

Six-dimensional $(2, 0)$ theory on tori

Måns Henningson

Department of Fundamental Physics, Chalmers University of Technology, S-412 96 Göteborg, Sweden

E-mail: mans@chalmers.se

Abstract. The six-dimensional $(2, 0)$ theories are a comparatively new and rather abstract type of quantum theory with important relations both to supersymmetric Yang-Mills theory in lower dimensions and to string- and M -theory in higher dimensions. After a short introduction to these theories, we focus on the case when they are considered on flat tori [1][2]. In particular, we give an example of how their ground state degeneracies can be computed, and also briefly discuss the spectrum of BPS-states. Finally, we comment on the automorphic transformation properties of the partition function of such a theory under the mapping class group of a six-torus.

The maximal dimension of a space(-time) which admits superconformal symmetry is $d = 1+5$. The symmetry algebra is then

$$osp(2, 6|2n) = so(2, 6) \oplus sp(2n) \oplus \text{odd generators}$$

for some $n = 1, 2, \dots$, where the first two terms are the conformal algebra in six-dimensions and the R -symmetry algebra [3].

Indications for the existence of a quantum theory with such symmetry (for $n = 2$) follows by considering type IIB string theory on a $(1 + 9)$ -dimensional space-time with a codimension four singularity of some ADE -type. A self-consistent six-dimensional theory without dynamical gravity on the locus of the singularity then decouples from the bulk theory [4]. These so called $(2, 0)$ theories are highly unique: Apart from their ADE -type, they have no other discrete or continuous parameters.

Some reasons to study the $(2, 0)$ theories:

- They are quite different from other theories we know of, and still rather mysterious. Understanding them is likely to lead to much new mathematics and physics.
- They give a good opportunity to learn about important aspects of string theory without having to deal with quantum gravity.
- They are related to Yang-Mills theory in lower dimensions. In particular, they give a geometric understanding of S -duality of $N = 4$ super Yang-Mills theory in four dimensions.
- The study of $(2, 0)$ theory might be our best way towards a rigorous definition of quantum theory with infinitely many degrees of freedom.

A way to introduce a parameter in a $(2, 0)$ theory is to consider it on a space-time of the form $M^{1,5} = M^{1,4} \times S^1$ with a compact direction of radius R . At longer distances, the effective



theory is then given by maximally supersymmetric Yang-Mills theory on $M^{1,4}$ with coupling constant $g = R^{1/2}$ and action

$$S = \frac{1}{R} \int_{M^{1,4}} \text{Tr} (F \wedge *F + \dots).$$

The gauge group is of the form G/C , where G is simply connected with center subgroup C :

type	G	C
A_{n-1}	$SU(n)$	\mathbb{Z}_n
D_{2k}	$Spin(4k)$	$\mathbb{Z}_2 \times \mathbb{Z}_2$
D_{2k+1}	$Spin(4k+2)$	\mathbb{Z}_4
E_6	E_6	\mathbb{Z}_3
E_7	E_7	\mathbb{Z}_2
E_8	E_8	1.

In this way, $(2,0)$ theory can be seen as providing an ultra-violet completion of the Yang-Mills theory. The negative power of R indicates that $(2,0)$ theory has no Lagrangian description [5].

Let $M^{1,4} = \mathbb{R} \times M^4$, where the first factor denotes time. Quantum states of the Yang-Mills theory on this space are characterized by their magnetic 't Hooft flux

$$m \in H^2(M^4, C)$$

which determines the topological class of the gauge bundle over M^4 , and their electric 't Hooft flux

$$e \in \text{Hom}(H^1(M^4, C), U(1)) \simeq H^3(M^4, C)$$

which determines the transformation properties under 'large' gauge transformations [6]. From the perspective of $(2,0)$ theory on $M^{1,5} = M^{1,4} \times S^1$, we instead have a self-dual 't Hooft flux

$$f = e + m \in H^3(M^4 \times S^1, C).$$

A particularly interesting case is to consider $M^{1,5} = \mathbb{R} \times M^4 \times S^1$ with $M^4 = T^4$ a flat four-torus, since this preserves 16 supersymmetries. We can think of this as Yang-Mills theory on $\mathbb{R} \times T^4$ or as $(2,0)$ theory on $\mathbb{R} \times T^5$ with

$$T^5 = T^4 \times S^1.$$

The quantum states are characterized by their

- self-dual 't Hooft flux $f \in H^3(T^5, C)$
- energy $E \in \mathbb{R}^+$
- spatial momentum $p \in H^1(T^5, \mathbb{R})$
(with the fifth component given by the Yang-Mills instanton number over T^4)
- $sp(4)$ R -symmetry representation R .

Supersymmetry implies the energy bound

$$E \geq |p|.$$

There are three classes of states:

- Vacuum states have $E = p = 0$ and are annihilated by all supercharges.
- BPS states have $E = |p| > 0$ and are annihilated by half of the supercharges.

- non-BPS states have $E > |p|$.

The spectra of vacua and BPS states are invariant under smooth deformations of the geometry of T^5 , and can thus be followed from weak to strong coupling in the Yang-Mills perspective. We will discuss the computation of the spectrum of vacua from the Yang-Mills perspective. (Similar reasoning may be applied to the BPS-states, although we do not yet have any non-trivial checks on the results.)

At weak coupling, vacuum states are localized at orbifold singularities of the moduli space of flat connections over T^4 . The low energy theory is given by maximally supersymmetric matrix quantum mechanics based on the subgroup S of the gauge group G/C left unbroken by the configuration at the singularity. This quantum mechanical model has a number n_S , depending on S , of normalizable ground states. Summing over the orbifold singularities gives the complete spectrum of vacua, which may be decomposed according to their electric and magnetic 't Hooft fluxes e and m .

Covariance under the $SL_4(\mathbb{Z})$ mapping class group of T^4 is manifest in the Yang-Mills formulation, but $(2,0)$ theory indicates covariance under the $SL_5(\mathbb{Z})$ mapping class group of $T^5 = T^4 \times S^1$. This leads to predictions that appear quite non-trivial from the Yang-Mills point of view.

As an example, we consider the D_{2k+1} $(2,0)$ theories. There are 6 $SL_5(\mathbb{Z})$ orbits of self-dual 't Hooft flux $f \in H^3(T^5, \mathbb{Z}_4)$. But a single orbit may be realized in different ways in the corresponding $Spin(4k+2)/\mathbb{Z}_4$ Yang-Mills theory. In this way we get alternative expressions for the generating functions

$$\mathcal{N}_f(q) = \sum_{k=0}^{\infty} N_f(D_{2k+1}) q^{4k+2}$$

of the number $N_f(D_{2k+1})$ of vacua with 't Hooft flux f . (Here q is a formal parameter.) E.g. for a certain $SL_5(\mathbb{Z})$ orbit of f , we have three alternative expressions (modulo q^{4k} -terms) for $\mathcal{N}_f(q)$:

$$\begin{aligned} \mathcal{N}_f(q) &= \frac{1}{8} (P_{\text{even}}^8(q) + P_{\text{odd}}^8(q)) \\ &= \frac{1}{4} P_{\text{even}}^4(q) P_{\text{odd}}^4(q) \\ &= Q^4(q) (P_{\text{odd}}^3(q^2) P_{\text{even}}^9(q^2) + 3 P_{\text{odd}}^5(q^2) P_{\text{even}}^7(q^2) \\ &\quad + 3 P_{\text{odd}}^7(q^2) P_{\text{even}}^5(q^2) + P_{\text{odd}}^9(q^2) P_{\text{even}}^3(q^2)) \\ &= q^6 + 10q^{10} + 67q^{14} + 350q^{18} + \dots \end{aligned}$$

Here

$$\begin{aligned} P_{\text{even}}(q) &= \frac{1}{2} \prod_{k=1}^{\infty} (1 + q^{2k-1}) + \frac{1}{2} \prod_{k=1}^{\infty} (1 - q^{2k-1}) \\ P_{\text{odd}}(q) &= \frac{1}{2} \prod_{k=1}^{\infty} (1 + q^{2k-1}) - \frac{1}{2} \prod_{k=1}^{\infty} (1 - q^{2k-1}) \\ Q(q) &= \prod_{k=1}^{\infty} (1 + q^{2k}). \end{aligned}$$

A promising approach to understand the complete spectrum of states is to consider the partition functions

$$Z_f = \text{Tr}_{\mathcal{H}_f} \exp(-tE + ix \cdot P + iAR),$$

where \mathcal{H}_f is the Hilbert space of states with self-dual 't Hooft flux $f \in H^3(T^5, C)$, and t , x , and A are some formal parameters. After continuation to Euclidean time, these partition functions can be seen as pertaining to a particular decomposition of a flat six-torus $T^6 = S^1 \times T^5$ defined by the T^5 geometry together with t and x , where the first factor denotes the 'time' direction. Twisting by R -symmetry in the spatial directions determines, together with the parameters A , a flat $sp(4)$ connection over this T^6 .

The set of partition functions Z_f for $f \in H^3(T^5, C)$ should have automorphic properties under the $SL_6(\mathbb{Z})$ mapping class group of T^6 . Indeed, these partition function can be regarded as components of an element Z of a certain vector space V . The space V furnishes an irreducible representation of a discrete Heisenberg algebra generated by elements Φ_v for $v \in H^3(T^6, C)$ subject to the relations

$$\Phi_v \Phi_w = \exp \left(2\pi i \int_{T^6} v \wedge w \right) \Phi_w \Phi_v.$$

To construct a basis of V , we break the covariance and decompose

$$H^3(T^6, C) \ni v = f + g \in H^3(T^5, C) \oplus H^2(T^5, C).$$

We then have a basis E_f , $f \in H^3(T^5, C)$ of V . This is such that

$$\begin{aligned} E_f &= \Phi_f E_0 \\ \Phi_g E_0 &= E_0. \end{aligned}$$

The Z_f are the components of Z relative to this basis.

E.g. under a continuous shift of the 'time' cycle of T^6 by an integer linear combination β of the spatial cycles, Z_f is multiplied by an f -dependent phase factor:

$$Z_f \mapsto Z_f \exp \left(\pi i \int_{T^5} f \wedge f[\beta] \right).$$

The Hamiltonian interpretation is that the spatial momentum $p \in H^1(T^5, \mathbb{R})$ obeys the shifted quantization law

$$p - f \cdot f \in H^1(T^5, \mathbb{Z}),$$

where $f \cdot f \in H^1(T^5, \mathbb{R}/\mathbb{Z})$. The best known example of this phenomenon is the possible non-integrality of the fifth component of p (i.e. the instanton number over T^4) for a non-trivial G/C bundle (i.e. of non-trivial magnetic 't Hooft flux m). Another example is the possible non-integrality of the four spatial components of p in situations where both the electric and magnetic 't Hooft fluxes e and m are non-trivial.

Acknowledgments

This research is supported by grants from the Swedish Research Council and the Göran Gustafsson foundation.

References

- [1] M. Henningson, "The low-energy spectrum of (2,0) theory on $T^5 \times \mathbb{R}$," JHEP **0811**, 028 (2008) [arXiv:0809.4156 [hep-th]].
- [2] M. Henningson, "BPS states in (2,0) theory on $\mathbb{R} \times T^5$," JHEP **0903**, 021 (2009) [arXiv:0901.0785 [hep-th]].
- [3] W. Nahm, "Supersymmetries and their representations," Nucl. Phys. B **135**, 149 (1978).
- [4] E. Witten, "Some comments on string dynamics," arXiv:hep-th/9507121.
- [5] E. Witten, "Conformal Field Theory In Four And Six Dimensions," arXiv:0712.0157 [math.RT].
- [6] G. 't Hooft, "A Property Of Electric And Magnetic Flux In Nonabelian Gauge Theories," Nucl. Phys. B **153**, 141 (1979).