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Non-relativistic AdS/CFT and the GCA

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Abstract. We construct a non-relativistic limit of the AdS/CFT conjecture by taking, on the boundary side, a parametric group contraction of the relativistic conformal group. This leads to an algebra with the same number of generators called the Galilean Conformal Algebra (GCA). The GCA is to be contrasted with the more widely studied Schrodinger algebra which has fewer generators. The GCA, interestingly, can be given an infinite dimensional lift for any dimension of spacetime and this infinite algebra contains a Virasoro Kac-Moody sub-algebra. We comment briefly on potential realizations of this algebra in real-life systems. We also propose a somewhat unusual geometric structure for the bulk gravity dual to the GCA. This involves taking a Newton-Cartan like limit of Einstein's equations in anti de Sitter space which singles out an AdS_2 comprising of the time and radial direction. The infinite dimensional GCA arises out of the contraction of the bulk Killing vectors and is identified with the (asymptotic) isometries of this Newton-Cartan structure.

1. Introduction

Even after more than a decade, the AdS-CFT conjecture [1] continues to throw up rich, new avenues of investigation. One such recent direction has been to consider extensions of the conjecture from its original relativistic setting to a non-relativistic context. This opens the door to potential applications of the spirit of gauge-gravity duality to a variety of real-life strongly interacting systems. It was pointed out in [2] that the Schrodinger symmetry group [3, 4, 5], a non-relativistic version of conformal symmetry, is relevant to the study of cold atoms. A gravity dual possessing these symmetries was then proposed in [6, 7].

Instead of the Schrodinger group here we consider an *alternative* non-relativistic realization of conformal symmetry in the context of the AdS/CFT conjecture [8]. This symmetry will be obtained by considering the nonrelativistic group contraction of the relativistic conformal group $SO(d+1,2)$ in $d+1$ space-time dimensions. This Galilean conformal group is to be contrasted with the more studied Schrodinger group from which it differs in some crucial respects as we shall go on to outline.

However, the most interesting feature of the GCA seems to be its natural extension to an infinite dimensional symmetry algebra. This is somewhat analogous to the way in which the finite conformal algebra of $SL(2,C)$ in two dimensions extends to two copies of the Virasoro algebra. Our algebra contains one copy of a Virasoro together with an $SO(d)$ current algebra (on adding the appropriate central extension). We comment on the potential realization of this extended symmetry.

In addition to possible applications to non-relativistic systems, one of the motivations for studying the contracted $SO(d+1,2)$ conformal algebra is to examine the possibility of a new tractable limit of the parent AdS/CFT conjecture. In fact, the BMN limit [9] of the AdS/CFT

conjecture is an instance where, as result of taking a particular scaling limit, one obtains a contraction of the original $SO(4,2) \times SO(6)$ (bosonic) global symmetry. In our case, the non-relativistic contraction is obtained by taking a similar scaling limit on the parent theory. Like in the BMN case, taking this limit would isolate a closed subsector of the full theory. There are, however, some important differences here from a BMN type limit. Normally the BMN type scaling leads to a Penrose limit of the geometry in the vicinity of some null geodesic. These are typically pp-wave like geometries whose isometry is the same as that of the contracted symmetry group on the boundary. The non-relativistic scaling limit that leads to the GCA on the boundary is at first sight more puzzling to implement in the bulk. This is because, under the corresponding scaling, the bulk metric degenerates in the spatial directions x_i. We propose a novel non-metric Newton-Cartan like formulation of the bulk theory, which has a geometry with an AdS_2 base and the spatial R^d fibred over it. As a check of this proposal, we will see that the infinite dimensional GCA symmetries are realized in this bulk geometry as asymptotic isometries. These generators will also be seen to reduce to the generators of the GCA on the boundary.

2. Non-Relativistic Conformal Symmetries

2.1. Schrodinger Symmetry

The Schrodinger symmetry group in $(d+1)$ dimensional spacetime, $Sch(d,1)$ has been studied as a non-relativistic analogue of conformal symmetry. Its name arises from being the group of symmetries of the free Schrodinger wave operator in $(d+1)$ dimensions. In other words, it is generated by those transformations that commute with the operator $S = i\partial_t + \frac{1}{2m}\partial_i^2$. However, this symmetry is also believed to be realized in interacting systems, most recently in cold atoms at criticality.

The symmetry group contains the usual Galilean group (denoted as $G(d,1)$) with its central extension.

$$\begin{array}{rcl} [J_{ij}, J_{rs}] & = & so(d), \quad [J_{ij}, B_r] = -(B_i\delta_{jr} - B_j\delta_{ir}) \\ [J_{ij}, P_r] & = & -(P_i\delta_{jr} - P_j\delta_{ir}), \quad [J_{ij}, H] = 0, \quad [P_i, P_j] = 0 \\ [B_i, B_j] & = & 0, \quad [B_i, P_j] = m\delta_{ij} \quad [H, P_i] = 0, \quad [H, B_i] = -P_i. \end{array} \tag{2.1}$$

Here J_{ij} $(i,j = 1 \ldots d)$ are the usual $SO(d)$ generators of spatial rotations. P_r are the d generators of spatial translations and B_j those of boosts in these directions. Finally H is the generator of time translations. The parameter m is the central extension and has the interpretation as the non-relativistic mass (which also appears in the Schrodinger operator S).

As vector fields on the Galilean spacetime $R^{d,1}$, they have the realization (in the absence of the central term)

$$J_{ij} = -(x_i\partial_j - x_j\partial_i), \quad H = -\partial_t, \quad P_i = \partial_i, \quad B_i = t\partial_i \tag{2.2}$$

In addition to these Galilean generators there are *two* more generators which we will denote by $\tilde{K}, \tilde{D}$. $\tilde{D} = -(2t\partial_t + x_i\partial_i)$ is a dilatation operator, which unlike the relativistic case, scales time and space differently. $(x_i \to \lambda x_i, \quad t \to \lambda^2 t.)$ $\tilde{K}$ acts something like the time component of special conformal transformations. It has the form $\tilde{K} = -(tx_i\partial_i + t^2\partial_t)$ and generates the finite transformations $x_i \to \frac{x_i}{(1+\mu t)}, \quad t \to \frac{t}{(1+\mu t)}$.

These two additional generators have non-zero commutators

$$\begin{array}{rcl} [\tilde{K}, P_i] & = & B_i, \quad [\tilde{K}, B_i] = 0, \quad [\tilde{D}, B_i] = -B_i \\ [\tilde{D}, \tilde{K}] & = & -2\tilde{K}, \quad [\tilde{K}, H] = -\tilde{D}, \quad [\tilde{D}, H] = 2H. \end{array} \tag{2.3}$$

Note that there is no analogue in the Schrodinger algebra of the spatial components K_i of special conformal transformations. Thus we have a smaller group compared to the relativistic conformal group (12 generators +1 central extension in $(3+1)$ dimensions as opposed to 15.).

2.2. Contraction of the Relativistic Conformal Group

We know that the Galilean algebra $G(d,1)$ arises as a contraction of the Poincare algebra $ISO(d,1)$. Physically this comes from taking the non-relativistic scaling

$$t \to \epsilon^r t \qquad x_i \to \epsilon^{r+1} x_i \tag{2.4}$$

with $\epsilon \to 0$. This is equivalent to taking the velocities $v_i \sim \epsilon$ to zero (in units where $c = 1$). For the process of group contraction the parameter r will play no role apart from modifying an over all factor which is unimportant. Hence we will mostly take $r = 0$.

Starting with the expressions for the Poincare generators ($\mu, \nu = 0, 1 \ldots d$)

$$J_{\mu\nu} = -(x_\mu \partial_\nu - x_\nu \partial_\mu) \qquad P_\mu = \partial_\mu, \tag{2.5}$$

the above scaling gives us the Galilean vector field generators of (2.2)

$$\begin{array}{rclll} J_{ij} & = & -(x_i\partial_j - x_j\partial_i) & & P_0 = H = -\partial_t \\ P_i & = & \partial_i & J_{0i} = B_i = t\partial_i. & \end{array} \tag{2.6}$$

They obey the commutation relations (without central extension) of (2.1). To obtain the Galilean Conformal Algebra, we simply extend the scaling (2.4) to the rest of the generators of the conformal group $SO(d+1,2)$. Namely to

$$D = -(x \cdot \partial) \qquad K_\mu = -(2x_\mu (x \cdot \partial) - (x \cdot x)\partial_\mu) \tag{2.7}$$

where D is the relativistic dilatation generator and K_μ are those of special conformal transformations. The non-relativistic scaling in (2.4) now gives (see also [10])

$$D = -(x_i\partial_i + t\partial_t), \;\; K = K_0 = -(2tx_i\partial_i + t^2\partial_t), \;\; K_i = t^2\partial_i. \tag{2.8}$$

Note that the dilatation generator D is the *same* as in the relativistic theory and scales space and time in the same way. Therefore it is different from the dilatation generator $\tilde{D}$ of the Schrodinger group. Similarly, the temporal special conformal generator K in (2.8) is different from $\tilde{K}$. Finally, we now have spatial special conformal transformations K_i which were not present in the Schrodinger algebra.

Since the usual Galilean algebra $G(d,1)$ for the generators (J_{ij}, P_i, H, B_i) is a subalgebra of the GCA, we will not write down their commutators. The other non-trivial commutators of the GCA are [10]

$$\begin{array}{rcl} [K, K_i] & = & 0, \quad [K, B_i] = K_i, \quad [K, P_i] = 2B_i \\ [J_{ij}, K_r] & = & -(K_i\delta_{jr} - K_j\delta_{ir}), \quad [J_{ij}, K] = 0, \quad [J_{ij}, D] = 0 \\ [K_i, K_j] & = & 0, \quad [K_i, B_j] = 0, \quad [K_i, P_j] = 0, \quad [H, K_i] = -2B_i, \\ [D, K_i] & = & -K_i, \quad [D, B_i] = 0, \quad [D, P_i] = P_i, \\ [D, H] & = & H, \quad [H, K] = -2D, \quad [D, K] = -K. \end{array} \tag{2.9}$$

We can compare the relevant commutators in (2.9) with those of (2.3) and we notice that they are different. Thus the Schrodinger algebra and the GCA only share a common Galilean subgroup and are otherwise different. In fact, one can verify using the Jacobi identities for (D, B_i, P_j) that the Galilean central extension in $[B_i, P_j]$ is *not* admissible in the GCA. This is another difference from the Schrodinger algebra, which as mentioned above, does allow for the central extension. Thus in some sense, the GCA is the symmetry of a "massless" (or gapless) nonrelativistic system.

3. The Infinite Dimensional Extended GCA

The most interesting feature of the GCA is that it admits a very natural extension to an infinite dimensional algebra of the Virasoro-Kac-Moody type. To see this we denote

$$L^{(-1,0,1)} = H, D, K, \quad M_i^{(-1,0,1)} = P_i, B_i, K_i. \tag{3.10}$$

The finite dimensional GCA which we had in the previous section can now be recast as

$$\begin{aligned} [L^{(m)}, L^{(n)}] &= (m-n)L^{(m+n)}, \quad [L^{(m)}, M_i^{(n)}] = (m-n)M_i^{(m+n)} \\ [J_{ij}, M_k^{(m)}] &= -(M_i^{(m)}\delta_{jk} - M_j^{(m)}\delta_{ik}), \quad [M_i^{(m)}, M_j^{(n)}] = 0, \quad [J_{ij}, L^{(n)}] = 0. \end{aligned} \tag{3.11}$$

The indices $m, n = 0, \pm 1$. We have made manifest the $SL(2, R)$ subalgebra with the generators $L^{(0)}, L^{(\pm 1)}$. In fact, we can define the vector fields

$$L^{(n)} = -(n+1)t^n x_i \partial_i - t^{n+1}\partial_t, \quad M_i^{(n)} = t^{n+1}\partial_i \tag{3.12}$$

with $n = 0, \pm 1$. These (together with J_{ij}) are then exactly the vector fields in (2.2) and (2.8) which generate the GCA (without central extension).

If we now consider the vector fields of (3.12) for *arbitrary* integer n, and also define

$$J_a^{(n)} \equiv J_{ij}^{(n)} = -t^n(x_i\partial_j - x_j\partial_i) \tag{3.13}$$

then we find that this collection obeys the current algebra

$$\begin{aligned} [L^{(m)}, L^{(n)}] &= (m-n)L^{(m+n)} \qquad [L^{(m)}, J_a^{(n)}] = -nJ_a^{(m+n)} \\ [J_a^{(n)}, J_b^{(m)}] &= f_{abc}J_c^{(n+m)} \qquad [L^{(m)}, M_i^{(n)}] = (m-n)M_i^{(m+n)}. \end{aligned} \tag{3.14}$$

The index a labels the generators of the spatial rotation group $SO(d)$ and f_{abc} are the corresponding structure constants. We see that the vector fields generate a $SO(d)$ Kac-Moody algebra without any central terms. In addition to the Virasoro and current generators we also have the commuting generators $M_i^{(n)}$ which function like generators of a global symmetry. We can, for instance, consistently set these generators to zero. The presence of these generators therefore do not spoil the ability of the Virasoro-Kac-Moody generators to admit the usual central terms in their commutators.

We wish to understand what these symmetries mean. There is a simple interpretation for the generators $M_i^{(n)}, L^{(n)}, J_a^{(n)}$. We know that $M_i^{(-1,0,1)}$ generate uniform spatial translations, velocity boosts and accelerations respectively. From (3.12) one can see that $M_i^{(n)}$ generate arbitrary time dependent (but spatially independent) accelerations. $x_i \to x_i + b_i(t)$. Similarly $J_{ij}^{(n)}$ generate arbitrary time dependent, space-independent rotations: $x_i \to R_{ij}(t)x_j$. These two set of generators together generate the Coriolis group: the biggest group of "isometries" of "flat" Galilean spacetime [15].

The action of generators $L^{(n)}$ can read this off from (3.12): $t \to f(t)$, $x_i \to \frac{df}{dt}x_i$. It amounts to a reparametrisation of the absolute time t. Under this reparametrisation the spatial coordinates x_i act as vectors (on the worldline t). It seems as if this is some kind of "conformal isometry" of the Galilean spacetime, rescaling coordinates by the arbitrary time dependent factor $\frac{df}{dt}$.

Given that the Galilean limit can be obtained by taking a definite scaling limit within a relativistic theory, we expect to see the GCA (and perhaps its extension) as a symmetry of some subsector within every relativistic conformal field theory. For instance, in the best studied case of $\mathcal{N} = 4$ Yang-Mills theory, we ought to be able to isolate a sector with this symmetry.

One clue is the presence of the $SL(2,R)$ symmetry together with the preservation of spatial rotational invariance. One might naively think this should be via some kind of conformal quantum mechanics obtained by considering only the spatially independent modes of the field theory. But this is probably not totally correct for the indirect reasons explained in the next paragraph.

Recently, the nonrelativistic limit of the relativistic conformal hydrodynamics, which describes the small fluctuations from thermal equilibrium, have been studied [11, 12, 13]. One recovers the non-relativistic incompressible Navier-Stokes equation in this limit. The symmetries of this equation were then studied by [12] (see also [13]). One finds that all the generators of the finite GCA are indeed symmetries except for the dilatation operator D .The generator K acts trivially. In particular it has the K_i as symmetries. It is not surprising that the choice of a temperature should break the scaling symmetry of D. The interesting point is that the arbitrary accelerations $M_i^{(n)}$ are also actually a symmetry [14] (generating the so-called the Milne group [15]). Thus we have a part of the extended GCA as a symmetry of the non-relativistic Navier-stokes equation which should presumably describe the hydrodynamics in every nonrelativistic field theory.

Coming back to the Navier-Stokes equation, if the viscosity is set to zero, one gets the incompressible Euler equations In this case one has the entire finite dimensional GCA being a symmetry since D is now also a symmetry. It is the viscous term which breaks the symmetry under equal scaling of space and time. This shows that one can readily realize "gapless" non-relativistic systems in which space and time scale in the same way!

4. The Bulk Dual

Now we wish to turn our attention to the string theory side. Here it should be possible to take a similar scaling limit along the lines of the non-relativistic limit studied in [16, 17, 18]. Below we will only consider features of this scaling limit when the parent bulk theory is well described by gravity. This will already involve some novel features. This has to with the fact that the usual pseudo-Riemannian metric degenerates when one takes a non-relativistic limit. Nevertheless, there is a well defined, albeit somewhat unfamiliar, geometric description of gravity in such a limit [19]. In the (asymptotically) flat space case this is known as the Newton-Cartan theory of gravity which captures Newtonian gravity in a geometric setting. This is a *non-metric* gravitational theory. One can generalise this to the case of a negative cosmological constant as well. A variant of this is what we propose below as the right framework for the gravity dual of systems with the GCA. In the next subsection we will briefly review features of the Newton-Cartan theory and then go onto describe the case with a negative cosmological constant.

4.1. Newton-Cartan Theory of Gravity and its modification in AdS

In the Newton-Cartan description of gravity, the $(d+1)$ dimensional spacetime M has a time function t on it which foliates the spacetime into d dimensional spatial slices. Stated more precisely (see for example [20]): one defines a contravariant tensor $\gamma = \gamma^{\mu\nu}\partial_\mu \otimes \partial_\nu$ $(\mu, \nu = 0 \dots d)$ such there is a time 1-form $\tau = \tau_\mu dx^\mu$ which is orthogonal to γ in the sense that $\gamma^{\mu\nu}\tau_\mu = 0$. The metric γ, which has three positive eigenvalues and one zero eigenvalue, will be the non-dynamical spatial metric on slices orthogonal to the worldlines defined by τ. There is no metric on the spacetime as a whole. In fact, its geometric structure is that of a fibre bundle with a one dimensional base (time) and the d dimensional spatial slices as fibres.

The dynamics is encoded in a torsion free affine connection $\Gamma^\mu_{\nu\lambda}$ on M. We will demand that this connection is compatible with both γ and τ i.e.

$$\nabla_\rho \gamma^{\mu\nu} = 0 \qquad \nabla_\rho \tau_\nu = 0. \tag{4.15}$$

This enables one to define a time function t ("absolute time") since we have $\tau_\mu = \nabla_\mu t$. Unlike the Christoffel connections which are determined by the spacetime metric in Einstein's theory, this Newton-Cartan connection is not fixed by just these conditions. One has to impose some additional relations. Defining $R^{\mu\nu}_{\lambda\sigma} = \gamma^{\nu\alpha}R^{\mu}_{\lambda\alpha\sigma}$, one can define a Newtonian connection as one which obeys the additional condition $R^{\mu\nu}_{\lambda\sigma} = R^{\nu\mu}_{\sigma\lambda}$.

Now, we would like to parametrically carry out the non-relativistic scaling on the bulk AdS_{d+2} which would capture the physics of the nonrelativistic limit in the $(d+1)$ dimensional boundary theory. In the next section we will describe the bulk scaling in more detail. Here we will simply motivate its qualitative features and give the resulting Newton-Cartan like description of the bulk geometry.

We know that the boundary metric degenerates in the nonrelativistic limit with the d spatial directions scaling as $x_i \propto \epsilon$ while $t \propto \epsilon^0$. We expect this feature to be shared by the bulk metric. One expects that the geometry on constant radial sections to have such a scaling. Since the radial direction of the AdS_{d+2} is an additional dimension, we have to fix its scaling. The radial direction is a measure of the energy scales in the boundary theory via the holographic correspondence. We therefore expect it to also scale like time i.e. as ϵ^0. This means that in the bulk the time and radial directions of the metric *both* survive when performing the scaling. Together these constitute an AdS_2 sitting inside the original AdS_{d+2}.

What this implies for the dynamics is that we should have a Newton-Cartan like description but with the special role of time being replaced by an AdS_2. The geometric structure, in analogy with that of the previous section, is that of a fibre bundle with AdS_2 base and the d dimensional spatial slices as fibres.

Accordingly, we will consider a ("spatial") metric $\gamma = \gamma^{\mu\nu}\partial_\mu \otimes \partial_\nu$ $(\mu, \nu = 0 \dots d+1)$ which now has *two* zero eigenvalues corresponding to the time and radial directions. (In a canonical choice of coordinates these directions will correspond to $\mu = 0, d+1$). Mathematically the two null eigenvectors will be taken to span the space of left invariant 1-forms of AdS_2. These will also define the AdS_2 metric $g_{\alpha\beta}$ in the usual way (This is the analogue of the time metric defined in the previous subsection).

We will once again have dynamical, torsion free affine connections $\Gamma^{\mu}_{\nu\lambda}$ which are compatible with both the spatial and AdS_2 metrics

$$\nabla_\rho \gamma^{\mu\nu} = 0 \qquad \nabla_\rho g_{\alpha\beta} = 0. \tag{4.16}$$

There will also be Christoffel connections from the AdS_2 and spatial metrics which will not be dynamical if we do not allow these metrics, specifically $g_{\mu\nu}$, to fluctuate.

4.2. GCA in the Bulk

In this section we will carry out the non-relativistic scaling limit on the AdS_5 piece of the bulk. We will also do this for the $SO(4,2)$ isometries of AdS_5 and obtain the same contracted algebra as before. Consider the metric of AdS_5 in Poincare coordinates

$$ds^2 = \frac{1}{z'^2}(\eta_{\mu\nu}dx^\mu dx^\nu - dz'^2) = \frac{1}{z'^2}(dt'^2 - dz'^2 - dx_i^2) \tag{4.17}$$

The nonrelativistic scaling limit that we will be considering is, as motivated in the previous section $t', z' \to \epsilon^0 t', \epsilon^0 z'$ $x_i \to \epsilon^1 x_i$.

In this limit we see that only the components of the metric in the (t', z') directions survive to give the metric on an AdS_2. The d dimensional spatial slices parametrised by the x_i are fibred over this AdS_2. The Poincare patch has a horizon at $z' = \infty$ and to extend the coordinates beyond this we will choose to follow an infalling null geodesic, in an analogue of the Eddington-Finkelstein coordinates. Therefore define $z = z'$ and $t = t' + z'$. In these coordinates

$$ds^2 = \frac{1}{z^2}(-2dtdz + dt^2) = \frac{dt}{z^2}(dt - 2dz). \tag{4.18}$$

In the infalling Eddington-Finkelstein coordinates, the Killing vectors of AdS_5 read as

$$\begin{aligned} P_i &= \partial_i,\ B_i = (t-z)\partial_i - x_i\partial_t \\ K_i &= (t^2 - 2tz - x_j^2)\partial_i + 2tx_i\partial_t + 2zx_i\partial_z + 2x_ix_j\partial_j \\ J_{ij} &= -(x_i\partial_j - x_j\partial_i),\ D = -t\partial_t - z\partial_z - x_i\partial_i \\ H &= -\partial_t,\ K = -(t^2 + x_i^2)\partial_t - 2z(t-z)\partial_z - 2(t-z)x_i\partial_i \end{aligned} \tag{4.19}$$

Carrying out the scaling mentioned before we obtain the contracted Killing vectors

$$\begin{aligned} P_i &= \partial_i,\ B_i = (t-z)\partial_i, \quad K_i = (t^2 - 2tz)\partial_i, \quad J_{ij} = -(x_i\partial_j - x_j\partial_i) \\ H &= -\partial_t, \quad D = -t\partial_t - z\partial_z - x_i\partial_i, \quad K = -t^2\partial_t - 2(t-z)(z\partial_z + x_i\partial_i) \end{aligned} \tag{4.20}$$

We see that at the boundary $z = 0$ these reduce to the contracted Killing vectors of the relativistic conformal algebra. It can also be checked that these obey the same algebra as (2.1) and (2.9) or equivalently (3.11) after the relabeling of (3.10).

The interpretation of most of the generators is straightforward. We note that the H, K, D are scalars under the spatial $SO(d-1)$ and generate, as before, an $SL(2,R)$. We identify this as the isometry group of the AdS_2 base of our Newton-Cartan theory. We can again define an infinite family of vector fields in the bulk

$$\begin{aligned} M_i^{(m)} &= (t^{m+1} - (m+1)zt^m)\partial_i, \quad J_{ij}^{(n)} = -t^n(x_i\partial_j - x_j\partial_i) \\ L^{(n)} &= -t^{n+1}\partial_t - (n+1)(t^n - nzt^{n-1})(x_i\partial_i + z\partial_z) \end{aligned} \tag{4.21}$$

These vector fields reduce on the boundary to (3.12) and (3.13).

It is rather remarkable that these vector fields also obey the commutation relations of the Virasoro-Kac-Moody algebra, the same as in the boundary theory

$$\begin{aligned} [L^{(m)}, L^{(n)}] &= (m-n)L^{(m+n)} \qquad [L^{(m)}, J_a^{(n)}] = -nJ_a^{(m+n)} \\ [J_a^{(n)}, J_b^{(m)}] &= f_{abc}J_c^{(n+m)} \qquad [L^{(m)}, M_i^{(n)}] = (m-n)M_i^{(m+n)}. \end{aligned} \tag{4.22}$$

How do we interpret all these additional vector fields from the point of view of the bulk? Firstly, notice that the vector fields $M_i^{(n)}$ and $J_a^{(n)}$ only act on x_i. From the viewpoint of the fibre bundle structure, these are simply rotations and translations on the spatial slices dependent on time as well as z. These are the isometries of the spatial metric. They are also trivially isometries of the AdS_2 metric since they do not act on those coordinates. Now we come to the action of the Virasoro generators, $L^{(n)}$. These turn out to be the asymptotic isometries of the AdS_2 in the sense of [21] and ordinary symmetries of the flat fibres. Thus the $L^{(n)}, J_a^{(n)}, M_i^{(N)}$ together generate (asymptotic) isometries of the spatial and AdS_2 metrics γ^{ij} and g_{ab} [8]. Therefore it seems natural to consider the action of these generators on the Newton-Cartan like geometry.

5. Concluding Remarks

We have seen that the nonrelativistic conformal symmetry obtained as a scaling limit of the relativistic conformal symmetry has several novel features which make it a potentially interesting case for further study. The GCA, we have argued, is different from the Schrodinger group which has been studied recently. It also has the advantage of being embedded within the relativistic theory. Hence we ought to have realizations of the GCA in every interacting relativistic conformal field theory. The obvious question is to understand this sector in a particular case such as $\mathcal{N} = 4$ Super Yang-Mills theory and to see whether the infinite dimensional extension can be dynamically realized. We have provided indications why this might be the case generically.

Using this algebra on the boundary, we can construct representations and these turn out to be labelled by the boosts and the dilations [22]. One can, like in the case of the relativistic conformal

theory, use the finite algebra to determine the two and three point correlation functions upto numerical factors. There has also been recent progress in understanding the quantum nature of GCA in two dimensions [23], where the infinite algebra arises out of contraction of linear combinations of the two copies of the Virasoro algebra. This indicates that the GCA might be thought of as a high spin sector of the relativistic theory. There is also a recent realization of the quantum two-dimensional GCA in a cosmological topologically massive three-dimensional gravity theory [24].

A straightforward generalization of the results in the above discussion would be to a supersymmetric extension of the Kac-Moody algebra. These have been investigated in [25, 26].

The bulk description in terms of a Newton-Cartan like geometry is somewhat unfamiliar and it would be good to understand it better. In particular, one needs a precise bulk-boundary dictionary to characterize the duality. At least implicitly this is determined by taking the parametric limit of the relativistic duality.

In the case of the Schrodinger symmetry the dual gravity theory is proposed to live in *two* higher dimensions than the field theory. This also provided the route for embedding the dual geometry in string theory. It is interesting to ask if there is something analogous in our case, whereby the GCA is realized as a standard isometry of a higher dimensional geometry (e.g. $(d+3)$ dimensional for a $(d+1)$ dimensional field theory).

Coming back to the boundary theory, it is interesting to ask whether there are intrinsically non-relativistic realizations of the GCA, perhaps in a real life system. It is encouraging in this context that the incompressible Euler equations concretely realize the GCA, providing an example of a gapless non-relativistic system.

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