also noted and quantified how this quality is reduced with time. Note also the oscillations of the errors, also with shorter period for the error in inclination. It means that the second order averaged approximations loose accuracy, recovered it again and so on. The averaged models are very good when $e' = 0.1$. The single-averaged model is again better than the double-averaged model, for the same reasons explained before.

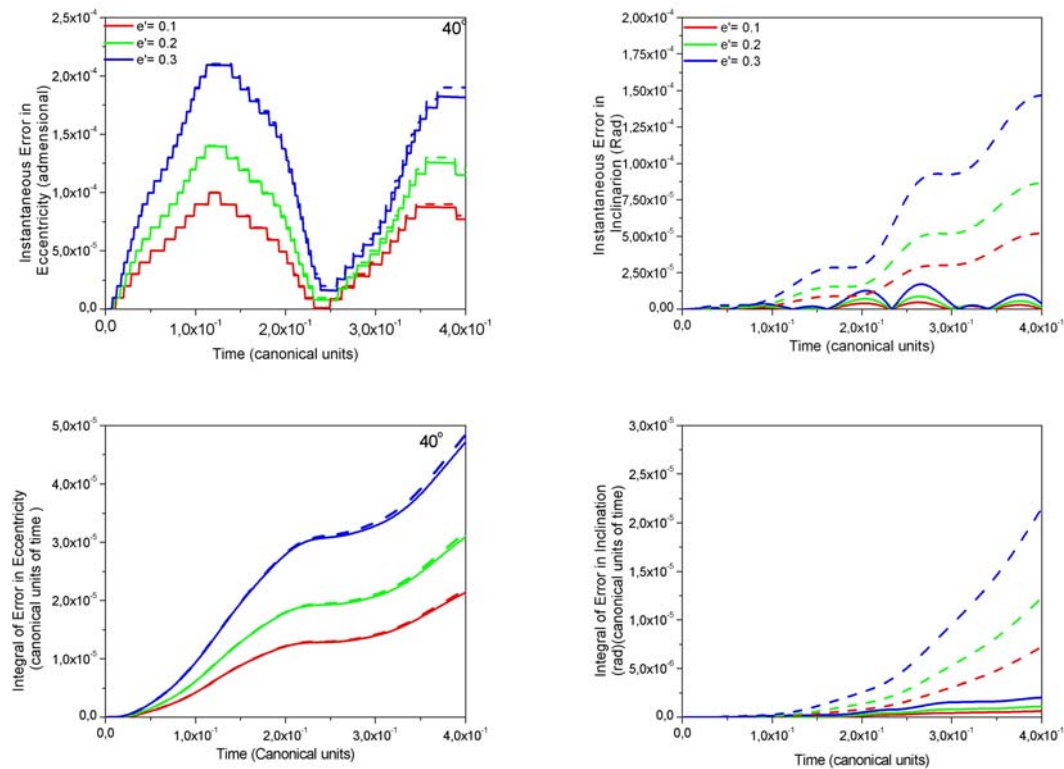


Figure 2. Instantaneous and integral errors in eccentricity and inclination as a function of time for High-Earth orbits and initial inclinations of 40° . The eccentricity of the perturbing body (e') is shown in colour codes: 0.1 (red), 0.2 (green) and 0.3 (blue). The lines are: single averaged model (solid line) and double-averaged model (dashed line).

Figure 3 shows the results for an initial inclination of 60 degrees, which is above the critical angle. Of course the decrease of the quality of the second order averaged approximations with the increase of the eccentricity of the primaries is also evident by this type of analysis. The oscillations of the errors also occur for this inclination. Again, the averaged models are excellent when $e' = 0.1$ and the double averaged model is better than the single-averaged model.

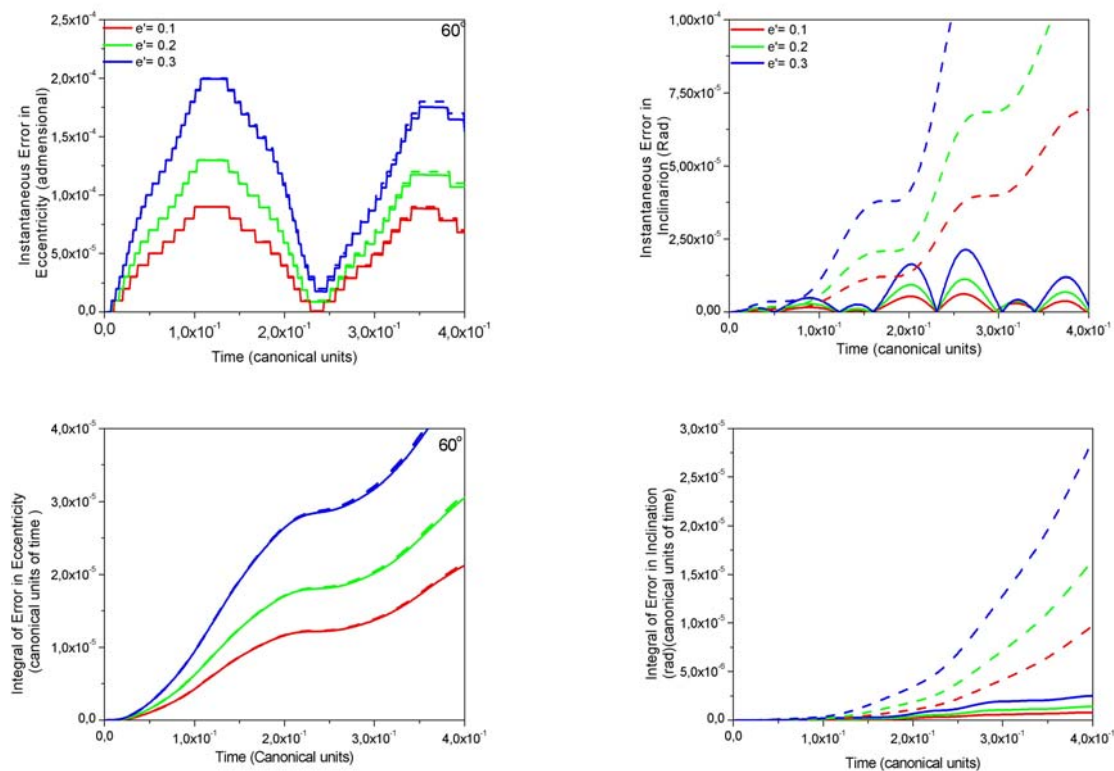


Figure 3. Instantaneous and integral errors in eccentricity and inclination as a function of time for *High-Earth Orbits* and initial inclinations of 60° . The eccentricity of the perturbing body (e') is shown in color codes: 0.1 (red), 0.2 (green) and 0.3 (blue). The lines are: single average model (solid line) and double averaged model (dashed line).

4. Conclusions

A comparison of the double and single-averaged models up to the second-order with the full restricted elliptic problem was performed. The second order averaged models are accurate for a perturbing body in a circular orbit. This accuracy decreases with the increase of the eccentricity of the perturbing body. The second order averaged models are not accurate and should not be used for eccentricities of the perturbing body of 0.3 or larger.

Then, a new approach that consists of studying the integral of the errors between the models over the time was made. It showed and quantified the better accuracy of the single-averaged model for the time span of two days used in the present research. This approach showed to be an interesting form of quantifying those differences.

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