

Regularized Blind Deconvolution with Poisson Data

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Abstract. We propose easy-to-implement algorithms to perform blind deconvolution of nonnegative images in the presence of noise of Poisson type. Alternate minimization of a regularized Kullback-Leibler cost function is achieved via multiplicative update rules. The scheme allows to prove convergence of the iterates to a stationary point of the cost function. Numerical examples are reported to demonstrate the feasibility of the proposed method.

1. Introduction

Alternating multiplicative update rules have been popularized recently by Lee and Seung [1] for the task of nonnegative matrix factorization, i.e. given Y a $n \times m$ nonnegative matrix of observations, the task of finding a $n \times p$ nonnegative matrix K and a $p \times m$ nonnegative matrix X such that $Y = KX$. These rules can be derived by minimizing either a least-squares criterion or the following generalized Kullback-Leibler divergence between Y and KX :

$$KL(Y, KX) = \sum_{i=0}^{m-1} \sum_{j=0}^{n-1} \left[Y_{i,j} \ln \left(\frac{Y_{i,j}}{(KX)_{i,j}} \right) - Y_{i,j} + (KX)_{i,j} \right].$$

The minimization of this latter cost function is the usual model for the case where the data Y are affected by photon-counting noise, i.e. obey a Poisson distribution. The minimization is performed alternately with respect to K and X with the following update rules, for $k = 0, 1, \dots$, starting from arbitrary but strictly positive matrices $K^{(0)}$ and $X^{(0)}$:

$$K^{(k+1)} = \frac{K^{(k)}}{\mathbf{1}_{n \times m} (X^{(k)})^T} \circ \frac{Y}{K^{(k)} X^{(k)}} (X^{(k)})^T; \quad X^{(k+1)} = \frac{X^{(k)}}{(K^{(k+1)})^T \mathbf{1}_{n \times m}} \circ (K^{(k+1)})^T \frac{Y}{K^{(k+1)} X^{(k)}},$$

where $\mathbf{1}_{n \times m}$ is a $n \times m$ matrix of ones, X^T is the transpose of X and we use the Hadamard (entrywise) product \circ and division. Monotonic decrease of the cost function is guaranteed by the fact that the algorithm can be derived as a Majorization-Minimization (MM) scheme, through the use of surrogate cost functions [1].

In the present paper, we focus on a special instance of this problem namely blind deconvolution in incoherent optical imaging, where the original image X , the blurred image Y and the space-invariant point spread function (PSF) K are nonnegative light intensities. In this case the matrix multiplications are to be replaced by circular convolutions (assuming



$m = p = n$). If the PSF K is known, the above algorithm is well known in astronomy as Richardson-Lucy's one [2, 3] and in medical imaging as (EM)ML (Expectation-Maximization Maximum Likelihood). For the special case of blind deconvolution, the above alternate minimization algorithm has been discussed previously in the literature by different authors dating back to [4, 5].

To prevent the algorithm to converge to the trivial solution where the PSF is a Dirac delta peak and the image is the original blurred one, we consider regularized versions of the alternating Richardson-Lucy algorithm, adding appropriate penalty terms to the Kullback-Leibler divergence. In addition we impose a flux normalization condition on the PSF. We can then establish the convergence of the iterates to a stationary point of the corresponding cost function which is separately but not jointly convex with respect to the image and the PSF. Note that for this problem convergence results are rare in the literature. Recent exceptions are the blind imaging methods based on the SGP algorithm of [6] and proposed in [7] for space-variant blind image restoration and in [8] for post-adaptive-optics imaging in astronomy, as well as the inexact block coordinate descent method of [9]. These algorithms and convergence results, however, are not exactly of the same kind as ours. The alternate proximal algorithm of [10] applies to the case of Gaussian noise (least-squares discrepancy) but could also be applied to Poisson noise when a quadratic approximation of the Kullback-Leibler divergence is used.

2. Blind deconvolution algorithms

Let X be the $n \times n$ matrix formed by the 2D image intensities $X_{i,j}$ in pixel (i, j) and K the $n \times n$ matrix containing the values of the space-invariant PSF. By $K * X$ we denote their 2D circular convolution which can be computed by going to Fourier space by means of a FFT. Besides $\tilde{X}_{i,j} \equiv X_{n-i, n-j}$. Assuming $Y_{i,j} \geq 0, \forall i, j$, we want to solve the following minimization problem:

$$\text{find } \underset{K, X}{\operatorname{argmin}} F(K, X) = KL(Y, K * X) + \frac{\mu}{2} \|K\|_F^2 + \lambda \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} X_{i,j} + \frac{\nu}{2} \|X\|_F^2$$

under the constraints $X_{i,j} \geq 0$, $K_{i,j} \geq 0$ and $\sum_{i=0}^{n-1} \sum_{j=0}^{n-1} K_{i,j} = 1$, with nonnegative regularization parameters λ, μ, ν . Notice that we penalize the squared Frobenius norm $\|K\|_F^2 = \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} K_{i,j}^2$ of the PSF and both the Frobenius and the sparsity-enforcing ℓ^1 -norm of the image. This allows to cover the cases of pure quadratic (Tikhonov) regularization when $\lambda = 0$, of the so-called "lasso" regularization when $\nu = 0$ as well as of a combination of both penalties which goes under the name "elastic-net" after [11]. As we noticed in practice, the latter strategy seems to perform better in the recovery of neighboring pixel values which are correlated.

To solve the optimization problem, we use the following alternating minimization algorithm which can be easily derived using the usual surrogate for the Kullback-Leibler divergence and the fact that the penalties are separable, i.e. are a sum of terms, each of which depends only on a single unknown scalar component:

- 1) initialize $X^{(0)}$ and $K^{(0)}$ with arbitrary strictly positive values, satisfying the normalization

$$\text{constraint } \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} K_{i,j}^{(0)} = 1 ;$$

- 2) at each iteration

- i) introduce a Lagrange parameter α for the normalization constraint and, for $\mu \neq 0$, determine its value by solving the following equation

$$2\mu + n^2 \left(\alpha + \sum_{l=0}^{n-1} \sum_{m=0}^{n-1} X_{l,m}^{(k)} \right) - \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} \sqrt{\left(\alpha + \sum_{l=0}^{n-1} \sum_{m=0}^{n-1} X_{l,m}^{(k)} \right)^2 + 4\mu K_{i,j}^{(k)} \left(\frac{Y}{K^{(k)} * X^{(k)}} * \tilde{X}^{(k)} \right)_{i,j}} = 0$$

- by means of a few Newton-Raphson iterations, whereas for $\mu = 0$,
- use $\alpha = \sum_{l=0}^{n-1} \sum_{m=0}^{n-1} Y_{l,m} - \sum_{l=0}^{n-1} \sum_{m=0}^{n-1} K_{l,m}^{(k)}$,
- ii) compute $A^{(k)} = K^{(k)} \circ \left(\frac{Y}{K^{(k)} * X^{(k)}} * \tilde{X}^{(k)} \right)$,
- iii) compute $B^{(k)} = \mathbf{1}_{n \times n} X^{(k)} \mathbf{1}_{n \times n} + \alpha \mathbf{1}_{n \times n}$,
- iv) set $K^{(k+1)} = \frac{2A^{(k)}}{B^{(k)} + \sqrt{B^{(k)} \circ B^{(k)} + 4\mu A^{(k)}}}$,
- v) compute $C^{(k+1)} = X^{(k)} \circ \left(\tilde{K}^{(k+1)} * \frac{Y}{K^{(k+1)} * X^{(k)}} \right)$,
- vi) compute $D^{(k+1)} = \mathbf{1}_{n \times n} K^{(k+1)} \mathbf{1}_{n \times n} + \lambda \mathbf{1}_{n \times n}$,
- vii) set $X^{(k+1)} = \frac{2C^{(k+1)}}{D^{(k+1)} + \sqrt{D^{(k+1)} \circ D^{(k+1)} + 4\nu C^{(k+1)}}}$.

Notice that the nonnegativity of both the image and the PSF is automatically preserved throughout the multiplicative update scheme.

Building upon Zangwill's convergence theory [12], we can establish the following properties for the above algorithm.

Theorem

If Y is a nonnegative matrix and $X^{(0)}, K^{(0)}$ are two strictly positive matrices, with normalization $\sum_{i=0}^{n-1} \sum_{j=0}^{n-1} K_{i,j}^{(0)} = 1$, then

- $F(K^{(k+1)}, X^{(k+1)}) \leq F(K^{(k)}, X^{(k)})$;
- $F(K^{(k+1)}, X^{(k+1)}) < F(K^{(k)}, X^{(k)})$ if and only if $(K^{(k+1)}, X^{(k+1)}) \neq (K^{(k)}, X^{(k)})$;
- the sequence of iterates $(K^{(k)}, X^{(k)})$ converges to a stationary point (K^*, X^*) of $F(K, X)$.

The detailed proof of this convergence theorem is too long to be reported here. Hence we just sketch the key elements to be established:

- prove that all the fixed points of the algorithm are stationary points i.e. verify the first order Karush-Kuhn-Tucker conditions;
- prove that all the iterates lie in a compact set;
- using points a) and b) above, Zangwill's theorem allows to conclude that all the accumulation points of the sequence of iterates are stationary points;
- finally, show that the sequence cannot oscillate between different stationary points.

Our results can be extended to cover the case of the following smoothed total-variation (TV) penalty on the image

$$TV(X) = \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} \left| \nabla_{i,j} X \right|_{\varepsilon} \quad \text{with} \quad \left| \nabla_{i,j} X \right|_{\varepsilon} = \sqrt{\varepsilon^2 + (X_{i+1,j} - X_{i,j})^2 + (X_{i,j+1} - X_{i,j})^2}$$

using the separable surrogate for this penalty introduced in [13]. Moreover, we can derive similar algorithms and convergence results for the case of a Gaussian noise, i.e. when the Kullback-Leibler divergence is replaced by a quadratic Frobenius norm. To the best of our knowledge, also in the non-blind case, the algorithm with a smoothed TV penalty is new for the case of Poisson noise whereas our convergence results are original for both cases of Gaussian and Poisson noise.

3. Numerical simulations

We have investigated the numerical performance of our method by testing it on various image restoration problems and we report a few examples here. As test images, we use the spiral nebula NGC 7027 (256×256), the satellite (256×256) and the cameraman (512×512). The blurred image is generated by circular low-pass filtering, i.e. by convolution with an Airy pattern, and corrupted by Poisson noise which results in a root-mean-square error (RMSE) of 1% for the nebula and the satellite and of 2.5 % for the cameraman.



Figure 1. The original images and the original PSF

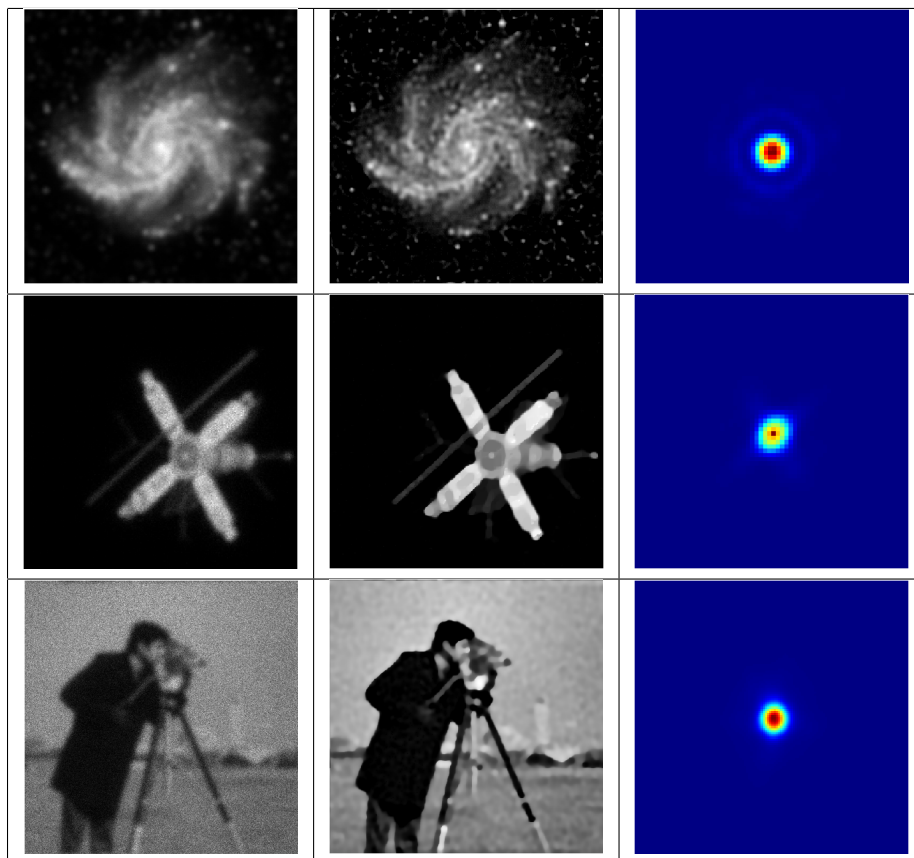


Figure 2. The blurred and noisy images, the reconstructed images and the reconstructed PSFs

For the satellite and the cameraman, we used a TV penalty on the image and for the nebula an elastic-net penalty. The algorithm was initialized with the blurred and noisy image and

with a uniform PSF. The original images and PSF are reported in Figure 1, the reconstructions in Figure 2 whereas the values of all parameters used are given in Table 1, together with the number of iterations performed, the computing time in MATLAB on a laptop and the relative RMS reconstruction errors on both the image X and the PSF K . For a better visualization of the PSF, only the central part of the image (zooming by a factor 4 in both dimensions) is displayed in false colors.

Image	μ	λ	ε / ν	It	Time	$RMSE_K$	$RMSE_X$
Nebula	10^9	10^{-7}	$6 \cdot 10^{-8}$	2000	12m37s	0.30	0.33
Satellite	$1.5 \cdot 10^6$	0.0485	$\sqrt{10^{-7}}$	200	1m46s	0.20	0.18
Camerman	10^7	0.03	$\sqrt{10}$	2000	54m30s	0.17	0.15

Table 1. Parameters used for the simulations

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