

Black Hole Bose Condensation

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Abstract. General consensus on the nature of the degrees of freedom responsible for the black hole entropy remains elusive despite decades of effort dedicated to the problem. Different approaches to quantum gravity disagree in their description of the microstates and, more significantly, in the statistics used to count them. In some approaches (string theory, AdS/CFT) the elementary degrees of freedom are indistinguishable, whereas they must be treated as distinguishable in other approaches to quantum gravity (eg., LQG) in order to recover the Bekenstein-Hawking area-entropy law. However, different statistics will imply different behaviors of the black hole outside the thermodynamic limit. We illustrate this point by quantizing the Bañados-Teitelboim-Zanelli (BTZ) black hole, for which we argue that Bose condensation will occur leading to a “cold”, stable remnant.

By now it seems clear that most approaches to quantum gravity are able to obtain some general features of black hole thermodynamics [1] from statistical mechanics in the thermodynamic limit. However, this seeming universality of black hole thermodynamics is belied by the fact that the leading candidates for a theory of quantum gravity differ from each other both in the microstates being counted and in the statistics applied to the elementary degrees of freedom. In string theory the black hole microstates are dual to weak field D-brane states [2], in the AdS/CFT approach they are taken to be the states of a particular horizon conformal field theory (CFT) [3] and in loop quantum gravity (LQG) they are represented by punctures of a spin network on the event horizon [4]. In string theory and AdS/CFT the elementary degrees of freedom are indistinguishable whereas they must be treated as distinguishable in LQG in order to recover the Bekenstein-Hawking law. We may expect that these different models will diverge in their descriptions of the black hole at very low temperatures and outside the thermodynamic limit, where statistics plays an important role.

In this contribution we ask what behavior one may expect outside the thermodynamic limit when Bose statistics is used to count the microstates. Since such a question may only be answered in a model dependent way, we pose it within the context of a model of the quantum black hole that arises naturally from a canonical quantization of gravitational collapse. We discuss the statistical mechanics of the BTZ black hole. (The importance of the BTZ black hole lies in the fact that many black holes in string theory have near horizon geometries of the form $BTZ \times \mathcal{M}$, where \mathcal{M} is a simple manifold, and their thermodynamics can be obtained directly or by duality from the entropy of the BTZ black hole [5]. This makes our discussion relevant to a very large class of interesting black hole solutions in higher dimensions.)

We will argue that Bose-Einstein condensation should occur at sufficiently low temperatures. We will determine the condensate temperature and the condensate fraction as well as the entropy. We show that the area law holds at high temperatures but breaks down below the critical



temperature, where the entropy becomes dominated by a mass independent term making the black hole cold and stable even though its mass remains non-vanishing.

Both canonical quantum gravity [6] and the AdS/CFT correspondence [7, 8] predict an equispaced mass spectrum for the BTZ black hole and agree on the indistinguishability of the elementary degrees of freedom. In canonical quantum gravity one obtains a stationary state solution of the Wheeler-DeWitt equation with the mass levels

$$M_j = \frac{\hbar}{l} \left(j + \frac{1}{2} \right), \quad (1)$$

where $l^2 = -\Lambda^{-1}$ is the AdS length. The macroscopic black hole is thought of as the end state of the collapse of many, say $\mathcal{N} = \sum_j \mathcal{N}_j$, matter shells, \mathcal{N}_j of which occupy level j of (1). When a boundary contribution from the origin is included [9], the total black hole mass is given as

$$M = M_0 + \sum_j \frac{\hbar}{l} \left(j + \frac{1}{2} \right) \mathcal{N}_j. \quad (2)$$

From this point of view, the elementary degrees of freedom are bosonic mass shells and a black hole microstate is a particular distribution of \mathcal{N} shells between the levels in (1). We begin with the grand partition function,

$$\Xi(\mu, \beta) = e^{-\beta M_0} \prod_j (1 - e^{-\beta(\varepsilon_j - \mu)})^{-1}, \quad (3)$$

where the chemical potential μ is determined from the constraint $\sum_j \langle \mathcal{N}_j \rangle = \mathcal{N}$ and ε_j is the energy of the level j . Introducing the fugacity, $\lambda^{-1} = e^{-\beta(\mu - \varepsilon_0)} > 1$, and $\Delta\varepsilon_j = \varepsilon_j - \varepsilon_0$ we then have

$$\Xi(\mu, \beta) = e^{-\beta M_0} \prod_j \frac{\lambda}{\lambda - e^{-\beta\Delta\varepsilon_j}}. \quad (4)$$

Since the average level occupation number is computed directly from $\Xi(\beta, \mu)$,

$$\langle N_j \rangle = -\frac{1}{\beta} \frac{\partial \ln \Xi}{\partial \varepsilon_j} = \frac{1}{e^{\beta\Delta\varepsilon_j} - \lambda}, \quad (5)$$

our condition for determining the chemical potential is

$$\mathcal{N} = \sum_{j=0}^{\infty} \frac{1}{e^{\beta\Delta\varepsilon_j} - \lambda} = \frac{\lambda}{1 - \lambda} + \sum_{j=1}^{\infty} \frac{\lambda}{e^{\beta\Delta\varepsilon_j} - \lambda}. \quad (6)$$

The first term on the right represents the ground state occupancy, N_0 (henceforth, average values will be understood and the angular brackets will be omitted), and the second represents the number of excited shells, N_{ex} .

The condensate fraction, f_C , is the ratio of the ground state occupancy to the total number of shells, $f_C = N_0/\mathcal{N}$. We will first determine f_C as a function of the temperature. For the spectrum of the BTZ black hole in (1), $\Delta\varepsilon_j = \hbar j/l$ so replacing the last sum in (6) by its power series expansion, we can express the total number of shells as

$$\mathcal{N} = \frac{\lambda}{1 - \lambda} + \sum_{j=1}^{\infty} \sum_{r=1}^{\infty} \lambda^r e^{-\beta \hbar r j / l}. \quad (7)$$

The most direct way to evaluate the double sum in (7) is to employ the Mellin-Barnes representation of the exponential function and re-express it as an integral over the complex plane [10]. This gives

$$\mathcal{N} = \frac{\lambda}{1-\lambda} + \frac{1}{2\pi i} \int_{\tau-i\infty}^{\tau+i\infty} dt \frac{\Gamma(t)\zeta(t)\text{Li}_t(\lambda)}{(\beta\hbar/l)^t}, \quad (8)$$

where $\zeta(t)$ is the zeta function and $\text{Li}_t(\lambda)$ is the polylogarithm. For $\lambda < 1$ the integral is governed by the simple pole of the ζ -function at $t = 1$, therefore

$$\mathcal{N} = \frac{\lambda}{1-\lambda} + \frac{\text{Li}_1(\lambda)}{\beta\hbar/l} \equiv \frac{\lambda}{1-\lambda} - \frac{1}{\beta\hbar/l} \ln(1-\lambda). \quad (9)$$

Equation (9) can be solved for λ in terms of the Lambert function, $\mathcal{W}(x)$,

$$\lambda(\beta, \mathcal{N}) = 1 - \frac{\beta\hbar/l}{\mathcal{W}(\beta\hbar/l e^{(\mathcal{N}+1)\beta\hbar/l})}, \quad (10)$$

and determines the ground state occupancy as well as the condensate fraction,

$$f_C(\beta, \mathcal{N}) = \frac{N_0}{\mathcal{N}} = \frac{\mathcal{W}(\beta\hbar/l e^{(\mathcal{N}+1)\beta\hbar/l})}{\mathcal{N}\beta\hbar/l} - \frac{1}{\mathcal{N}}, \quad (11)$$

in terms of the temperature. It is more illuminating, however, to express the condensate fraction as a function of the dimensionless ratio, $x = T/T_c$, where T_c is the critical temperature at which all the shells are excited.

To calculate the critical temperature, T_c , it is necessary to set $\lambda \approx 1$ and $N_{\text{ex}} = \mathcal{N}$, the total number of shells. Once again employing the Mellin-Barnes transformation, this time replacing the polylogarithm function $\text{Li}_t(\lambda)|_{\lambda \approx 1}$ by the ζ -function, we find [10]

$$N_{\text{ex}} = \mathcal{N} \approx \frac{1}{2\pi i} \int_{\tau-i\infty}^{\tau+i\infty} dt \frac{\Gamma(t)\zeta^2(t)}{(\beta_c\hbar/l)^t}, \quad (12)$$

but the integral now admits a double pole at $t = 1$ via the ζ -function and therefore

$$\mathcal{N} \approx \left(\frac{\Gamma(t)}{(\beta\hbar/l)^t} \right)' = -\frac{l}{\beta_c\hbar} \ln(e^\gamma \beta_c\hbar/l), \quad (13)$$

where γ is the Euler-Mascheroni constant. Solving for the inverse critical temperature, we determine

$$\beta_c\hbar/l = \frac{\mathcal{W}(e^{-\gamma}\mathcal{N})}{\mathcal{N}}, \quad (14)$$

which is the exact version of the result in [11]. The Lambert function increases linearly for small values of its argument and slower than the log function for larger values. Thus, for a few shells the critical temperature is more or less constant whereas it increases roughly as $\mathcal{N}/\ln\mathcal{N}$ when \mathcal{N} grows large. When the black hole is composed of many shells, most of them will in fact lie in the ground state. For a small number of shells the ground state occupancy can be a large percentage of the total number of shells in a significant interval of temperatures.

Using (14) to re-express the condensate fraction in terms of the variable $x = T/T_c$, we easily arrive at

$$f_C(x, \mathcal{N}) = \frac{x}{\mathcal{W}_N} \mathcal{W}\left(\frac{\mathcal{W}_N}{\mathcal{N}x} e^{\frac{(\mathcal{N}+1)\mathcal{W}_N}{\mathcal{N}x}}\right) - \frac{1}{\mathcal{N}}, \quad (15)$$

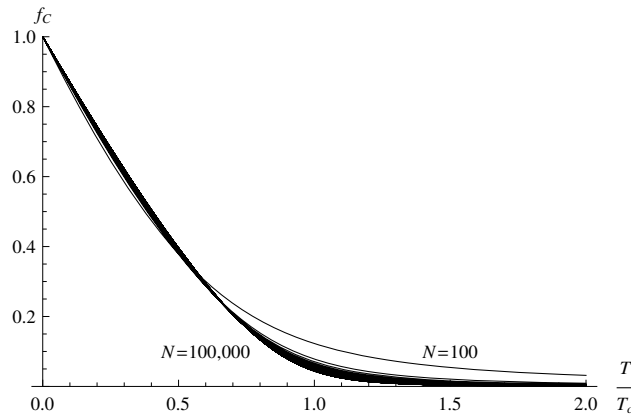


Figure 1. f_C as a function of T/T_c .

where we have defined $\mathcal{W}_N = \mathcal{W}(e^{-\gamma}\mathcal{N})$. At high temperatures,

$$f_C \stackrel{x \rightarrow \infty}{\approx} \frac{1}{\mathcal{N}} + \frac{\mathcal{W}_N}{\mathcal{N}x} + \mathcal{O}(x^{-2}) \quad (16)$$

whereas, at low temperatures,

$$f_C \stackrel{x \rightarrow 0}{\approx} 1 - \frac{\ln \mathcal{N}}{\mathcal{W}_N} x + \mathcal{O}(x^2) \quad (17)$$

Figure 1 shows the behavior of f_C as a function of x for \mathcal{N} ranging from one hundred to one hundred thousand shells. It is small, increasing slowly to the left when $x \gg 1$ but significantly faster when $x \ll 1$ and approaching unity in the limit as $x \rightarrow 0$. Again, the slope of the condensate fraction, $\partial f_C / \partial x$, approaches $\ln \mathcal{N} / \mathcal{W}_N$ for small x and it is vanishingly small for large x . Thus increasing the number of shells saturates the slope of the condensate fraction. This behavior suggests that the critical temperature may be thought of as the condensation temperature.

The average energy of the system, which is associated with the black hole mass, can be expressed in terms of the level occupancy in the usual way and the sum rewritten in the integral representation we have used so far. For $\lambda < 1$ there is only a simple pole coming from the ζ -function at $t = 2$, so the black hole mass can be given in terms of the temperature by

$$M(\beta, \mathcal{N}) = M_0 + \mathcal{N}\varepsilon_0 + \frac{\text{Li}_2(\lambda)}{\beta^2 \hbar / l} \quad (18)$$

and the heat capacity, expressed in terms of $x = T/T_c$, is

$$C(x, \mathcal{N}) = \frac{2\mathcal{N}x\text{Li}_2(\lambda)}{\mathcal{W}_N} - \frac{\mathcal{N}x^2}{\lambda\mathcal{W}_N} \ln(1 - \lambda) \left(\frac{\partial \lambda}{\partial x} \right)_{\mathcal{N}}, \quad (19)$$

where the prime on the polylogarithm represents a derivative with respect to λ . The specific heat, C/\mathcal{N} , is displayed in figure (2) for one hundred to one hundred thousand shells. The uppermost curve is for $\mathcal{N} = 100$. The specific heat is seen to decrease with increasing \mathcal{N} , but there is no true phase transition in this system. At low temperatures, the fugacity in (10)

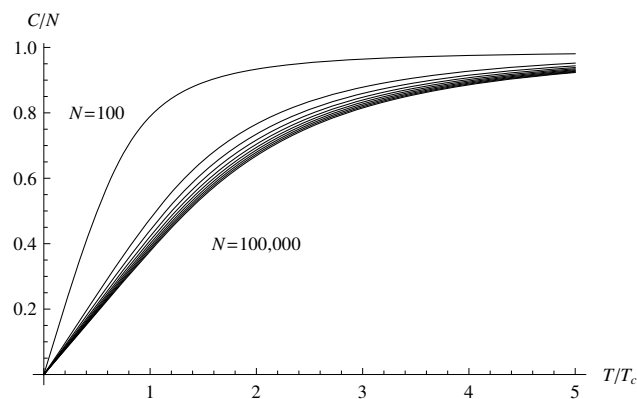


Figure 2. C/\mathcal{N} as a function of T/T_c .

behaves roughly as $1 - \mathcal{N}^{-1}$ and the second term in the expression for the heat capacity in (19) becomes vanishingly small. Thus in the limit as $\beta \rightarrow \infty$, using $\text{Li}_2(1) = \pi^2/6$ we find

$$\frac{C}{\mathcal{N}} \approx \frac{\pi^2 kT}{3\mathcal{N}\hbar/l}. \quad (20)$$

(see [12, 13]). On the other hand, at high temperatures the fugacity behaves as

$$\lambda = 1 - e^{-N\beta\hbar/l} \quad (21)$$

and the specific heat approaches unity in the limit as $\beta \rightarrow 0$.

We now turn to the entropy, which may be computed from

$$S = \ln \Xi + \beta U - \mu\beta\mathcal{N}. \quad (22)$$

Expanding $\ln \Xi$ in a power series and applying the transformation in Mellin-Barnes transformation, we arrive at

$$\ln \Xi = -\beta M_0 - \ln(1 - \lambda) + \frac{\text{Li}_2(\lambda)}{\beta\hbar/l} \quad (23)$$

and combine this expression with (18). Then the entropy function can be given in terms of the ratio $x = T/T_c$ as

$$S(\beta, \mathcal{N}) = \frac{2x\mathcal{N}}{\mathcal{W}_N} \text{Li}_2(\lambda) - \ln(1 - \lambda) - \mathcal{N} \ln \lambda. \quad (24)$$

The specific entropy is shown in figure (3) for one hundred to one hundred thousand shells. The uppermost curve represents one hundred shells and the specific entropy also decreases with increasing \mathcal{N} .

So far we have considered the thermodynamic potentials as functions of the number of shells, \mathcal{N} , and the temperature, T , but let us now consider this function in terms of its natural variables, M and \mathcal{N} . For a black hole that is far below the critical temperature, *i.e.*, as $x \rightarrow 0$, (24) is approximated by

$$S \approx \frac{\pi^2}{3\beta\hbar/l} + \ln \mathcal{N} + 1 \quad (25)$$

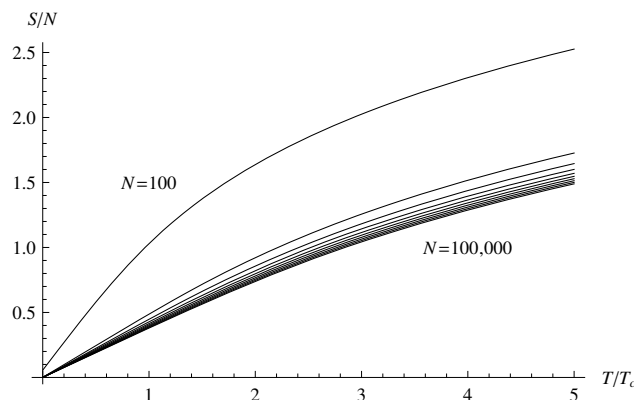


Figure 3. S/\mathcal{N} as a function of T/T_c .

This is not vanishing, but the specific entropy, S/\mathcal{N} , does vanish in the thermodynamic limit as the temperature approaches zero and so it is consistent with a generalized version of the third law. In this same limit we find from (18) that

$$\frac{\pi}{\beta\hbar/l} \approx \sqrt{\frac{6(M - M_0 - \mathcal{N}\varepsilon_0)}{\hbar/l}} \quad (26)$$

so we can give the entropy in terms of the black hole mass and the number of shells as

$$S \approx \frac{\pi}{3} \sqrt{\frac{6(M - M_0 - \mathcal{N}\varepsilon_0)}{\hbar/l}} + \ln \mathcal{N} + 1. \quad (27)$$

This is where Bose condensation plays a dominant role. At temperatures well below the critical temperature, where a large fraction of the shells are in the ground state, the first term on the right becomes negligible and the entropy is dominated by its logarithmic behavior. Thus even though a quantization of Einstein's gravity yields the expected results in the thermodynamic limit and at high temperatures ($M, \mathcal{N} \rightarrow \infty$, $M \gg \mathcal{N}\varepsilon_0$), its behavior outside this regime presents interesting features peculiar to quantum statistics. In particular, we see that BTZ black holes turn cold once condensation sets in. This is the “small” black hole or short distance regime in 2+1 dimensions.

We suggest that this is in fact the regime in which “small” black holes and more generally quantum gravity at short distances should be studied. Here, the entropy becomes dominated by a mass independent term so that the black hole becomes cold and (hence) stable even though the mass remains non-vanishing at $M \approx M_0 + \mathcal{N}\varepsilon_0$. Furthermore, because the thermodynamics of many higher dimensional black holes are derivable from the BTZ black hole, one may reasonably expect that this feature persists in more than three dimensions. It suggests that the universe may be full of stable, Bose condensed black hole relics, which contribute to the dark matter.

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References

- [1] J. D. Bekenstein, Ph.D. thesis, Princeton University (1972); *ibid* Lett. Nuovo Cimento **4** (1972) 737; *ibid* Phys. Rev. D **7** (1973) 2333.

- J. M. Bardeen, B. Carter, and S. W. Hawking, *Comm. Math. Phys.* **31** (1973) 161.
S. W. Hawking, *Comm. Math. Phys.* **43** (1975) 199.
T. Jacobson, *Phys. Rev. Lett.* **75** (1995) 1260.
T. Padmanabhan, *AIPConf. Proc.* 939 (2007) 114.
- [2] A. Strominger and C. Vafa, *Phys. Lett. B* **379** (1996) 99.
G.T. Horowitz and J. Polchinski, *Phys. Rev. D* **55** (1997) 6189.
A. Dabholkar, *Phys. Rev. Lett.* **94** (2005) 241301.
- [3] S. Hyun, W.T. Kim, and J. Lee, *Phys. Rev. D* **59** (1999) 084020.
R. Emparan, *JHEP* 9906 (1999) 036.
S. W. Hawking, J. Maldacena, and A. Strominger, *JHEP* 0105 (2001) 001.
S. A. Mukherji and S. S. Pal, *JHEP* 0205 (2002) 026.
S. K. Chakrabarti, K.S. Gupta, and S. Sen, *Int. J. Mod. Phys. A* **23** (2008) 2547.
- [4] C. Rovelli, *Phys. Rev. Lett.* **77** (1996) 3288.
A. Ashtekar, J. Baez, A. Corichi, and K. Krasnov, *Phys. Rev. Lett.* **80** (1998) 904.
M. Domagala and J. Lewandowski, *Class. Quant. Grav.* **21** (2004) 5233.
K. A. Meissner, *Class. Quant. Grav.* **21** (2004) 5245.
A. Corichi, J. Díaz-Polo and E. Fernández-Borja, *Phys. Rev. Lett.* **98** (2007) 181301.
I. Agulló, J. F. Barbero G., J. Díaz Polo, E. Fernández-Borja and E. J. S. Villaseñor, *Phys. Rev. Lett.* **100** (2008) 211301.
- [5] S. Nampuri, P.K. Tripathy and S.P. Trivedi, *JHEP*0807:072 (2008),
D. Birmingham and S. Carlip, *Phys. Rev. Letts.* **92** (2004) 111302,
M. Cvetič, Finn Larsen, *Phys. Rev. Letts.* **82** (1999) 484,
S. Carlip, *Class.Quant.Grav.* **15** (1998) 3609 (*and refs. therein.*)
- [6] C. Vaz, S. Gutti, C. Kiefer, T.P. Singh and L.C.R. Wijewardhana, *Phys. Rev. D* **77** (2008) 064021.
- [7] S. Carlip, *Phys. Rev. D* **51** (1995) 632.
- [8] A. Strominger, *JHEP* 9802 (1998) 009.
- [9] R. B. Mann and S. F. Ross, *Phys. Rev. D* **47**, 3319 (1993).
S. Gutti, *Class. Quant. Grav.* **22** (2005) 3223.
- [10] Martin Holthaus, Eva Kalinowski and Klaus Kirsten, *Ann. Phys.* **270** 1 (1998) 198.
- [11] F. Dalfovo, S. Giorgini, L.P. Pitaevskii and S. Stringari, *Rev. Mod. Phys.* **71** (1999) 463.
- [12] W. Ketterle and N.J. van Druten, *Phys. Rev. A* **54** (1996) 656.
- [13] G-L. Ingold and A. Lambrecht, *Eur. Phys. Jour. D* **1** (1998) 29.