

Spiky strings in the $SL(2)$ Bethe Ansatz

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Abstract. We consider spiky strings in the context of the $SL(2)$ Bethe ansatz equations. We obtain an asymmetric distribution of Bethe roots along one cut which determines the all loop anomalous dimension at leading and subleading orders in a large S expansion. At leading order in strong coupling (large λ) the energy of such states is given, in terms of the spin S and the number of spikes n by $E - S = \frac{n\sqrt{\lambda}}{2\pi} \left(\ln \frac{4\pi S}{\sqrt{\lambda}} + \ln \frac{4}{n} \sin \frac{\pi}{n} - 1 \right) + \mathcal{O} \left(\frac{\ln S}{S} \right)$. This result agrees perfectly with the same expansion obtained from the known spiky string classical solution. We also discuss a two cut spiky string Bethe root distribution at one-loop in the $SL(2)$ Bethe ansatz.

1. Introduction

The study of the $SL(2)$ sector of $\mathcal{N} = 4$ SYM gauge theory has lead to a remarkable progress in the understanding of the AdS/CFT correspondence. Much of this progress relates to twist two gauge theory operators of the type $\text{tr}(\Phi D_+^S \Phi)$. On the string side they are described by the folded string solution [1] as can also be verified by an alternative computation in terms of the cusp anomaly [2]. In the field theory an all loop Bethe ansatz was proposed in [3] which can be used to compute their anomalous dimension at all loops in the planar approximation. Computations of quantum corrections to the folded string solution [4] at leading order in large S as well as its generalization to $AdS_3 \times S^1$ [5, 6, 7] were recently used to successfully check such all loop asymptotic Bethe ansatz [8, 9, 10, 11]. The all loop asymptotic Bethe ansatz computation was further extended to the subleading orders in the large S expansion [11, 12, 13] and exact matching with the corresponding string theory computations [14] was found.

Beyond twist two, the $SL(2)$ sector contains operators of higher twist. For example operators of the type

$$\mathcal{O} = \text{tr} \left(D_+^{\frac{S}{n}} \Phi D_+^{\frac{S}{n}} \Phi \dots D_+^{\frac{S}{n}} \Phi \right) \quad (1.1)$$

are described, on the string side, by the spiky string solutions [15]. By analogy with the twist two operators, the full understanding of these operators from the field theory point of view requires the construction of the Bethe ansatz solution that describes them. In this paper we provide a proposal to describe the spiky string solution in the all loop Bethe ansatz.

In [16, 17] and later in [18] such operators were described by a spin chain with a number of sites n being the same as the number of spikes. At leading order in large S the spiky string solution touches



the boundary of AdS_3 . It was shown [19] that at this order the spiky string solution can be mapped to the folded string solution by a conformal transformation, and thus the all loop energy at leading order in large S is given by

$$E - S = \frac{n}{2} f(\lambda) \ln S + \mathcal{O}(S^0, n) \quad (1.2)$$

where $f(\lambda)$ is a universal scaling function known as the cusp anomaly. Here we obtain this result from the field theory side by an all loop Bethe ansatz computation.

An important ingredient in the construction of the Bethe ansatz solution is a set of integers n_k which may be interpreted as the bosonic mode numbers of the waves propagating in the spin chain. The folded string solution in AdS_3 reduces [20] to the folded string solution in flat space in the limit $\frac{S}{\sqrt{\lambda}} \ll 1$. In flat space we can quantize the string exactly in terms of right and left moving waves propagating on the string. The folded string has the same number of right and left moving excitations, i.e. $n_L = n_R$, with the same wave numbers $k_L = k_R = 1$. In AdS_3 , for all values of spin S and coupling λ , it was proposed that in a Bethe ansatz computation the folded string solution is described by two sets of modes, one with $n_k = -1$ and the other with $n_k = 1$. These numbers were used then as the input in the one loop $SL(2)$ Bethe ansatz equations for two cuts (corresponding to $n_k = \pm 1$), and a symmetric Bethe root distribution was found to describe the folded string solution [21]. Similar bosonic quantum numbers were used later [22, 3] to find the corresponding solution to the all loop Bethe ansatz.

In our case, we look at the spiky string solution in flat space, find the left and right wave numbers and use them as an input to find the corresponding Bethe root equation that we then proceed to solve. It turns out that the bosonic quantum numbers that should be used in the Bethe ansatz equations are $n_L = -1$ and $n_R = n - 1$. Using this input we find an asymmetric distribution of Bethe roots spread over one cut along the real axis at one loop in weak coupling and leading order in the large S expansion. We then extend the computation to all loops and further obtain the all loop result also for the subleading order in large S . Remarkably, the result expanded at strong coupling precisely matches the result known from expanding the classical string solution [14], that is, at order $\ln S$ and also S^0 .

We then proceed to consider a two cut solution to the one-loop Bethe ansatz equations. Again we use the “spiky string” bosonic quantum numbers for the integers n_k , i.e. one cut has $n_k = -1$ and the other $n_k = n - 1$ where n is the number of spikes. The solution has four real parameters $d < c < a < b$ which represent the position of the two cuts given by the segments $[d, c]$ and $[b, a]$. The solution obtained allows us to compute the energy E , spin S , R-charge J and number of spikes n in terms of these parameters, allowing to express the energy as a function of the other physical quantities $E = E(S, J, n)$.

2. Spiky strings in flat space

In flat space $ds^2 = -dt^2 + dx^2 + dy^2$ we can quantize the theory exactly, and it turns out that the spiky solutions are just a superpositions of a left and a right moving waves:

$$x = A \cos((n-1)\sigma_+) + A(n-1) \cos(\sigma_-) \quad (2.1)$$

$$y = A \sin((n-1)\sigma_+) + A(n-1) \sin(\sigma_-) \quad (2.2)$$

$$t = 2A(n-1)\tau = A(n-1)(\sigma_+ + \sigma_-) \quad (2.3)$$

where $\sigma_+ = \tau + \sigma$, $\sigma_- = \tau - \sigma$, n is the number of spikes and A is constant that determines the size of the string. Here, (τ, σ) parameterize the world-sheet of a string. The solutions are periodic in σ with period 2π and satisfy the equations of motion in conformal gauge, $(\partial_\tau^2 - \partial_\sigma^2)X^\mu = 0$, as well as the constraints $(\partial_+ X)^2 = (\partial_- X)^2 = 0$. Quantum mechanically the state has $n_R = A^2(n-1)^2$ right moving excitations of wave number $k_R = 1$ and $n_L = A^2(n-1)$ left moving excitations with wave number $k_L = n-1$ (satisfying the level matching condition $n_R k_R = n_L k_L$). All excitations carry one

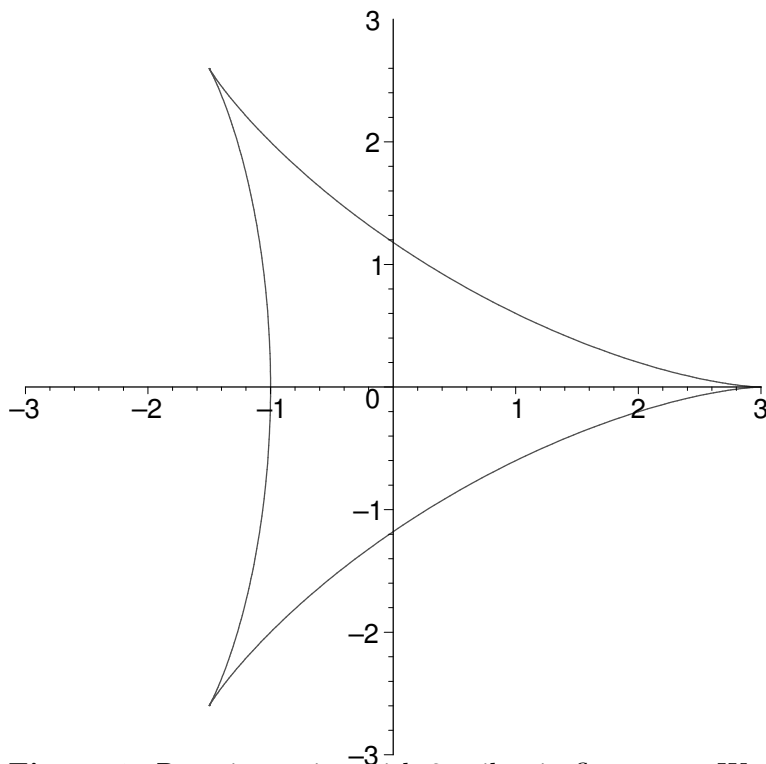


Figure 1. Rotating string with 3 spikes in flat space. We plot (x, y) in eq.(2.3) parametrically as a function of $\sigma = 0 \rightarrow 2\pi$ for $n = 3$, $A = 1$ and $\tau = 0$.

unit of angular momentum and therefore the total angular momentum and energy are given by

$$S = n_L + n_R = A^2 n(n-1), \quad E = \sqrt{2(n_L k_L + n_R k_R)} = 2A(n-1), \quad E = 2\sqrt{\frac{n-1}{n}}S \quad (2.4)$$

which agrees with a classical computation. For $n = 2$ we recover the standard Regge trajectory $E = \sqrt{2S}$ and for $n > 2$ we get a Regge trajectory of modified slope. At fixed time, the shape of the string for $n = 3$ is shown in fig. 1.

3. Spiky strings in the one-loop Bethe Ansatz: 1-cut solution

To obtain the $\ln S$ dependence from the Bethe ansatz at large S we follow the same prescription used in [22]. In the $SL(2)$ sector the Bethe roots are on the real axis and, for large S , they accumulate on various segments or cuts. For the folded string with $J = 2$ and $S \gg 1$, there is only one cut and the root distribution is symmetric. In this case, namely for the lowest twist, there is only one state whereas for $J > 2$ there is more than one state. We consider excited energy states which describe the n -spike solution in the large S limit for a fixed finite J .

We first consider the one loop $SL(2)$ spin chain. The one loop Bethe ansatz equations corresponding to a $XXX_{-\frac{1}{2}}$ nearest neighbor spin chain are

$$\left(\frac{u_k + \frac{i}{2}}{u_k - \frac{i}{2}} \right)^J = \prod_{j \neq k}^S \frac{u_k - u_j - i}{u_k - u_j + i}, \quad k = 1, 2, \dots, S \quad (3.1)$$

Taking the logarithm of (3.1), and further expanding in $\frac{1}{u}$ as appropriate for large S gives [22]

$$\frac{J}{u_k} = 2\pi n_k - 2 \sum_{j=1, j \neq k}^S \frac{1}{u_k - u_j} \quad (3.2)$$

where u_k are the Bethe roots and J is the length of the corresponding spin chain. The total momentum should vanish due to the cyclicity of the trace which implies

$$\prod_{j=1}^S \frac{u_j + \frac{i}{2}}{u_j - \frac{i}{2}} = 1 \quad (3.3)$$

The one loop energy is given by

$$\frac{E - S - J}{2g^2} = \sum_{k=1}^S \frac{1}{u_k^2 + \frac{1}{4}} \quad (3.4)$$

At this point one should solve eq.(3.2) for the real numbers u_k subject to the condition (3.3). When the number of roots S is very large, the roots accumulate in cuts on the real axis and one can conveniently approximate the problem by defining a density of roots. Thus we introduce the root distribution function $\rho_0(u) = \sum_k \delta(u - u_k)$. Furthermore, for large S , the cut grows as S and we can make the approximation that $u \gg 1$. Therefore, at leading order in the large S expansion (3.2), we need to solve the equations

$$\oint du \frac{\rho_0(u)}{u' - u} = \pi n_w(u') \quad (3.5)$$

$$\int du \rho_0(u) = S \quad (3.6)$$

$$\oint du \frac{\rho_0(u)}{u} = 0, \quad (3.7)$$

which are, the Bethe equation, the normalization of the density, and the zero momentum condition. The function $\rho_0(u)$ is non-zero on a finite interval that needs to be determined as part of the solution and that we take to extend from $d < x < a$ with $d < 0$ and $a > 0$.

To solve (3.5) we also need the bosonic numbers n_w . Following the intuition from the folded string at all-loops here we identify n_w with the left and right wave numbers of the corresponding flat space solution, i.e. $n_w = -1$ and $n_w = n - 1$ where n is the number of spikes. More precisely we take the function $n_w(u)$ as

$$n_w(u) = \begin{cases} -1, & d < u < 0 \\ n - 1, & 0 < u < a \end{cases} \quad (3.8)$$

To solve (3.5) we introduce a function $F(w) = \sqrt{w - a}\sqrt{w - d}$ which has a cut on the real axis extending precisely from d to a . Furthermore, right above (below) the cut it is purely imaginary with positive (negative) imaginary part. Using this function we define the resolvent as (see [23] for details)

$$G(w) = \frac{1}{\pi} \int_d^a \frac{F(w)}{F(u + i0^+)} \frac{n_w(u)}{u - w} du \quad (3.9)$$

On the cut we get

$$G(x \pm i\epsilon) = \pm \rho(x) + i n_w(x), \quad x \in C, \quad \epsilon \rightarrow 0^+. \quad (3.10)$$

with $\rho_0(u)$ determined to be

$$\rho_0(u) = \frac{1}{\pi} \oint_d^a \left| \frac{F(u)}{F(u')} \right| \frac{n_w(u')}{u' - u} du' \quad (3.11)$$

It turns out that for (3.11) to be a solution of (3.5) we need to satisfy the consistency condition

$$\text{Res} \left[\frac{G(w)}{w - u}, \infty \right] = -G_0 = 0 \quad (3.12)$$

where G_0 is the leading coefficient of the expansion of $G(w)$ at large w

$$G(w) \simeq G_0 + G_1 \frac{1}{w} + \dots, \quad w \rightarrow \infty \quad (3.13)$$

Explicitly for 1-cut we obtain

$$G_0 = -\frac{1}{\pi} \int_d^a \frac{n_w(u)}{F(u + i0^+)} du = i \frac{n-2}{2} + \frac{in}{\pi} \arcsin \left(\frac{a+d}{a-d} \right) \quad (3.14)$$

$$G_1 = -\frac{a+d}{2} G_0 - \frac{1}{\pi} \int_d^a \frac{u n_w(u)}{F(u + i0^+)} du \quad (3.15)$$

On the other hand, equation (3.6) gives

$$S = \int_d^a \rho_0(u) du = \frac{1}{2} \oint_C G(w) dw = i\pi \text{Res} [G(w), \infty] = -i\pi G_1 \quad (3.16)$$

and eq.(3.7) is

$$0 = \oint_d^a \frac{\rho_0(u)}{u} du = \frac{1}{2} \oint_C \frac{G(w)}{w} dw = i\pi \text{Res} \left[\frac{G(w)}{w}, \infty \right] = i\pi G_0 \quad (3.17)$$

which is satisfied if (3.12) is. In the previous equations we used a contour C that encircles the cut in a clockwise direction. Using (3.14) we then find

$$-\frac{a}{d} = \tan^2 \left(\frac{\pi}{2n} \right) \quad (3.18)$$

Using (3.15) and (3.16) gives

$$S = n\sqrt{-ad} \quad (3.19)$$

We now need to compute the energy which, within the Bethe ansatz is given by

$$\frac{E - S - J}{2g^2} = \int_d^a \frac{\rho_0(u)}{u^2 + \frac{1}{4}} du = \pi \left(G\left(\frac{i}{2}\right) - G\left(-\frac{i}{2}\right) \right) = n \ln \left| \frac{\left(\sqrt{a(\frac{i}{2} - d)} + \sqrt{(a - \frac{i}{2})(-d)} \right)^2}{-\frac{i}{2}(a - d)} \right| \quad (3.20)$$

For fixed n and large S we see from eqs.(3.18) and (3.19) that $a \sim d \sim S \gg 1$ so we can approximate the last equation as

$$\frac{E - S - J}{2g^2} \cong n \ln \left(-\frac{8ad}{a - d} \right), \quad S \gg 1 \quad (3.21)$$

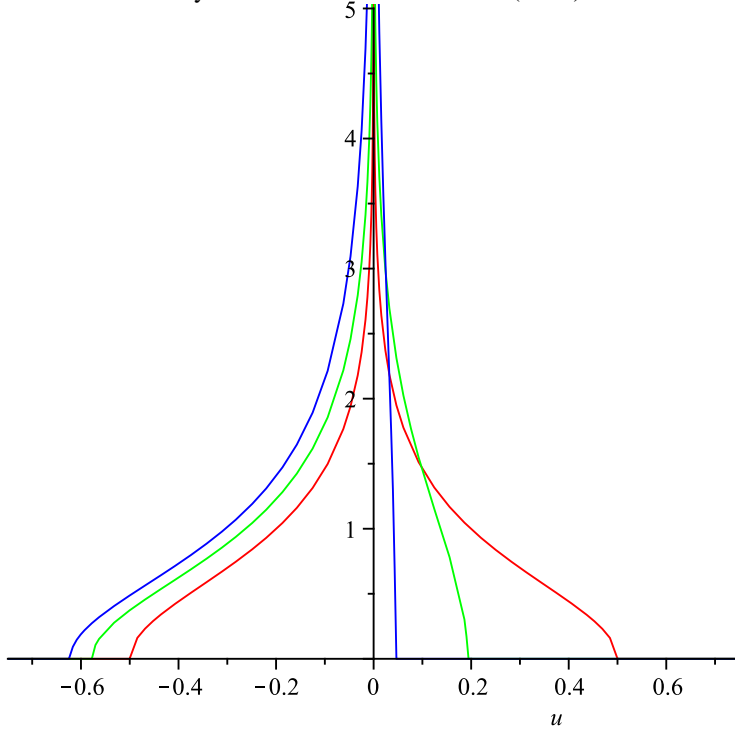


Figure 2. Root density $\rho_0(\bar{u}) = \rho_0(\frac{u}{S})$ for $n = 2$, $n = 3$ and $n = 6$. The larger the n the more skewed the distribution becomes.

Therefore, through eqs.(3.19), (3.21), we have solved the problem by writing E and S in terms of two parameters a and d related by the condition (3.18). Explicitly we can write

$$a = \frac{S}{n} \tan \frac{\pi}{2n}, \quad d = -\frac{S}{n} \cot \frac{\pi}{2n}, \quad \frac{E - S - J}{2g^2} = 2n \ln \left(\frac{4S}{n} \sin \frac{\pi}{n} \right) \quad (3.22)$$

The root distribution is asymmetric. Using eq.(3.11) it can be written as

$$\rho_0(u) = \frac{n}{\pi} \ln \left| \frac{\sqrt{a(u-d)} + \sqrt{-d(a-u)}}{\sqrt{a(u-d)} - \sqrt{-d(a-u)}} \right| \quad (3.23)$$

$$(3.24)$$

Typical plots of the root distribution are shown in Figure 2.

4. Spiky strings in the all loop Bethe ansatz: 1-cut solution

We proceed to find a 1-cut solution to the all-loop Bethe equations. We will follow the procedure used in [22]. The asymptotic all loop Bethe equations in the $SL(2)$ sector read

$$\begin{aligned} 2J \arctan(2u_k) + iJ \ln \frac{1 + g^2/(x_k^-)^2}{1 + g^2/(x_k^+)^2} &= 2\pi \tilde{n}_k - 2 \sum_{\substack{j=-S/2 \\ j \neq 0}}^{S/2} \arctan(u_k - u_j) \\ &+ 2i \sum_{\substack{j=-S/2 \\ j \neq 0}}^{S/2} \ln \frac{1 - g^2/x_k^+ x_j^-}{1 - g^2/x_k^- x_j^+} - 2 \sum_{\substack{j=-S/2 \\ j \neq 0}}^{S/2} \theta(u_k, u_j), \end{aligned} \quad (4.1)$$

where $\theta(u, v)$ denotes the dressing phase and $x^\pm(u)$ are defined through $u \pm \frac{i}{2} = x^\pm(u) + \frac{g^2}{x^\pm(u)}$.

As in the one-loop Bethe ansatz, in the large S limit we introduce a root density $\rho(u)$. The all loop equations can then be written as

$$\begin{aligned} & \frac{J}{u^2 + 1/4} + iJ \frac{d}{du} \ln \frac{1 + g^2/(x^-(u))^2}{1 + g^2/(x^+(u))^2} = 2\pi\rho(u) - 2 \int_a^d \frac{\rho(u')}{(u - u')^2 + 1} \\ & + \int_a^d du' \rho(u') \frac{d}{du} \left(2i \ln \frac{1 - g^2/x^+(u)x^-(u')}{1 - g^2/x^+(u')x^-(u)} - 2\theta(u, u') \right) \end{aligned} \quad (4.2)$$

For the highest excited state J is equal to the number of cusps, n .

To simplify the presentation of the all-loop computation we will omit the dressing phase and only restore its contribution in the end. We can now proceed with the all loop equation by splitting off the one-loop piece of the density, $\rho(u) = \rho_0(u) + 2g^2\tilde{\sigma}(u)$,

$$\begin{aligned} 0 &= 2\pi\tilde{\sigma}(u) - 2 \int_{-\infty}^{\infty} \frac{\tilde{\sigma}(u')}{(u - u')^2 + 1} + \frac{1}{2g^2} \int_d^a du' \rho_0(u') \frac{d}{du} \left(2i \ln \frac{1 - g^2/x^+(u)x^-(u')}{1 - g^2/x^+(u')x^-(u)} \right) \\ &+ \int_{-\infty}^{\infty} du' \tilde{\sigma}(u') \frac{d}{du} \left(2i \ln \frac{1 - g^2/x^+(u)x^-(u')}{1 - g^2/x^+(u')x^-(u)} \right). \end{aligned} \quad (4.3)$$

In the above we have extended the integral boundaries to $\pm\infty$ in the terms containing the higher loop density, $\tilde{\sigma}(u)$.

Using the one-loop resolvent at large S , $G(\pm\frac{i}{2}) = \pm\frac{n}{\pi} \ln S + \dots$, we find

$$\begin{aligned} 0 &= 2\pi\tilde{\sigma}(u) - 2 \int_{-\infty}^{\infty} \frac{\tilde{\sigma}(u')}{(u - u')^2 + 1} + n \ln S \frac{d}{du} \left(\frac{1}{x^+(u)} + \frac{1}{x^-(u)} \right) \\ &+ \int_{-\infty}^{\infty} du' \tilde{\sigma}(u') \frac{d}{du} \left(2i \ln \frac{1 - g^2/x^+(u)x^-(u')}{1 - g^2/x^+(u')x^-(u)} \right). \end{aligned} \quad (4.4)$$

After Fourier transformation¹ and a redefinition of the density,

$$\hat{\sigma}(t) = -2ne^{t/2} \hat{\tilde{\sigma}}(t) \ln S, \quad (4.5)$$

the result is

$$\hat{\sigma}(t) = \frac{t}{e^t - 1} \left(K(2gt, 0) - 4g^2 \int_0^\infty dt' K(2gt, 2gt') \hat{\sigma}(t') \right) \quad (4.6)$$

Interestingly, the contribution from the dressing phase can be included in the kernel K

$$K(t, t') = K_0(t, t') + K_1(t, t') + K_d(t, t') \quad (4.7)$$

where

$$K_0(t, t') = \frac{2}{tt'} \sum_{n=1}^{\infty} (2n-1) J_{2n-1}(t) J_{2n-1}(t'), \quad K_1(t, t') = \frac{2}{tt'} \sum_{n=1}^{\infty} 2n J_{2n}(t) J_{2n}(t') \quad (4.8)$$

$$K_d(t, t') = 8g^2 \int_0^\infty dt'' K_1(t, 2gt'') \frac{t''}{e^{t''} - 1} K_0(2gt'', t'). \quad (4.9)$$

¹ We use the convention $\hat{f}(t) = \int_{-\infty}^{\infty} dw e^{-iwt} f(w)$. Details on the Fourier transforms used here can be found in [22].

Further we find that the energy is given by

$$E - S = 8g^2 n \hat{\sigma}(0) \ln S + \mathcal{O}(S^0). \quad (4.10)$$

which gives

$$E - S = \frac{n}{2} f(g) \ln S + \mathcal{O}(S^0) \quad (4.11)$$

which matches the string theory result at leading order in large S [19].

The results discussed in section 3 and 4 can be extended to the next order in the large S expansion. We obtain the following all loop result

$$E - S = n + f(g) \frac{n}{2} \left(\ln S + \gamma_E + \ln \left(\frac{2}{n} \sin \frac{\pi}{n} \right) \right) + \frac{n}{2} B_2(g) + \dots \quad (4.12)$$

where $B_2(g)$ denotes the virtual scaling function of twist 2 operators. Using the first orders in the expansions at strong coupling [12],

$$f(g) = 4g - \frac{3 \ln 2}{\pi} + \mathcal{O}(1/g) \quad (4.13)$$

$$B_2(g) = (-\gamma_E - \ln g) f(g) - 4g(1 - \ln 2) - \left(1 - \frac{6 \ln 2}{\pi} + \frac{3(\ln 2)^2}{\pi} \right) + \mathcal{O}(1/g) \quad (4.14)$$

we find

$$\begin{aligned} E - S &= 2ng \left(\ln \frac{S}{g} + \ln \left(\frac{4}{n} \sin \frac{\pi}{n} \right) - 1 \right) \\ &+ \frac{n}{2} \left(1 + \frac{6 \ln 2}{\pi} - \frac{3 \ln 2}{\pi} \ln \left(\frac{4}{n} \sin \frac{\pi}{n} \right) - \frac{3 \ln 2}{\pi} \ln \frac{S}{g} \right) + \mathcal{O}(1/g) \end{aligned} \quad (4.15)$$

We see that the leading strong coupling result is in agreement with the known string theory result [14]. For $n = 2$ we recover the result for the folded string [12], this result is in full agreement with the one-loop computation from string theory [14]. For arbitrary n (4.15) is expected to be in agreement with the one-loop result computed from the sigma model [24].

5. Spiky strings in the one-loop Bethe Ansatz: 2-cuts solution

We start again with equation (3.2) but we no longer drop the term proportional to J

$$\oint du' \frac{\rho_0(u')}{u - u'} = \pi n_w(u) - \frac{J}{2u} \equiv \pi N_w(u) \quad (6.1)$$

The condition that the total momentum should vanish, and the normalization of the root density give

$$\int du \rho_0(u) \ln \frac{u + \frac{i}{2}}{u - \frac{i}{2}} = 0, \quad \int du \rho_0(u) = S \quad (6.2)$$

The 1-loop anomalous dimension is

$$\frac{E - S - J}{2g^2} = \int du \frac{\rho_0(u)}{u^2 + \frac{1}{4}} \quad (6.3)$$

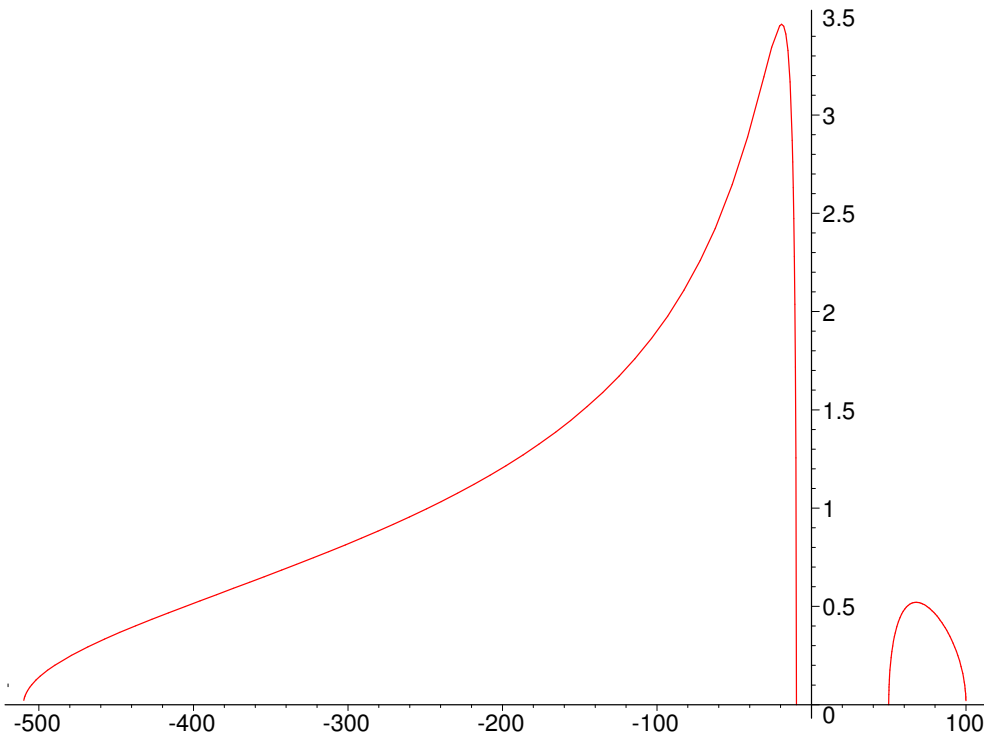


Figure 3. Root density $\rho_0(\bar{u}) = \rho_0(\frac{u}{s})$ for $n = 3$ and $a = 100, b = 50, c = -9.771958775, d = -510.0091708$.

Let us consider two cuts $d \leq c \leq b \leq a$. Extending the bosonic wave number distribution from the previous section we take

$$n_w(u) = \begin{cases} -1, & d < u < c \\ n-1, & b < u < a \end{cases} \quad (6.4)$$

Then we find the solution for the root distribution function is ($r = \frac{(a-b)(c-d)}{(a-c)(b-d)}$)

$$\begin{aligned} \rho_{0L}(u') &= -\frac{2n}{\pi} \sqrt{\frac{(c-u')(b-u')}{(u'-d)(a-u')}} \frac{1}{\sqrt{(a-c)(b-d)}} \left((a-d) \Pi\left[\frac{(b-a)(u'-d)}{(b-d)(u'-a)}, r\right] + (u'-a) K[r] \right) \\ &+ \frac{J}{2\pi^2 u'} \sqrt{|F(u')|} (A_4 - A_5) \end{aligned} \quad (6.5)$$

for the left interval $u' \in [d, c]$, and

$$\begin{aligned} \rho_{0R}(u') &= \frac{2n}{\pi} \sqrt{\frac{(u'-c)(u'-b)}{(u'-d)(a-u')}} \frac{1}{\sqrt{(a-c)(b-d)}} \left((d-a) \Pi\left[\frac{(d-c)(u'-a)}{(a-c)(u'-d)}, r\right] + (u'-d) K[r] \right) \\ &+ \frac{J}{2\pi^2 u'} \sqrt{|F(u')|} (-A_4 + A_5) \end{aligned} \quad (6.6)$$

for the right one $u' \in [b, a]$. We show an example of a 2-cut root distribution in figure 3. The existence

of this solution implies two relationships among the arbitrary parameters a, b, c, d

$$nA_1 + \frac{J}{2\pi}(A_4 - A_5) = 0, \quad A_2 + (n-1)A_3 = 0 \quad (6.7)$$

Normalization and momentum conditions give two additional equations

$$\frac{J}{2\pi}(A_2 - A_3) + A_6 + (n-1)A_7 = S, \quad \sqrt{abcd}[A_4 + (n-1)A_5 + \frac{J}{2\pi}(A_8 - A_9)] = 0 \quad (6.8)$$

A_i are a, b, c, d -dependent elliptic integrals defined in [23]. The four equations can be solved in principle for the four parameters a, b, c, d in terms of S, J, n and thus the one-loop dimension can be obtained as $E = E(n, S, J)$. The 1-cut solution of section 3 can be recovered by taking the limit $b = \epsilon$, $c = -\eta\epsilon$ with $\epsilon \rightarrow 0_+$.

Acknowledgments

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