

Spaces of Quantum Field Theories

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Abstract. The concept of a “space of quantum field theories” or “theory space” was set out in the 1970’s in work of Wilson, Friedan and others. This structure should play an important role in organizing and classifying QFTs, and in the study of the string landscape, allowing us to say when two theories are connected by finite variations of the couplings or by RG flows, when a sequence of QFTs converges to another QFT, and bounding the amount of information needed to uniquely specify a QFT, enabling us to estimate their number. As yet we do not have any definition of theory space which can be used to make such arguments.

In this talk, we will describe various concepts and tools which should be developed for this purpose, inspired by the analogous mathematical problem of studying the space of Riemannian manifolds. We state two general conjectures about the space of two-dimensional conformal field theories, and we define a distance function on this space, which gives a distance between any pair of theories, whether or not they are connected by varying moduli.

1. Introduction

Quantum field theory (QFT) is a remarkably successful physical framework, describing particle physics, critical phenomena, and certain many-body systems. Superstring theory is also closely based on QFT. A central problem of theoretical physics is to classify the QFTs and understand the relations between them.

After more than fifty years of effort, we have no complete classification, nor is such a thing even on the horizon, except for a few special subclasses of theories. In the broadest terms, we can classify QFT’s by the dimension of space-time, and the presence of continuous symmetry and/or supersymmetry. Within such a class, to get a well posed problem, we should place some upper bound on the total number of degrees of freedom, under some precise definition.

The most basic question of classification would then be: is there any sense in which the total set of QFTs in this class is finite, or countable, or compact, or something we can in any way put definite limits on? To put this a different way, can we make a list, say of Lagrangians for definiteness, in which every QFT is guaranteed a place on the list? In itself this need not be the final answer – an entry on the list might need to satisfy further consistency conditions to be a QFT, or several entries might describe the same QFT. Still, given such a starting point, we could continue by addressing these difficulties to systematically map out the space of QFTs. But, except for a few special classes, we have no such list, and no answer to this question.

Now, given sufficiently restrictive assumptions, it can happen that some finite subset of choices of field content and coupling constants fixes the entire theory, making the classification problem



tractable. For example, if we are interested in four-dimensional theories with 16 supercharges,¹ there are good arguments that the only examples are the supersymmetric Yang-Mills theories. These are determined by a choice of gauge group G (a compact Lie group) and a complexified gauge coupling for each group factor, with certain known equivalences between theories (S-dualities).

Another class of QFTs which has been classified is the “minimal CFTs,” the two-dimensional conformal field theories with central charge $c < 1$. Their classification is a bit more complicated, with an “algebraic” part (the representation theory of the Virasoro algebra) determining the spectrum of primary fields, and a “global” part which determines the list of closed operator algebras satisfying modular invariance (see [16, 7] for an overview of this classification).

While there are a few more results of this type, the general classification problem, even for cases of central interest such as $d = 4$, $N = 1$ supersymmetric theories, seems so distant that it is not much discussed. This is because we have no answer to the question we posed above. More specifically, because we do not sufficiently well understand the relation between the description of a QFT (such as a bare Lagrangian) and the actual observables, it is not known how to reduce the problem to finite terms.

To illustrate this point, let us consider what might appear to be a simple analog to the minimal CFTs, namely the 2d CFT's with $c < 2$. As is well known, the representation theory of the Virasoro algebra does not get us very far here; the number of primary fields grows very quickly with dimension (superpolynomially) and we do not have any way to reduce this large amount of data to a manageable subset. While one can consider larger algebras and rational CFTs, this amounts to looking under a rather well inspected lamppost; we know that there are nonrational CFTs, and that crucial aspects of the problem (such as the fact that CFTs can have moduli) are not respected by this simplification.

There is no reason why we must take an algebraic approach, and one might try to describe the space of $c < 2$ CFTs in other ways. For example, one could start by considering the Lagrangians with two scalar fields and a nontrivial potential. By the c -theorem, we know all such theories will flow to CFTs (often trivial) with $c < 2$. While these could be listed, there is no reason to think this includes all the $c < 2$ theories. An example of a $c < 2$ theory which probably cannot be obtained this way is the direct sum of an Ising model ($c = 1/2$) with two tricritical Ising models ($c = 7/10$).

This particular theory can be obtained using three scalar fields, and it is a reasonable hypothesis that any $c < 2$ theory could be obtained in a similar way, as a flow from a starting point with more scalar fields. If this were the case, we could imagine solving the classification problem by simply listing Lagrangians, but we would need one more ingredient for this list to have an end: an “inverse c -theorem” that tells us that all theories with $c < 2$ can be obtained as RG flows from a set of theories with at most N scalar fields. At present we do not know whether this is so, nor is there any conjecture for the N required for such a claim to hold.

This is an example of a fairly concrete approach to classification, which might or might not work. But one of the points I want to make in this lecture is that we (physicists) have too simple an idea of what a classification should be. I believe we need to broaden this idea to make progress, in ways analogous to those which were developed to address similar problems in mathematics.

Intuitively, one classifies a set of mathematical objects by listing them, and giving an algorithm which, given an object X , determines which one it is. The compact Lie groups are a paradigmatic example in which the objects are classified by discrete structures. As with the minimal CFTs, this classification has an algebraic part, the classification of Lie algebras (say in terms of Dynkin diagrams), and a global part, the classification of discrete subgroups which

¹ In this talk, we only discuss unitary field theories without gravity.

can appear as the fundamental group. In other cases, one has families of objects depending on parameters. One then needs to understand the moduli spaces in which these parameters take values; familiar examples are the moduli spaces of Riemann surfaces.

These are cases in which the classification program was successful. On the other hand, many other superficially analogous mathematical problems have not been solved in such concrete terms, and we will argue that this includes problems quite analogous to classification of QFT. This did not stop the mathematicians, but rather inspired the development of other ways to think about classification problems, more abstract than listing objects or defining moduli spaces. Some are familiar in physics, such as the idea of a topological invariant. Others are not, including most of the ideas of metric geometry. [18]

Before proceeding, let us recall the best existing “classification” of QFTs, which is based on perturbation theory. A large set of QFTs (perhaps all of them) can be defined by quantizing a classical action, say by using a regularized functional integral, computing in perturbation theory, and performing renormalization. For these theories, the classification problem has three parts. First, we need to enumerate actions and the couplings they depend on. We also need to identify equivalences between theories and redundant couplings. We then need to study the renormalization group and decide which operators are relevant and marginal, so that we can take the continuum limit. In perturbation theory, these steps are by now fairly well understood.

Finally, we need to understand the relation between the perturbation theory and the actual observables of the theory. In some cases, this is fairly clear – quantities computed in perturbation theory are asymptotic to the true correlation functions or S-matrix elements as the couplings go to zero. If the physics at finite coupling is qualitatively similar to the limiting free theory, one can essentially regard the coefficients of the perturbative expansion as physical observables. If these agree between two theories, they are the same; if they are “close” in some sense the theories are close, and so on.

Although this map from the action to the observables is somewhat complicated, as physicists we learn how to work with it, and this is our working definition of a “space of QFTs.” Thus the stable particles correspond to fields, certain leading coefficients in scattering amplitudes or correlation functions correspond to couplings, and so on. The resulting moduli space of QFTs is a space with one component for each choice of field content, and with each component parameterized by a finite set of couplings in the Lagrangian.

One problem with this working definition is that it is based on perturbation theory, and there are many QFTs for which this is not a good description. Formally, we might distrust perturbation theory because the bare coupling is large, or because a coupling is relevant and becomes large at low energy. Now in itself this does not prove that the physics is qualitatively different from the picture given by perturbation theory; an example where it is not is the 2d Landau-Ginzburg models (*i.e.*, scalar fields with a potential with isolated critical points).

But in many other cases we know that it is. The perturbative excitations might be confined, so that only composite objects are observable. Even when the perturbative excitations are not confined, there might be solitons which are lighter, and in a physical sense are more fundamental. These thoughts lead to the concept of duality equivalences, and the idea that a large part of theory space might be described by patching together perturbative descriptions which are valid in different regimes.²

Of course, there might be other regions where no perturbative description is valid. Indeed, we do not know whether we yet know about all the types of QFT, or whether we have all of the concepts we will need to describe them. One test is that a local description of theory space must be complete, in the sense that any relevant or marginal perturbation keeps us within the space. One must also ask whether there are disconnected components of theory space, not obtained

² Perhaps the first significant use of this idea was in [22]. It was a common thread throughout the mid-90’s, as illustrated by [23, 9].

by perturbing the theories we know about. To address this question, one's basic definitions probably must place the QFTs within some larger set with a simpler description.

These are attractive questions and ideas, which are part of the general study of the space of QFTs. This concept seems to have come out of statistical field theory; it was discussed in Wilson and Kogut's famous review [24] and many works since. At least as an intuition, it provides a framework for a good deal of work on QFT. Nevertheless, we do not know very much about it, as illustrated by the question of our introduction, or more precise questions such as these:

- Is theory space connected (by paths)? Given two QFTs X and Y , are they in the same component? If so, is the path between them "finite length" or "infinite length" ?
- How "big" is theory space? Can we show that the number of QFTs satisfying some conditions is finite, and estimate the number?
- How many parameters (or measurements) do we need to characterize a QFT? At weak coupling, we can answer this, but more generally?

In trying to make these ideas and questions precise, one runs up against unsolved problems. Certain necessary concepts and tools have not yet been developed, and another goal of the talk is to explain what these are.

In previous talks on this subject [12] I expressed this point with the phrase that "to study the space of X s" (here meaning QFT), one must have a definition of an X . While I would still maintain this, having the definition is not enough; there are other equally important aspects of the problem. Let me list a few, which I will explain below:

- Topology and metric on theory space
- Embedding theorems
- Weak QFT

In explaining these ideas, we will follow the basic analogy between the classification of QFT and the classification of Riemannian manifolds, *i.e.* manifolds with metrics, which first emerged in Friedan's work on renormalization of 2d sigma models [14].

2. Topology and metric on theory space

The next point I want to make in this talk is that it is useful to make precise definitions of when two QFTs are "close." To give the basic idea, suppose we regard a QFT as defined by a list of all of the observables – S-matrix elements, or operator dimensions and o.p.e. coefficients. We then say that two theories are at a distance ϵ in "theory space" if the largest difference between these coefficients is ϵ . Of course, we could use this to define a distance in terms of the other definitions of QFT.

Now, one of the basic problems with regarding a QFT as defined by the list of all of its observables, is that since there are an infinite number of operators with an infinite number of correlation functions, there are an infinite set of observables, and this picture of theory space may seem rather unconstrained and useless. Of course, a first step in the discussion is to reduce this to simpler terms, and for 2d CFT it is known that we can derive all the other correlation functions from the three-point functions of primary fields. However, this is still an infinite amount of data.

This should not stop us. On reflection, one already has a milder version of the same sort of problem when thinking about spaces of ordinary functions, even for a real analytic function of a single real variable. Such a function is characterized by its Taylor expansion around a given point, an infinite set of coefficients. And, indeed, there is no *a priori* upper bound on the number of coefficients one needs to describe such a function. Nevertheless this point does not stop us; we have a good intuition for how to work with functions, and in "normal physics problems" we can

find finite dimensional spaces of functions which are good enough approximations to describe the functions of interest.

Our goal is to develop the analogous intuition here, which tells us what finite dimensional subset of this data is needed to actually determine a particular QFT. Again, there is a simple conceptual answer to this question – if we are perturbing around a given theory, we need to know the couplings for the relevant and marginal operators, a finite subset. The problem is to say this in some way which, unlike the standard discussions, does not depend so drastically on which theory we are perturbing around. To do this, we need to formulate a few basic concepts, such as what it means to “perturb around a given theory,” in a more general way.

Let us be a bit more concrete. Suppose we have a set of QFTs $T(\alpha)$, depending on various discrete or continuous parameters α . Each QFT has a space of local operators $\phi_i(x)$, and we grant that we know the n -point correlation functions for all the theories. The problem is to define a distance between pairs of theories, denoted

$$d(T_1, T_2), \quad (1)$$

which satisfies the axioms of a distance (or metric space³):

$$d(T_1, T_2) \geq 0 \quad (\text{with equality only if } T_1 \cong T_2). \quad (2)$$

$$d(T_1, T_2) = d(T_2, T_1), \quad (3)$$

$$d(T_1, T_2) + d(T_2, T_3) \geq d(T_1, T_3) \quad (\text{the triangle inequality}). \quad (4)$$

One can keep some concrete sets of QFTs in mind which to test a definition. The first is the two-dimensional sigma models with target space a torus of dimension d , a flat metric g_{ij} , and constant B field. These QFTs are CFTs and their moduli space is well known to be

$$SO(d, d; \mathbb{Z}) \backslash SO(d, d; \mathbb{R}) / SO(d; \mathbb{R}) \times SO(d; \mathbb{R}) \quad (5)$$

The second is the $c = 1$ CFTs, which were classified in [17]. The moduli space of such theories consists of several disconnected components. One is made up of two half-lines parameterizing the radius of the target spaces S^1 and S^1/\mathbb{Z}_2 , with the endpoint of the second half-line identified with a particular point on the first half-line. There are also three isolated “exceptional” theories which can be defined as quotients of the $SU(2)_1$ WZW model by the E -type discrete subgroups of $SU(2)$.

2.1. Zamolodchikov distance

There is a standard Riemannian metric on any moduli space on CFTs, the Zamolodchikov metric [26, 19]. It is defined as follows: a tangent vector t to the space of theories at a theory T corresponds to a marginal local operator O_t in the theory; the length of a tangent vector t is then

$$\|t\|_{Zam}^2 \equiv \langle O_t(0)O_t(1) \rangle_T. \quad (6)$$

This corresponds to the metric on field space in a space-time effective action for string theory compactified on T .

A Riemannian metric can be used to define a standard distance function – the distance between T_1 and T_2 is the length of the shortest path connecting them. In the special case of the torus target spaces, the Zamolodchikov metric is homogeneous, so any pair of theories T_1 and T_2 is connected by a unique shortest geodesic which can easily be found explicitly. We then take

³ In parts of the mathematics literature, such a distance function is referred to as a metric; the metric tensor familiar from general relativity is referred to as a Riemannian metric.

$d(T_1, T_2)$ to be the length of this geodesic; familiar geometric arguments show that this satisfies the stated axioms.

So far all this will probably be familiar, and the reader may be expecting us to propose the straightforward generalization of this. Any moduli space of CFTs will have a Zamolodchikov metric, and we can take the distance $d(T_1, T_2)$ to be the length of the shortest path between them. Again, this will satisfy the axioms.

While this metric is certainly very important in CFT and string theory, if we consider it as a candidate distance to use in defining theory space, it is not really what we want. The main problem with it is simply that it does not define a distance between every pair of CFTs (much less QFTs). For example, the $c < 1$ minimal models have no moduli and are thus isolated points in theory space. Consider the case $c = 4/5$; there are two unitary partition functions, one diagonal and the other that of the three-state Potts model. These are not connected within the space of CFTs, so they have no distance in this sense. But we do not want to say they are at infinite distance, as they can both be obtained as limits of spin systems related by simply varying bare couplings. Rather, a definition of the distance between them must be based on other concepts.

Disconnected components of the space of CFTs are common, such as the exceptional $c = 1$ theories. It is reasonable to expect many more examples, and our definition of theory space must be able to handle them.

2.2. Distance between CFTs

Let us propose a different definition of distance which can be applied to any pair of two-dimensional CFTs with the same central charge c . As explained in the founding paper [5], using the state-operator correspondence and the operator product expansion, all n -point functions can be expressed in terms of the spectrum of operator dimensions (h_i, \bar{h}_i) , and the 3-point functions

$$C_{ijk} = \langle \phi_i(z_1)\phi_j(z_2)\phi_k(z_3) \rangle \quad (7)$$

for some fixed z_1, z_2, z_3 , or equivalently the o.p.e. coefficients.⁴ Thus, the data we need to describe a CFT is this (admittedly infinite) list of numbers $(h_i, \bar{h}_i, C_{ijk})$. If we can define a distance purely in terms of this data, it will apply to any pair of CFTs.

Let us regard this data for a theory T as defining a vector $V(T)$ in the space of infinite sequences of real numbers; then we could define the distance between a pair of theories T_1 and T_2 as the distance between the corresponding vectors. Furthermore, since the space of sequences of real numbers is a linear space, we can define this distance in terms of a norm, so

$$d(T_1, T_2) = \|V(T_1) - V(T_2)\|. \quad (8)$$

A standard class of such norms are the ℓ_p norms,

$$\|V\|_p = \left(\sum_i |v_i|^p \right)^{1/p}, \quad (9)$$

where $p \geq 1$ is a real number. One can also take the limit $p \rightarrow \infty$ to get

$$\|V\|_\infty = \max_i |v_i|. \quad (10)$$

⁴ To simplify the discussion here, we normalize all the operators to have unit two-point function. This is actually not the best choice when considering a general family of theories, as it can lead to spurious divergences in the o.p.e. coefficients, but we will discuss this point elsewhere.

Thus, by choosing some p , we get a distance between T_1 and T_2 .

We will also need to choose a normalization convention for the vectors V . While at first it might seem natural to take $\|V\| = 1$, with this choice the maximum distance between any pair of points will be 2, which is not what we want. Later we will fix this choice by other considerations.

This style of definition has many disadvantages compared to the Zamolodchikov metric. It is less natural – we have to choose p , and we are combining the various CFT data in a somewhat arbitrary way. At first sight, it does not correspond to any observable in string theory. On the other hand, it has the great advantage that it defines a distance between **every** pair of CFTs, since every CFT corresponds to a vector $V(T)$.

2.3. Symmetries and dualities

Before discussing the appropriate choice of p , we need to deal with some problems with the definition as stated. One evident problem is that the vector $V(T)$ is not uniquely defined by the CFT, since it depends on a choice of basis. We might try to state a prescription for ordering the basis, such as taking the lowest dimension operators first. However, if there are operator degeneracies, and especially if the CFT has symmetries, such prescriptions will not lead to a unique choice of V .

A better way to deal with this problem is as follows. We consider the Hilbert spaces for the two theories \mathcal{H}_1 and \mathcal{H}_2 , and the set of unitary transformations $U : \mathcal{H}_1 \rightarrow \mathcal{H}_2$. Any such unitary transformation determines a corresponding linear transformation on the Hamiltonian $L_0 + \bar{L}_0$, the o.p.e. coefficients C_{ijk} , and on their matrix elements in an explicit basis. Thus, it defines an action on the space of linear sequences $V(T)$. Using this, we take the previous definition and minimize over all possible choices of U ,⁵

$$d(T_1, T_2) = \min_U \|U(V(T_1)) - V(T_2)\|. \quad (11)$$

While this obviously satisfies the other axioms, we need to check the triangle inequality. This can be seen by writing Eq. (4) as

$$\|U_{12}(V(T_1)) - V(T_2)\| + \|U_{32}(V(T_3)) - V(T_2)\| \geq \min_{U'} \|U' \cdot U_{12}(V(T_1)) - U_{32}(V(T_3))\|. \quad (12)$$

where U_{12} and U_{32} are the choices minimizing Eq. (11). Since the U 's all come from unitaries acting on \mathcal{H} , minimizing over U' in this expression is the same as minimizing over U in the expression Eq. (11) for $d(T_1, T_3)$. Now, already for $U' = 1$, this inequality is true, since it is just the ordinary triangle inequality. But the additional step of minimizing over U' can only decrease the right hand side, respecting the inequality.

In practice, if T_1 and T_2 are not very close, it may be difficult to find the minimizing U , and in this sense the definition is nonconstructive. Nevertheless it is a standard mathematical definition (used to define the Gromov-Hausdorff distance between manifolds) and it has a number of advantages. Most importantly, it takes account of symmetry and duality relations between theories. Suppose the theories T_1 and T_2 are T-dual; then there will exist a unitary transformation U such that the distance $d(T_1, T_2) = 0$, as it should be.

2.4. Problems with high dimension operators

While this definition is better than the first one, it still does not work. Consider the example of a free boson with radius R ; one set of operators in the two theories are the winding operators

$$V_n =: \exp in(\phi_L - \phi_R) : , \quad n \in \mathbb{Z} \quad (13)$$

⁵ Note that U is not a general unitary matrix acting on the space of vectors V (in which case the minimum would be determined by the lengths $\|V(T_1)\|$ and $\|V(T_2)\|$), since it is constructed as a tensor product of unitaries acting on \mathcal{H} .

with dimension $h = n^2 R^2$. Suppose we consider a pair of theories related by a small variation of R . Clearly the operator V_n in the first theory will correspond to V_n in the second, so we will have

$$h^{(R_1)} - h^{(R_2)} = n^2 (R_1^2 - R_2^2). \quad (14)$$

Thus, for any nonzero $R_1 - R_2$, as n becomes large this difference will become arbitrarily large, and the distance defined this way (for any p) will be infinite.

Looking at other examples, this problem is very general. For any definition of this type to make sense, the effect of varying high dimension operators and their o.p.e.'s must be suppressed.

Conversely, if there were to exist pairs of CFTs which agreed in all the o.p.e.'s of low dimension operators, and only differed for operators with large h , we would not be able to define a space of CFTs. Intuitively, we do not expect this, because the structure of the theory is determined by the relevant and marginal operators, but we do not know of any precise claims to this effect.

One step towards making this precise would be to demonstrate the following

Conjecture 1 *There exists an H , growing linearly with c , such that if two CFT's agree in their spectrum and o.p.e. coefficients for all operators with $h \leq H$ and $\bar{h} \leq H$, they must be the same CFT.*

Although the intuition based on relevant and marginal operators suggests $H = d$, this need not be the case. A related question which comes up in gauge-gravity duality [25], is to find theories with a large gap Δ_1 in the low lying spectrum of primary fields. Clearly, if there exist two such theories, H must be at or below the gap. Simple arguments show that, for a chiral CFT, the gap cannot be larger than $\Delta_1 = c/24 + 1$, and this bound is saturated for $c = 24$. Although it is not known whether such "extremal CFTs" exist for $c > 24$, from works such as [15] it seems plausible that Δ_1 grows linearly with c . This suggests the growth of H proposed in the conjecture.

While some property like this seems necessary for a theory space as we are envisioning it to exist at all, in itself it would not answer our previous questions. For example, it would not directly imply any bound on the number of such operators in a given CFT. Nevertheless we believe this would be an important claim about CFT.

2.5. Convergence and relation to comparing partition functions

To fix the problem we discussed in the previous subsection, we can put in a convergence factor which suppresses the contributions of high dimension operators. Thus, instead of Eq. (8), we use

$$d(T_1, T_2) = \left(\sum_i |e^{-t2\Delta_i^{(1)}} \Delta_i^{(1)} - e^{-t2\Delta_i^{(2)}} \Delta_i^{(2)}|^p + \sum_{i,j,k} |e^{-t(\Delta_i^{(1)} + \Delta_j^{(1)} + \Delta_k^{(1)})} C_{ijk}^{(1)} - e^{-t(\Delta_i^{(2)} + \Delta_j^{(2)} + \Delta_k^{(2)})} C_{ijk}^{(2)}|^p \right)^{1/p} \quad (15)$$

where $\Delta_i = h_i + \bar{h}_i$, and with some fixed $t > 0$. This particular choice of convergence factor is made to realize the axioms Eq. (3) and Eq. (4), and can be motivated in terms of the world-sheet origin of the data h_i and C_{ijk} . For example, C_{ijk} corresponds to an amplitude on a sphere with three punctures, and the regulated version is obtained by replacing each puncture with a boundary of length $2\pi t$ (obtaining the "pair of pants" diagram). Defining the operator

$$H \equiv \sum_a L_0^{(a)} + \bar{L}_0^{(a)} \quad (16)$$

where the sum over a represents a sum over the boundaries, we can write this as

$$d(T_1, T_2) = \min_U \|e^{-tH_1}UV(T_1) - e^{-tH_2}V(T_2)\|, \quad (17)$$

where we reintroduced the unitary transformation of Eq. (11).

This type of definition could also be expressed in terms of differences between the correlation functions of the two CFTs. By expanding in a sum over intermediate states, we can write a four-point function as

$$\langle \phi_1(x_1)\phi_2(x_2)\phi_3(x_3)\phi_4(x_4) \rangle = f(x) \sum_i C_{12}^i C_{34i} e^{-th_i - \bar{t}h_i}, \quad (18)$$

where (t, \bar{t}) are simple functions of the x_a (essentially the logarithms of cross-ratios), as is $f(x)$. At fixed t one gets a regulated sum of o.p.e. coefficients of the form we want, and clearly one could use the t dependence to extract the coefficients and construct an expression like Eq. (15).

This still leaves the need to minimize over unitaries U . One could reduce these issues by, instead of comparing individual correlation functions and then adding the results, taking a physical observable which already combines all the o.p.e. coefficients. Consider the genus two partition function; by thinking of this as obtained by attaching two “pairs of pants” one sees that it has an expansion of the form

$$Z_{g=2}(T; t_1, t_2, t_3) = \sum_{ijk} |C_{ijk}|^2 e^{-t_1\Delta_i - t_2\Delta_j - t_3\Delta_k} \quad (19)$$

and could also be used for this purpose. We will return to this idea shortly.

2.6. More physical definitions

Some of these mathematical prescriptions can be translated into more physical terms. A prototype for this is to consider boundary conditions in a single CFT. Given a pair of boundary conditions B_1 and B_2 , a natural way to define the distance between them is in terms of the lowest dimension open string operator between them [10, 11, 20]. Given the partition function Z_{12} for such open strings, we define

$$d^2(B_1, B_2) = \frac{c}{24} - \lim_{t \rightarrow \infty} \log Z_{12}(t) \quad (20)$$

in conventions in which an operator of dimension h contributes $\exp(c/24 - h)t$ to Z . The relation $d^2 = h$ is motivated by the standard formula for the dimension of a winding state in the large volume limit.

This will be zero if $B_1 \cong B_2$, since the identity operator will contribute. Naively it can also be zero if $B_1 \neq B_2$ but the branes intersect, but in a unitary CFT the only operator of dimension zero is the identity operator, which cannot exist in this Hilbert space. In fact there will be additional “twist” contributions to the ground state energy which raise the dimension. While this gives us two of the axioms, we do not know a proof of the triangle inequality for this definition (away from the large volume limit).

Let us now consider a distance between a pair of $d = 2$ CFTs. The direct analog of the preceding discussion would involve a hypothetical $d = 3$ theory which admitted $d = 2$ CFTs as boundary conditions. However another option, which could be applied to any pair of CFTs T_1 and T_2 , would be to make some physical construction in the tensor product theory $T_1 \otimes T_2$.

We then need a physical analog of the unitary operator U which relates the two Hilbert spaces. The simplest thing to consider would be the boundary states in the product theory.

Equivalently, we can consider a defect line $L(U)$ which separates a region of the world-sheet carrying theory T_1 from one carrying theory T_2 [4]. Given any world-sheet definition in which the two theories are related by perturbations and flows, a precise definition of defect line can be made, as discussed in many works such as [6].

If $T_1 \cong T_2$, we can find a “topological boundary state” which can be freely moved without changing the energy; in this sense it has zero tension. This suggests that we define $d(T_1, T_2)$ in terms of the tension of the defect, minimized over all such defects.

Let us now return to Eq. (17). For the special case $p = 2$, we can write the norm $\|V\|$ as the square root of an inner product (V, V) . Now, making the world-sheet identifications of the previous subsection, the inner product on the CFT Hilbert space \mathcal{H} , gives us an inner product on the space of coefficients V (for example, C_{ijk} lives in the dual of $\mathcal{H} \otimes \mathcal{H} \otimes \mathcal{H}$, etc.). Thus, we can rewrite the regulated difference between the o.p.e. coefficients appearing in Eq. (17) as

$$\begin{aligned} d_C(T_1, T_2)^2 &= \min_U (e^{-tH_1}UV(T_1) - e^{-tH_2}V(T_2), e^{-tH_1}UV(T_1) - e^{-tH_2}V(T_2)) \quad (21) \\ &= \|e^{-tH_1}V(T_1)\|^2 + \|e^{-tH_2}V(T_2)\|^2 - 2 \max_U (e^{-tH_1}UV(T_1), e^{-tH_2}V(T_2)) \\ &= Z_{g=2}(T_1, 2t, 2t, 2t) + Z_{g=2}(T_2, 2t, 2t, 2t) - 2 \max_U Z_{g=2}(T_1|L(U)|T_2; 2t, 2t, 2t) \end{aligned}$$

where the last term

$$Z_{g=2}(T_1|L(U)|T_2, 2t, 2t, 2t) = (e^{-tH_1}UV(T_1), e^{-tH_2}V(T_2)) \quad (22)$$

is defined by these equations. Now, the point is that this term also has a world-sheet interpretation. Since the unitary U maps \mathcal{H}_1 to \mathcal{H}_2 , we realize it as a defect line $L(U)$, separating two regions of the world-sheet carrying the two theories. Thus it is the partition function on a genus 2 Riemann surface, built by sewing together two pairs of pants, but with a the common defect line $L(U)$ along all three components of the junction.

A similar discussion shows that the regulated difference between the operator dimensions in the two theories can be written as the d/dt derivative (or equivalently an insertion of $L_0 + \bar{L}_0$) of a difference of genus one amplitudes,

$$d_h(T_1, T_2)^2 = -\frac{d}{dt} \left(Z_{g=1}(T_1, 2t) + Z_{g=1}(T_2, 2t) - 2 \max_U Z_{g=1}(T_1|L(U)|T_2; 2t) \right). \quad (23)$$

Finally, the squared distance $d^2(T_1, T_2)$ is defined as the sum of Eq. (21) and Eq. (23),

$$d(T_1, T_2)^2 = \min_U c_h(t)d_h(T_1, T_2)^2 + c_C(t)d_C(T_1, T_2)^2 \quad (24)$$

with the choice of defect line $L(U)$ which minimizes the total distance (so the argument around Eq. (12) will work) and where $c_h(t)$ and $c_C(t)$ are normalization factors chosen to cancel the leading divergences of the partition functions (for example, $c_h(t) \sim \exp ct/6$). By the previous formal arguments, this will satisfy the triangle inequality as well as the other axioms. Thus we have a physical interpretation of the distance Eq. (17) for the particular case $p = 2$.

There are some evident variations on this one could consider. For example, one could add higher genus terms, or even integrate over moduli space (in the critical c or with some suitable weight) to make this look more like a string theory observable. There are also a number of questions which we hope to clarify in subsequent work. One is whether to require that the defect lines be local on the world-sheet. If so, then for Eq. (17) and Eq. (24) to be the same, we would need to show that the minimizing U is local.

With some refinements, such as specifying the specific metric on the surfaces, the definition Eq. (24) might make sense for general 2d QFTs as well. This point is also under study.

2.7. Topology and boundedness

One role for a definition of distance between CFTs or QFTs is as a technical or mathematical tool, used to establish properties like the convergence of an approximate definition. We will discuss this idea in §4. But there are also interesting physical questions about it.

One property which is reasonable to ask, and which we would conjecture holds for the definition Eq. (17) or its physical counterpart Eq. (24), is that it is uniform with respect to the Zamolodchikov distance. In other words,

- A sequence of CFTs which converges in the Zamolodchikov metric, also converges in the metric Eq. (24). Similarly, if a sequence converges in Eq. (24), and all pairs of CFTs in the sequence have well-defined Zamolodchikov distances, it will converge there as well.
- One can also talk about an unbounded sequence, meaning a sequence T_i for which the distances $d(T, T_n)$ to some reference theory T go to infinity for large n . This notion should also be the same.

Because the definition of Eq. (24) is essentially linear, questions of convergence and especially of boundedness are likely to be much simpler to study.

A more specific relation to the Zamolodchikov metric can be found by computing derivatives of Eq. (24) with respect to the moduli. It seems plausible that, with a suitable choice of the coefficients $c(t)$ in Eq. (24), this distance will be equal to or less than the Zamolodchikov distance. These points are being checked and explicit examples worked out [13].

The boundedness property allows us to restate a conjecture of Kontsevich and Soibelman (as cited in [1]). Loosely stated, it is that the only infinite distance limits in the space of CFTs are the large volume limits. More precisely, it is that a family of CFTs with fixed c and a lower bound on the nonzero operator dimensions h_i , is totally bounded (or precompact). Besides determining the infinite distance limits, this implies that the space can be completed, meaning that the limit of any convergent sequence of CFTs can be regarded as defining a point in a larger completion of the space. There are many examples of such limit points obtained by shrinking volumes of cycles to zero, whose interpretation in string theory involves nonperturbative effects such as new massless states, instanton corrections, or both. Such limit points need not satisfy all of the axioms of CFT, and it would be interesting to have a general theory of them.

As a final comment, although we arrived at a nice definition with a concrete world-sheet interpretation, in mathematics one often uses more than one definition of distance and/or norm, even in the same problem, as each can have its own advantages with respect to the other structures in the problem.

3. Embedding theorems

Any approach based on observables such as the above faces the problem that the consistency conditions for observables to come from a QFT are very complicated. On the other hand, the consistency conditions on a Lagrangian which can be used to define a QFT by a functional integral are fairly simple. This suggests that we base the classification on the Lagrangian approach, however we do not know whether all QFTs can be obtained this way.

A very direct way to address this question would be to give a prescription which, given some subset of the observables of a QFT, produces a Lagrangian which is guaranteed to reproduce the original QFT. While at first this may sound implausible, what I have in mind is a prescription which is highly redundant, and perhaps only useful to make this conceptual point. For example, we might imagine a prescription in which every operator up to some very high dimension is represented by its own field in the Lagrangian. Let us consider the Landau-Ginzburg theories with a single real field ϕ and the Lagrangian

$$\mathcal{L} = (\partial\phi)^2 - \sum_{i=1}^n t_i \phi^i, \quad (25)$$

then we would introduce operators $O_k =: \phi^k$ up to $k = n - 2$, and possibly further operators involving derivatives of ϕ . We would then introduce couplings which enforce the relations between these fields, such as Lagrange multipliers

$$\lambda(O_2 - C_{112}O_1^2) \quad (26)$$

where C_{112} is an o.p.e. coefficient, and so on. The idea would be to have so many couplings that we could tune as many dimensions and o.p.e. coefficients as we needed to reproduce the desired QFT.

Is this a reasonable goal? Even if it is possible, it might turn out to have the problem we mentioned in the introduction, that even after putting some upper bound on the number of degrees of freedom of our QFT, we might need an arbitrarily large number of fields in order to reproduce it. If so, this approach would probably not be worth the trouble to develop.

Restricting attention to 2d CFT, we are thus led to conjecture the “inverse c -theorem” of the introduction,

Conjecture 2 *All 2d CFTs of a given central charge c can be obtained as flows from the theory of N free bosons, for some definite N growing as the square of the central charge c .*

What reason is there to believe this, and what should we expect N to be? There are many directly analogous questions in mathematics which can guide us.

Suppose we want to classify the manifolds of a given dimension D . Might it be that all such manifolds can be obtained as submanifolds of \mathbb{R}^N , the Euclidean N -dimensional space? More explicitly, we are asking for an embedding, a function $X : M \hookrightarrow \mathbb{R}^N$ such that $X(p) = X(q)$ implies $p = q$, and the Jacobian $\partial_i X^I$ has maximal rank D .

This question was asked and answered early on; for smooth manifolds this is the Whitney Embedding Theorem, and one needs $N = 2D$. Thus, if all CFTs could be obtained as sigma models with target spaces of dimension equal to the central charge, and we did not need to tune the metric or couplings, the conjecture would hold with $N = 2c$.

This is of course a drastic oversimplification of the real situation. A closer analog might be to ask that we can reproduce the relevant and marginal couplings of the sigma model, meaning the potential, metric and B -field via the embedding. Denoting these as (V, g_{ij}, B_{ij}) , the mathematical question is to find a function f on some \mathbb{R}^N such that $f = 0$ on an embedding $X : M \hookrightarrow \mathbb{R}^N$ and is positive elsewhere, so that

$$g_{ij} = \partial_i X^I \partial_j X^J \delta_{IJ}; \quad B_{ij} = \partial_i X^I \partial_j X^J \omega_{IJ}$$

on M . The total potential would then be $Cf + \tilde{V}$, where \tilde{V} is a function on \mathbb{R}^N which restricts to V on M , and C is some large constant.

Looking into the mathematics, finding f and \tilde{V} is not hard.⁶ To realize the metric as the restriction of the Euclidean metric on \mathbb{R}^N , and the B -field as the pullback of a constant two-form, since together these have D^2 coefficients, one might think that it would be possible given D^2 adjustable functions, so that $N = D^2$.

Showing that this can be done globally (*i.e.* over all of M) is much more difficult. Nevertheless, the Nash embedding theorem states that this is possible for a smooth metric; one needs $N = (D + 2)(D + 3)/2$ (see [2] for a recent review). While I don't know of a similar theorem for the combined g and B problem, there is no obstruction to realizing B as a pullback, so it seems plausible that this can also be done with N of order D^2 .

This at least suggests that the conjecture could hold with N of order the square of the central charge c . Of course, one needs to discuss renormalization and many other issues to follow this

⁶ One can take for f the sums of squares of functions which define M locally, and combine these with V using a partition of unity on \mathbb{R}^N .

geometric route. Clearly the problem will be far more constrained given supersymmetry. For the case of (2, 2) SCFT, the relevant analogy is to the Kodaira embedding theorem, which suggests a general prescription for taking data from the topological open sector of an SCFT, and constructing a gauged linear sigma model which flows back to the same SCFT [3].

4. Weak QFTs

Our final approach is to try to sit the problem of constructing and classifying QFTs, in some larger problem obtained by relaxing some of the many axioms and consistency conditions. The hope is of course that this problem will be easier to solve, and that we can then go on to enforce the additional conditions required to get actual or “honest” QFTs. In other words, we start by classifying “weak QFTs” which obey some but not all of the axioms of QFT.

The term⁷ is by analogy to the idea in mathematics of a “weak solution” of a partial differential equation. This is a solution which need not be a function but might be a distribution or something else. Even if the goal is to find a solution which is a function, often it can be easier to first show that such a weak solution exists, and then show that it is continuous, smooth or satisfies other desired properties. The basic example is the solution of linear PDE’s by Fourier transform, which does not naturally produce a function but rather an “ L^2 function,” which is only defined up to a set of measure zero.

When stated in such generality, of course this approach has been used extensively, one might even say universally, in the problem of defining QFTs. For example, a lattice theory or a regulated functional integral might be regarded as defining a “weak” form of a Euclidean or Lorentz invariant local QFT. Of course, in this case one has good reasons to expect the desired symmetry and locality properties to be recovered in the long distance limit.

The class of weak theories we have in mind here would be one in which we can define the weak QFT in terms of a finite amount of data, and then express the additional axioms needed for it to be an honest QFT as some finite set of equations which the data must satisfy.

A prototype for this is the classification of the minimal $c < 1$ CFTs. Such a CFT is defined by giving a finite list of primary fields and the finite set of their o.p.e. coefficients. This data then must satisfy the BPZ (associativity) and Cardy conditions to define an honest CFT.

It would be interesting to be able to generalize this to $c > 1$. The main problem is of course that we cannot work with the actual list of primary fields. We need to know something like Conjecture 1 to cut this down to size.

Suppose we knew the data of Conjecture 1; *e.g.* the complete list of primary fields and o.p.e. coefficients for all operators with dimension $h < H$ and $\bar{h} < H$. Could we use this to define a “weak QFT” ?

An interesting related idea appears in the work [21] of Rattazzi *et al.* They show that a CFT with a scalar operator ϕ of dimension h_ϕ , must contain an operator $:\phi^2:$ in the o.p.e.

$$\phi\phi \rightarrow 1 + :\phi^2: + \dots \quad (27)$$

with an upper bound on its dimension $h_{\phi^2} \leq f(h_\phi) < 2h_\phi$. Interestingly, the argument does not use Virasoro representation theory, only the finite-dimensional conformal group $SO(d, 2)$, and thus could work in any space-time dimension. One needs explicit expressions for the conformal blocks, which were found for $d = 2$ and $d = 4$ in [8].

The argument relies on associativity of the 4-point function, schematically

$$\sum_i \langle \phi(z_1)\phi(z_2)|i\rangle \langle i|\phi(z_3)\phi(z_4)\rangle = \sum_j \langle \phi(z_2)\phi(z_3)|j\rangle \langle j|\phi(z_4)\phi(z_1)\rangle. \quad (28)$$

⁷ suggested to me by Edward Witten

It is important that every term on both sides is positive, by unitarity (reflection positivity). Using conformal symmetry, one can write this (in any d) as a relation between functions of one complex variable z , a generalization of the cross ratio in $d = 2$ which maps the coincident limits $z_1 \rightarrow z_2, z_3, z_4$ to $z \rightarrow 0, \infty, 1$ respectively. Then, while this is a functional relation, one focuses on the behavior at the symmetric point $z = 1/2$. Although it is not at all obvious, from numerical study it turns out that the difference of the conformal blocks entering the two sides of this equation, behaves differently at $z = 1/2$ for intermediate operators O_i with $h_i < f(h_\phi)$ and those with $h_i > f(h_\phi)$. This can be expressed as a sign of a second derivative which must vanish for Eq. (28) to hold, and thus both types of operators must be present in the sum.

In the terms we are using here, one can think of this as a “weak CFT” argument. One is considering a class of theories which has most of the structure of CFT, such as symmetry and positivity, but need not satisfy associativity. Within this class, one can write down four-point functions which match an assumed part of the spectrum and o.p.e. Eq. (27). One then imposes associativity and gets a constraint.

Let us return to our earlier question: how can we use the data of Conjecture 1 to define a “weak QFT” ? What we clearly can do, is construct approximate four-point functions by summing over intermediate operators with $h, \bar{h} \leq H$, such as

$$\sum_{m \text{ such that } h \leq H} \langle \phi_i(z_1) \phi_j(z_2) | m \rangle \langle m | \phi_k(z_3) \phi_l(z_4) \rangle \quad (29)$$

While these will reproduce the data, they will not be associative.

Then, we define a function U which expresses how far these approximate four-point functions are from associativity, schematically

$$U = \sum \|F - F'\| \quad (30)$$

where $F - F'$ is the difference between the two sides of Eq. (28), and the sum is over all correlation functions of operators with $h, \bar{h} \leq H$.

Of course, associativity of the four-point functions is not the only consistency condition on a CFT, there is also modular invariance at genus one, and perhaps others (since we did not impose associativity of all four-point functions). But we could go on and add other terms to measure the failure of these properties to hold.

We can then define a flow which decreases U . Given a choice of parameters to vary, say the dimensions and o.p.e. coefficients, and given a metric on this space, we have the gradient flow

$$\dot{V}^i = -G^{ij} \frac{\partial U}{\partial V^j}. \quad (31)$$

Thus, starting with a weak QFT, we flow to an honest QFT.

This scheme is admittedly somewhat complicated. Worse, it has two major problems:

- If we do not put in all the consistency conditions, we may end up at $U = 0$ with a set of correlation functions which do not correspond to any QFT.
- The function U might have local minima with $U > 0$. Indeed, as we put in more and more consistency conditions, the function U becomes complicated, and is likely to have more and more local minima.

So, we should probably not rush to implement this scheme numerically just yet. But the general idea might be a good one, analogous to constructing CFTs via RG flow but making different aspects of the problem manifest. Given enough analytic understanding of the space of CFTs one is trying to describe, one might be able to make good choices for U and the metric which avoid the problems we just raised.

References

- [1] B. S. Acharya and M. R. Douglas, “A finite landscape?,” arXiv:hep-th/0606212.
- [2] B. Andrews, “Notes on the isometric embedding problem and the Nash-Moser implicit function theorem,” 2002.
- [3] Paul. S. Aspinwall and M. R. Douglas, work in progress.
- [4] C. Bachas, J. de Boer, R. Dijkgraaf and H. Ooguri, “Permeable conformal walls and holography,” JHEP **0206**, 027 (2002) [arXiv:hep-th/0111210].
- [5] A. A. Belavin, A. M. Polyakov and A. B. Zamolodchikov, “Infinite conformal symmetry in two-dimensional quantum field theory,” Nucl. Phys. B **241**, 333 (1984).
- [6] I. Brunner and D. Roggenkamp, “Defects and Bulk Perturbations of Boundary Landau-Ginzburg Orbifolds,” JHEP **0804**, 001 (2008) [arXiv:0712.0188 [hep-th]].
- [7] A. Cappelli and J. B. Zuber, “A-D-E Classification of Conformal Field Theories,” arXiv:0911.3242 [hep-th].
- [8] F. A. Dolan and H. Osborn, “Conformal partial waves and the operator product expansion,” Nucl. Phys. B **678**, 491 (2004) [arXiv:hep-th/0309180].
- [9] M. R. Douglas, “Superstring dualities, Dirichlet branes and the small scale structure of space,” arXiv:hep-th/9610041.
- [10] M. R. Douglas, “D-branes and matrix theory in curved space,” Nucl. Phys. Proc. Suppl. **68**, 381 (1998) [arXiv:hep-th/9707228].
- [11] M. R. Douglas, “Two lectures on D-geometry and noncommutative geometry,” arXiv:hep-th/9901146.
- [12] M. R. Douglas, talk given at the Stony Brook Symposium in Honor of Chen Ning Yang and James H. Simons, March 27–28, 2008.
- [13] M. R. Douglas, work in progress.
- [14] D. H. Friedan, “Nonlinear Models In Two + Epsilon Dimensions,” Annals Phys. **163**, 318 (1985).
- [15] M. R. Gaberdiel, S. Gukov, C. A. Keller, G. W. Moore and H. Ooguri, “Extremal $N=(2,2)$ 2D Conformal Field Theories and Constraints of Modularity,” arXiv:0805.4216 [hep-th].
- [16] T. Gannon, “Monstrous moonshine and the classification of CFT,” arXiv:math/9906167.
- [17] P. H. Ginsparg, “Curiosities at $c = 1$,” Nucl. Phys. B **295**, 153 (1988).
- [18] *Metric structures for Riemannian and non-Riemannian spaces*, M. Gromov, M. Katz, P. Pansu, and S. Semmes, 2001, Birkhäuser ISBN: 978-0-8176-4582-3.
- [19] D. Kutasov, “Geometry on the space of conformal field theories and contact terms,” Phys. Lett. B **220**, 153 (1989).
- [20] G. W. Moore, “K-theory from a physical perspective,” arXiv:hep-th/0304018.
- [21] R. Rattazzi, V. S. Rychkov, E. Tonni and A. Vichi, “Bounding scalar operator dimensions in 4D CFT,” JHEP **0812**, 031 (2008) [arXiv:0807.0004 [hep-th]].
- [22] N. Seiberg and E. Witten, “Monopole Condensation, And Confinement In $N=2$ Supersymmetric Yang-Mills Theory,” Nucl. Phys. B **426**, 19 (1994) [Erratum-ibid. B **430**, 485 (1994)] [arXiv:hep-th/9407087].
- [23] C. Vafa, “Lectures on strings and dualities,” arXiv:hep-th/9702201.
- [24] K. G. Wilson and J. B. Kogut, “The Renormalization group and the epsilon expansion,” Phys. Rept. **12**, 75 (1974).
- [25] E. Witten, “Three-Dimensional Gravity Revisited,” arXiv:0706.3359 [hep-th].
- [26] A. B. Zamolodchikov, “Irreversibility of the Flux of the Renormalization Group in a 2D Field Theory,” JETP Lett. **43**, 730 (1986)