

Graviton Scattering Made Simple(r)¹

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Abstract. I summarize a few recent developments in unlocking the hidden structure of graviton scattering amplitudes. In particular: (1) I show how the ‘bonus relations’ for graviton amplitudes may be used to provide simple analytic proofs of the equivalence of various distinct formulas for the n -graviton MHV amplitude which have appeared in the literature (and, in particular, providing a direct proof of the original BGK formula); (2) review the ‘tree formula’ for MHV graviton scattering, which has several nice properties including the fact that it manifests both S_{n-2} symmetry as well as $1/z^2$ falloff at large z ; and (3) comment on the solution of the on-shell recursion relation for arbitrary tree-level graviton amplitudes.

1. Introduction

The past few years have witnessed dramatic progress in our understanding of the mathematical structure of scattering amplitudes. These developments have led to purely theoretical insights and also, as a pleasant spin-off, to powerful new methods for carrying out previously intractable calculations. The overwhelming majority of recent progress has been in maximally supersymmetric gauge theory (SYM), where we now have a pretty good understanding of scattering amplitudes at both weak and strong coupling, and in fact at all values of the coupling for $n = 4, 5$ particles. There are reasons to expect graviton scattering amplitudes to have even richer structure [4], but progress on that front has been much slower.

In this talk I discuss some recent progress on even the simplest graviton amplitudes at tree level. In fact I argue that there is still a lot of room for improvement on the simplest of simplest graviton amplitudes—maximally-helicity violating (MHV) amplitudes at tree level.

MHV amplitudes are the simplest non-vanishing amplitudes in both Yang-Mills theory and in gravity. These describe for example the production of any number of positive helicity particles from a pair of scattering negative helicity particles, as depicted in fig. 1. Amplitudes become more and more complicated as the number of additional negative helicity particles in the outgoing state increases.

2. The Parke-Taylor and Berends-Giele-Kuijf Formulas for MHV Amplitudes

The MHV n -gluon scattering amplitude is given by the formula

$$A(1, \dots, 2) = \frac{1}{\langle 12 \rangle \langle 23 \rangle \cdots \langle n1 \rangle} \quad (1)$$

¹ Based on work done in collaboration with J. Drummond, D. Nguyen, A. Volovich, and C. Wen in [1, 2, 3].



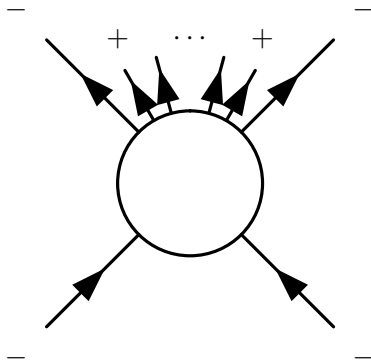


Figure 1. A maximally helicity violating (MHV) amplitude.

conjectured by Parke and Taylor [5] and proven by Berends and Giele in 1988 [6]. Here and in what follows we use the spinor helicity notation:

$$p_{a\dot{a}} = \lambda_a \tilde{\lambda}_{\dot{a}} \quad \langle i j \rangle = \epsilon_{ab} \lambda_i^a \lambda_j^b \quad [i j] = \epsilon_{\dot{a}\dot{b}} \tilde{\lambda}_i^{\dot{a}} \tilde{\lambda}_j^{\dot{b}} \quad (2)$$

$$(p_i + p_j)^2 = \langle i j \rangle [i j] \quad [i|p_j + p_k + \cdots |p_l] = [i j] \langle j l \rangle + [i k] \langle k l \rangle + \cdots . \quad (3)$$

The existence of the formula (1) has implications far beyond simply knowing the answer for a certain scattering amplitude, which one may or may not be particularly interested in. Rather, the existence of such a simple formula for something which would otherwise require enormously tedious Feynman diagram calculations is telling us that the theory has a rich hidden structure, and moreover that if one actually is interested in computing some particular amplitude, it behooves one to understand this structure.

For MHV graviton amplitudes, Berends, Giele and Kuijf conjectured in 1987 the formula [7]

$$\mathcal{M}_n = \sum_{\mathcal{P}(2, \dots, n-2)} \frac{[1 2][n-2 \ n-1]}{\langle 1 \ n-1 \rangle} \left(\prod_{i=1}^{n-3} \prod_{j=i+2}^{n-1} \langle i j \rangle \right) \prod_{k=3}^{n-3} [k|p_{k+1} + \cdots + p_{n-1}|n] . \quad (4)$$

The BGK formula serves perfectly well if one's interest is to know the answer for any particular graviton scattering amplitude, but it serves poorly as a guide towards discovering hidden structure in gravity, for several reasons:

- It contains vestigial references \cdots to an ordering of the gravitons (this is usually a consequence of having used KLT relations [8]), even though gravitons carry no color structure so the amplitude should be independent of the ordering;
- it only manifests a subgroup S_{n-3} of the full S_n symmetry that the amplitude should have under permutation of all n gravitons; and finally,
- even that piece of the symmetry is realized in a fairly ‘brutal’ way (by an explicit sum over the permutations \mathcal{P} of $(2, \dots, n-1)$).

In essence, the formula is ‘not sufficiently gravitational’.

3. A Tale of Two Families of Formulas [1]

Several variants of the BGK formula have appeared in the literature in recent years (see for example [9, 10, 11]), and the complexity of the various formulas is such that in general it is not possible to prove directly that two different formulas are in fact equivalent to each other.

Rather one must resort to numerical experimentation to convince oneself that they are. In fact the original 1987 BGK formula itself was only very recently proven to be correct by Mason and Skinner [12].

Some forms, like BGK, have manifest S_{n-3} symmetry, while others have a slightly larger manifest S_{n-2} symmetry, which however comes at a price I will describe shortly.

3.1. A High- z Limit

Four-dimensional amplitudes of gluons or gravitons are functions of the spinor helicity variables $(\lambda_i, \tilde{\lambda}_i)$. All gravity amplitudes have exceptionally soft behavior, falling off as $1/z^2$ (see [9, 13, 14, 15, 10, 16, 17] and [4] for the most complete treatment) as two particles are taken to infinity in a particular complex direction

$$\lambda_1 \rightarrow \lambda_1 + z\lambda_2, \quad \tilde{\lambda}_2 \rightarrow \tilde{\lambda}_2 - z\tilde{\lambda}_1, \quad z \rightarrow \infty \quad (5)$$

This is a property of amplitudes as a whole—individual Feynman diagrams can diverge in this limit, but miraculously their sum magically conspires to enable $1/z^2$ falloff. In contrast, gluon amplitudes vanish only as $1/z$ in the same limit [18].

3.2. On-Shell Recursion from $1/z$ Falloff

Falloff as $1/z$ for large z , a feature of both Yang-Mills theory and gravity, is already sufficiently strong to imply the on-shell recursion relation [18]. To see this, consider the contour integral

$$0 = \oint dz \frac{A(z)}{z} \quad (6)$$

with a contour circling infinity. Then deform the contour inward to write it as a sum over residues,

$$0 = A(0) + \sum_k \frac{1}{z_k} \text{Res}_{z=z_k} A(z). \quad (7)$$

Now the only poles in a tree-level amplitude are those which come from an internal propagator going on-shell. Moreover the residue at any such pole is a product of two tree amplitudes. Therefore this formula gives us a relation schematically of the form

In words: any amplitude may be expressed as a sum over all on-shell factorizations in which the two ‘special’ legs 1 and 2 are separated from each other. Note that this remarkable on-shell recursion only works for theories with spin. The mantra is that scalar field theories may have simple Lagrangians, but their amplitudes are quite messy. In contrast gauge and gravity theories have horrendous Lagrangians, but very nice amplitudes.

3.3. Solving the Recursion for MHV Amplitudes

For MHV amplitudes only a single type of factorization can appear, which means that the on-shell recursion takes the extremely simple form

$$\text{MHV}(1, 2, 3, \dots, n) = \sum_{j=3}^n \text{MHV}(j, \hat{2}) \text{MHV}(3, \hat{1})$$

which can easily be solved explicitly. Depending on how one chooses to organize the calculation one can put the result into slightly different forms (see for example [9, 11]). Of particular importance to us will be a particular formula of Elvang and Freedman [11] which can be written as

$$M_n^{\text{MHV}} = \sum_{\mathcal{P}(3, \dots, n)} [A^{\text{MHV}}(1, 2, \dots, n)]^2 G^{\text{MHV}}(1, 2, \dots, n) \quad (8)$$

where $A(1, 2, \dots, n)$ is nothing but the MHV gluon scattering amplitude shown above in eq. (1), and $G(1, 2, \dots, n)$ is a particular ‘gravity dressing factor.’ The sum is taken over the $(n-2)!$ permutations of the external lines which preserve the two ‘special’ legs. It is a generic feature of MHV formulas derived from the on-shell recursion that they have $(n-2)!$ terms. In contrast, recall that the BGK formula has $(n-3)!$ terms.

3.4. Bonus Relations from $1/z^2$ Falloff

The vanishing of amplitudes like $1/z^2$, which is a special feature of spin-2 (i.e., gravity), implies the existence of nontrivial hidden relations between amplitudes. To see this, consider the contour integral

$$0 = \oint dz M(z),$$

again with a contour circling infinity. Deforming the contour to write it as a sum over residues gives the relation

$$0 = \sum_k \text{Res}_{z=z_k} M(z) \quad (9)$$

which can be expressed graphically as

$$0 = \sum_P \frac{P^2}{[1|P|2]} \text{MHV}(1, 2, P, 3, 4)$$

For MHV amplitudes in particular we have simply

$$0 = \sum_{j=3}^n \frac{\langle 1j \rangle}{\langle 2j \rangle} \text{MHV}(j, \hat{2}) \text{MHV}(3, \hat{1})$$

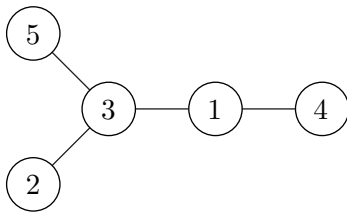


Figure 2. An example of a labeled tree diagram for $n = 7$ graviton scattering.

This linear relation can be used to eliminate any one of the $n - 2$ terms in the on-shell recursion for the MHV graviton amplitude. When applied recursively, this can be used to systematically reduce $(n - 2)!$ -term formulas to $(n - 3)!$ -term formulas.

3.5. Summary and Wishlist

All known formulas in the literature are of two types: they have either $(n - 2)!$ or $(n - 3)!$ terms. Until recently, all of the former had been proven by using the on-shell recursion, while all of the latter (like the original BGK formula) were just conjectures (like BGK) that could only be numerically compared to the former. Using the ‘bonus relation’ for gravity amplitudes we have demonstrated the relation between these two classes of formulas and provided the first direct proof that the 1987 BGK formula satisfies the on-shell recursion [1]. The $(n - 2)!$ -term formulas have a larger manifest symmetry, but they do not manifest the $1/z^2$ falloff: the sum over $(n - 2)!$ permutations contains redundant information which can be ‘squeezed’ out, using the bonus relation, leaving a sum over “only” $(n - 3)!$ terms.

Ideally, we might hope to expect that a truly gravitational formula should have:

- manifest $1/z^2$ falloff, term by term,
- no vestigial reference to \dots ,
- manifest symmetry that is naturally S_{n-2} or larger!), and finally,
- “simplicity and beauty.”

4. Twistor Space

In search of such a formula we now turn our attention to twistor space. Arkani-Hamed et. al. have recently shown [19] that amplitudes have some nice properties when expressed in ‘mixed’ twistor space representations. This means that for some particles we choose to Fourier transform $\lambda_i \rightarrow \mu_i$ while for others we choose to Fourier transform $\tilde{\lambda}_J \rightarrow \tilde{\mu}_J$. For a judicious choice of variables, the on-shell recursion becomes a simple integral equation roughly of the form (see [20] for a closely related formulation)

$$M_n = \int \sum_{k=3}^{n-1} M_k M_{n-k+2}.$$

In [3] our goal was to solve this ‘simple’ integral equation, already for the simplest example of MHV amplitudes. One very nice feature of twistor space is that cancellations which are far from obvious in physical space seem to happen automatically in twistor space.

5. The ‘Tree Formula’ for MHV Graviton Amplitudes [3]

By working in the ‘link representation’ introduced in [19], one can write down and prove a formula for the n -particle MHV graviton amplitude as a sum over all labeled tree diagrams with

vertices labeled $\{1, \dots, n-2\}$ (note one must choose two of the particles to treat as special; a sample diagram for $n=7$ is shown in fig. 2).² The value of a diagram is simply

$$\left[\frac{1}{\langle n-1 \, n \rangle^2} \prod_{a=1}^{n-2} \frac{1}{(\langle a \, n-1 \rangle \langle a \, n \rangle)^2} \right] \prod_{\text{edges } ab} \frac{[ab]}{\langle a \, b \rangle} \langle a \, n-1 \rangle \langle b \, n-1 \rangle \langle a \, n \rangle \langle b \, n \rangle \quad (10)$$

Features of this formula include

- No vestigial reference to any cyclic ordering of the gravitons,
- manifest S_{n-2} symmetry
- without the need for an explicit sum over permutations, and
- manifest $1/z^2$ falloff, term by term.

Finally we note that the formula only has $(n-2)^{n-4}$ terms involving the basic structures $\langle i \, j \rangle \dots [k \, l] \dots$, an enormous savings over BGK-like formulas which have of order $n!^2$ terms.

The tree formula has a particularly nice property of making the soft limits manifest. In 1965 Weinberg showed that graviton amplitudes have a universal soft limit [22] which does not receive quantum corrections to any loop order. In the limit as the momentum $p_1 \rightarrow 0$, the amplitude must behave as

$$\lim_{p_1 \rightarrow 0} \frac{M(1^+, 2^+, \dots, n^-)}{M(2^+, \dots, n^-)} = \sum_{i=2}^{n-2} g(i^+), \quad g(i^+) = \frac{\langle i \, n-1 \rangle \langle i \, n \rangle [1 \, i]}{\langle 1 \, n-1 \rangle \langle 1 \, n \rangle \langle 1 \, i \rangle}. \quad (11)$$

The tree formula beautifully manifests this fact: the only diagrams which survive this limit are those in which vertex “1” is attached to the rest of the diagram via a single line. Taking the limit amputates all these appendages, leaving precisely the diagrams contributing to the $n-1$ graviton amplitude, times the indicated factor.

6. All Tree-Level Graviton Amplitudes [2]

It is actually possible to solve the on-shell recursion for all tree-level graviton amplitudes [2], using two ingredients. The first is the solution for all tree-level SYM amplitudes due to Drummond and Henn [23] which schematically takes the form

$$A^{\text{all}} = A^{\text{MHV}} \sum_{\alpha} R_{\alpha}(\lambda_i, \tilde{\lambda}_i, \eta_i) \quad (12)$$

where the R_{α} are certain specific dual-conformally invariant objects. The second ingredient is the Elvang-Freedman formula for MHV graviton amplitudes, which can be cast into the form

$$M^{\text{MHV}} = \sum_{\mathcal{P}(3, \dots, n)} [A^{\text{MHV}}(1, 2, \dots, n)]^2 G^{\text{MHV}}(\lambda_i, \tilde{\lambda}_i), \quad (13)$$

where G^{MHV} is a particular quantity which we call the ‘gravity dressing factor’.

Inspired by these two results we were able to solve the on-shell recursion for all gravity amplitudes, obtaining a formula which naturally merges eqs. (12) and (13) into the form

$$M^{\text{all}} = \sum_{\mathcal{P}(3, \dots, n)} [A^{\text{MHV}}(1, 2, \dots, n)]^2 \sum_{\alpha} [R_{\alpha}(\lambda_i, \tilde{\lambda}_i, \eta_i)]^2 G(\lambda_i, \tilde{\lambda}_i). \quad (14)$$

² The tree formula can be shown to be equivalent (in a re-arranged form) to a formula presented by Bern, Dixon, Perelstein and Rozowsky using ‘half-soft’ factors [21].

Explicit formulas for some gravity dressing factor G may be found in [2]; they generalize the quantity G^{MHV} appearing in the MHV amplitude eq. (13).

A notable feature of eq. (14) is that the R 's are precisely the same ones that appear in gluon scattering, except they are now literally squared—this can be understood simply as a result of moving up from $\mathcal{N} = 4$ to $\mathcal{N} = 8$ on-shell superspace using the rule

$$\delta^4(\theta) \rightarrow [\delta^4(\theta)]^2 \equiv \delta^8(\theta). \quad (15)$$

Other than this minor modification, all of the super-structure of graviton scattering amplitudes is identical to that of gluon amplitudes; the gravity dressing factors G depend only on the spinor helicity variables $(\lambda_i, \bar{\lambda}_i)$ and not on the supercoordinates η_i .

The above formulas for non-MHV amplitudes, while explicit, are almost certainly not the end of the story since they do not possess any of the nice features of the MHV tree formula. So I am sure that we are still quite far from fully unlocking the structure of general graviton amplitudes, even at tree level!

Acknowledgements

This work was supported in part by the US Department of Energy under contract DE-FG02-91ER40688 and the US National Science Foundation under grants PHY-0638520 and PHY-0643150.

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