

# Associativity, Jacobi, Bremner, and All That

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**Abstract.** I discuss various aspects of multi-linear algebras related to associativity.

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## 1. Introduction

Nambu introduced a multilinear operator  $N$ -bracket in the context of a novel formulation of mechanics [15]:

$$[A_1 A_2 \cdots A_N] = \sum_{\sigma \in S_N} \text{sgn}(\sigma) A_{\sigma_1} \cdots A_{\sigma_N} , \quad (1)$$

where the sum is over all  $N!$  permutations of the operators. For example, the operator 3-bracket is

$$[ABC] = ABC - ACB + BCA - BAC + CAB - CBA . \quad (2)$$

The operator product here is assumed to be *associative*.

The same construction independently appeared in the mathematical literature more than 50 years ago [12, 13]. The theory of such multi-operator products, as well as their “classical limits” in terms of multivariable Jacobians, has been studied extensively [1, 2, 3, 5, 6, 7, 8, 9, 10, 11, 14, 16, 17, 18, 19, 20].

From an algebraic point of view, it is natural to seek *the analogue of the Jacobi identity* for operator  $N$ -brackets. For the case of even  $N$ -brackets, the obvious generalization where one  $N$ -bracket acts on another leads to a true identity. However, for odd  $N$ -brackets this usually does *not* work. For instance, it is almost always true that

$$[[ABC] DE] - [[ADE] BC] - [A [BDE] C] - [AB [CDE]] \neq 0 . \quad (3)$$

That is to say, the so-called FI (“fundamental identity”) *fails*. There is one especially notable exception:  $su(2)$ , as described by Nambu [15].

Fortunately, even brackets are not odd. They need only act twice to yield an identity. Namely [6, 11],

$$[ B_1 \cdots B_{N-1} [B_N \cdots B_{2N-1}] ] = 0 \quad \text{for } N \text{ even.} \quad (4)$$

Here, total antisymmetrization of all the  $B$ s is understood. When  $N = 2$  this is the familiar Jacobi identity. The proof is by direct calculation and follows as a consequence of associativity.

Unfortunately, an odd  $N$ -bracket acting on just one other odd  $N$ -bracket does *not* vanish even when totally antisymmetrized over all entries, but rather produces a  $(2N - 1)$ -bracket [6, 5]. Therefore the simplest identity obeyed by odd brackets of only one type, that does not introduce higher-order brackets, requires that they act at least *thrice*.



### Bremner Identity and GBIs

Bremner [1] proved an identity (henceforth the “BI”) for associative operator 3-brackets acting thrice.

$$[ [A [bcd] e] f g ] = [ [Abc] [def] g ] , \quad (5)$$

where it is understood that all *lower* case entries are totally antisymmetrized by implicitly summing over all  $6! = 720$  *signed* permutations of them.

The BI can be proven through a resolution of both LHS and RHS as a series of canonically ordered words. By direct calculation we find

$$\begin{aligned} [[A [bcd] e] f g] &= 24 Abcdefg - 36 bAcdefg + 36 bcAdefg \\ &\quad - 24 bcdAefg + 36 bcdeAfg - 36 bcdefAg + 24 bcdefgA . \end{aligned} \quad (6)$$

The same expansion holds for  $[[Abc] [def] g]$ , again by direct calculation. That is to say, both  $[[A [bcd] e] f g]$  and  $[[Abc] [def] g]$  can be rendered as a 7-bracket plus another 3-bracket containing 3-brackets:

$$[[A [bcd] e] f g] = \frac{1}{20} [Abcdefg] - \frac{1}{6} [A [bcd] [efg]] = [[Abc] [def] g] . \quad (7)$$

Thus the BI amounts to the combinatorial statement that there are two distinct ways to write a 7-bracket in terms of nested 3-brackets.

Xiang Jin, Luca Mezincescu, and I proved that a similar identity holds for any odd-order bracket acting thrice [4]. For odd  $N = 2L + 1$ , this generalized BI (“GBI”) is

$$\begin{aligned} &[ [AB_1 \cdots B_{2L}] [B_{2L+1} \cdots B_{4L+1}] B_{4L+2} \cdots B_{6L} ] \\ &= [ [A [B_1 \cdots B_{2L+1}] B_{2L+2} \cdots B_{4L}] B_{4L+1} \cdots B_{6L} ] . \end{aligned} \quad (8)$$

Again, this identity is a consequence of only associativity. Thus all odd brackets built from associative products of operators need only act thrice to yield an identity. Given an hypothesized closed algebra of odd  $N$ -brackets, the GBI provides the simplest test for consistency with an underlying associative product.

To prove the GBI, we again expanded in terms of canonically ordered words. By direct calculation,

$$[[A [B_1 \cdots B_{2L+1}] B_{2L+2} \cdots B_{4L}] B_{4L+1} \cdots B_{6L}] = \sum_{n=0}^{6L} (-1)^n m_n B_1 \cdots B_n A B_{n+1} \cdots B_{6L} , \quad (9)$$

where it is implicit that one is to totally antisymmetrize over all the  $B$ s. All the coefficients in the resolution are integers. Explicitly,

$$\begin{aligned} m_n &= (2L+1)! (2L)! (2L-1)! \times c_n , \\ c_n &= \begin{cases} (n+1)(4L-n)/2 & \text{for } 0 \leq n \leq 2L \\ 10L^2 - 6Ln + L + n^2 & \text{for } 2L+1 \leq n \leq 3L \\ c_{6L-n} & \text{for } 3L+1 \leq n \leq 6L \end{cases} . \end{aligned} \quad (10)$$

The same expansion holds for  $[[AB_1 \cdots B_{2L}] [B_{2L+1} \cdots B_{4L+1}] B_{4L+2} \cdots B_{6L}]$ , again by direct calculation. Hence the GBI.

For example,  $L = 1$  gives the previous coefficients (6), while  $L = 2$  gives

$$\begin{pmatrix} m_0 \\ m_1 \\ m_2 \\ m_3 \\ m_4 \\ m_5 \\ m_6 \\ m_7 \\ m_8 \\ m_9 \\ m_{10} \\ m_{11} \\ m_{12} \end{pmatrix} = 5!4!3! \times \begin{pmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \\ c_6 \\ c_7 \\ c_8 \\ c_9 \\ c_{10} \\ c_{11} \\ c_{12} \end{pmatrix} = 5!4!3! \times \begin{pmatrix} 4 \\ 7 \\ 9 \\ 10 \\ 10 \\ 7 \\ 6 \\ 7 \\ 10 \\ 10 \\ 9 \\ 7 \\ 4 \end{pmatrix} = \begin{pmatrix} 69\,120 \\ 120\,960 \\ 155\,520 \\ 172\,800 \\ 172\,800 \\ 120\,960 \\ 103\,680 \\ 120\,960 \\ 172\,800 \\ 172\,800 \\ 155\,520 \\ 120\,960 \\ 69\,120 \end{pmatrix} . \quad (11)$$

For simplicity, I emphasize the 3-bracket case in the following.

### 3-Algebras

For a 3-algebra with linearly independent operators  $T_a$  that obey

$$[T_a T_b T_c] = i F_{abc}{}^d T_d , \quad (12)$$

the BI becomes, with implicit total antisymmetrization of the six  $b_j$  indices,

$$F_{ab_1b_2}{}^x F_{b_3b_4b_5}{}^y F_{xyb_6}{}^z = F_{b_1b_2b_3}{}^x F_{axb_4}{}^y F_{yb_5b_6}{}^z . \quad (13)$$

Alternatively, after renaming and cycling indices,

$$F_{b_1b_2b_3}{}^x \left( F_{axb_4}{}^y F_{yb_5b_6}{}^z - F_{ab_4b_5}{}^y F_{yxb_6}{}^z \right) = 0 . \quad (14)$$

This *trilinear relation* is a condition on the structure constants required by an underlying associativity for any posited 3-algebra.

**Exercise** (25 points; show all details; due Friday): Use (14) to prove a classification theorem for 3-algebras.

To be more specific, consider now any *closed bilinear algebra* where all commutators and anticommutators are also elements of the algebra, as given by

$$[T_a T_b] = i f_{ab}{}^c T_c , \quad \{T_a T_b\} = g_{ab}{}^c T_c . \quad (15)$$

For example, for  $u(N)$  with the  $T_a$  given by  $N \times N$  matrices, the second RHS involves the well-known  $d_{ab}{}^c$  symbol, as well as Kronecker delta terms. Or, with a bit of freedom of interpretation, one may think of the operator product expansion of any CFT in this way.

For a bilinear algebra of this form, the corresponding 3-algebra is also completely determined, or “induced.” This follows from

$$2 \times [ABC] = \{[AB] C\} + \{[BC] A\} + \{[CA] B\} . \quad (16)$$

The induced 3-algebra structure constants are given in terms of the  $f$  and  $g$  symbols by

$$2 F_{abc}{}^x = f_{ab}{}^u g_{uc}{}^x + f_{bc}{}^u g_{ua}{}^x + f_{ca}{}^u g_{ub}{}^x . \quad (17)$$

Thus the BI conditions on the induced 3-algebra structure constants can be re-expressed in terms of  $f$  and  $g$ . Again with implicit antisymmetrizations, the BI conditions become

$$f_{b_1 b_2}^u g_{ub_3}^x \left( F_{axb_4}^y F_{yb_5 b_6}^z - F_{ab_4 b_5}^y F_{yxb_6}^z \right) = 0 . \quad (18)$$

These conditions are indeed obeyed when the  $f$  and  $g$  symbols satisfy the conditions wrought by associativity.

The Jacobi identity (JI),

$$[A [BC]] + [B [CA]] + [C [AB]] = 0 , \quad (19)$$

is a consequence of associativity but it is *not* equivalent to it, even when augmented with the super Jacobi identity (SJI),

$$\{[AB] C\} = \{A [BC]\} + \{B [AC]\} . \quad (20)$$

Here, we have used the usual (anti)commutator notation, sans commas,  $\{AB\} = AB + BA$  and  $[AB] = AB - BA$ .

However, there is *another* trilinear identity which, when paired with the SJI, *is* equivalent to associativity. Namely,

$$[A [BC]] = \{\{AB\} C\} - \{\{AC\} B\} . \quad (21)$$

For want of a more compelling name, we will refer to this third relation as the “super-duper Jacobi identity” (SDJI)<sup>1</sup>. Note that the JI follows from the SDJI.

For a closed bilinear algebra the SJI and SDJI identities require the following conditions to be obeyed by the structure constants:

$$g_{ab}^u f_{uc}^x = g_{au}^x f_{bc}^u + g_{bu}^x f_{ac}^u , \quad f_{bc}^u f_{au}^x = g_{ab}^u g_{uc}^x - g_{ac}^u g_{ub}^x . \quad (22)$$

The more familiar JI conditions on the bilinear algebra structure constants follow from the second of these.

As I mentioned already, the BI conditions for the induced 3-algebra structure constants, expressed in terms of  $f$  and  $g$ , follow from these two conditions. Moreover, the structure constants  $F$  for *any induced  $N$ -bracket algebra* can be expressed in terms of  $f$  and  $g$  for such closed bilinear algebras, and it can be shown that the conditions on  $F$  imposed by associativity are indeed satisfied as a consequence of these same two conditions on  $f$  and  $g$ .

As an *aside*, there are deformed versions of these identities involving the “quommutators”

$$[AB]_\lambda \equiv \lambda AB - \lambda^{-1} BA . \quad (23)$$

These naturally lead to 3-brackets. For example, the “Jaquobi identity”:

$$[[AB]_\lambda C]_\mu + [[BC]_\lambda A]_\mu + [[CA]_\lambda B]_\mu = \frac{1}{2} \left( \lambda + \frac{1}{\lambda} \right) \left( \mu - \frac{1}{\mu} \right) [ABC] + \frac{1}{2} \left( \lambda - \frac{1}{\lambda} \right) \left( \mu - \frac{1}{\mu} \right) \{ABC\} . \quad (24)$$

Etc.<sup>2</sup> Here, we have also used the totally symmetrized 3-bracket:  $\{ABC\} = ABC + ACB + BCA + BAC + CAB + CBA$ .

<sup>1</sup> Apologies to Irving Berlin.

<sup>2</sup> The *ultimate Jacquobi identity* would involve six complex parameters,  $\lambda_j$  and  $\mu_j$ ,  $j = 1, 2, 3$ , as in:  $\left[ [AB]_{\lambda_1} C \right]_{\mu_1} + \left[ [BC]_{\lambda_2} A \right]_{\mu_2} + \left[ [CA]_{\lambda_3} B \right]_{\mu_3} = 0$ . Requiring that this vanish gives six equations for the parameters. Assuming  $\mu_1 \neq 0$ ,  $\lambda_3 \neq 0$ ,  $\mu_3 \neq 0$ , the *generic* solution is:  $\lambda_1 = \mu_1 \lambda_3 \mu_3$ ,  $\lambda_2 = \mu_1 \lambda_3$ , and  $\mu_2 = \frac{1}{\mu_1 \mu_3}$ . So the solution manifold has complex dimension three, including an overall complex scale. For fixed scale, it is in fact a geometrically *ruled surface*, and it must contain the usual Jacobi, the super Jacobi, and the super-duper Jacobi identities. Indeed, it does. The symmetric group is a symmetry of the solution manifold, so other solutions are given by permutations of 1, 2, 3. There is also parity: Another solution is obtained from the generic one just by flipping the signs of all the parameters.

## Classical Manifolds

There are also interesting questions for classical manifold theory that arise in this context. A *classical* 3-bracket is defined by

$$[A, B, C] = \omega^{abc} \partial_a A \partial_b B \partial_c C, \quad (25)$$

with antisymmetric but otherwise arbitrary 3-tensor  $\omega^{abc}$ . The combination that constitutes the so-called FI is

$$\begin{aligned} & [E, F, [A, B, C]] - [[E, F, A], B, C] - [A, [E, F, B], C] - [A, B, [E, F, C]] \\ &= \left( \omega^{abc} \omega^{def} - \omega^{dbc} \omega^{aef} - \omega^{adc} \omega^{bef} - \omega^{abd} \omega^{cef} \right) \partial_d (\partial_a A \partial_b B \partial_c C \partial_e E \partial_f F) \\ &+ (\partial_a A \partial_b B \partial_c C \partial_e E \partial_f F) \left( \omega^{def} \partial_d \omega^{abc} - \omega^{dbc} \partial_d \omega^{aef} - \omega^{adc} \partial_d \omega^{bef} - \omega^{abd} \partial_d \omega^{cef} \right). \end{aligned} \quad (26)$$

In the literature, when  $\omega$  is such that this vanishes, this is called a *Nambu-Poisson manifold*. This gives two types of bilinear constraints on  $\omega$ , obviously.

But, to conclude this talk, it seems more reasonable (to me at least) that one should impose, instead of the FI, a classical analogue of the BI. For a classical N-bracket involving  $n \geq N$  (odd) independent variables, with antisymmetric but otherwise arbitrary N-tensor  $\omega^{a_1 \dots a_N}$ ,

$$[B_{i_1}, B_{i_2}, \dots, B_{i_N}] = \omega^{a_1 \dots a_N} \partial_{a_1} B_{i_1} \dots \partial_{a_N} B_{i_N}, \quad (27)$$

we define a *Bremner-Poisson manifold* as one for which the BI holds. This leads to requirements on the  $\omega$  tensor that differ from those imposed by the FI. So defined, Bremner-Poisson and Nambu-Poisson manifolds are different, in general. We will discuss this in more detail elsewhere.

Perhaps N-brackets and algebras have an important role to play in physics, as originally suggested by Nambu. Recently there has been considerable interest in N-brackets, especially 3-brackets, as expressed in the physics literature (see [2] and references therein). These ideas await further development.

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- [1] M R Bremner, “Identities for the ternary commutator” J Algebra **206** (1998) 615–623; M R Bremner and L A Peresi, “Ternary analogues of Lie and Malcev algebras” Linear Algebra and its Applications **414** (2006) 1–18. (Similar results were independently found by J Nuyts, *unpublished*, 2008.)
- [2] T Curtright, D Fairlie, X Jin, L Mezincescu, and C Zachos, “Classical and Quantal Ternary Algebras” Phys Lett **B675** (2009) 387–392 [arXiv:0903.4889 [hep-th]].
- [3] T Curtright, D Fairlie, and C Zachos, “Ternary Virasoro-Witt Algebra” Phys Lett **B666** (2008) 386–390 [arXiv:0806.3515 [hep-th]].
- [4] T Curtright, X Jin, and L Mezincescu, “Multi-operator brackets acting thrice” J Phys A **42** (2009) 462001, [arXiv:0905.2759 [math-ph]].
- [5] T L Curtright and C K Zachos, “Classical and quantum Nambu mechanics” Phys Rev **D68** (2003) 085001 [arXiv:hep-th/0212267].
- [6] J A de Azcárraga, and J C Pérez Bueno, “Higher-order simple Lie algebras” Commun Math Phys **184** (1997) 669–681 [arXiv:hep-th/9605213].
- [7] C Devchand, D Fairlie, J Nuyts, and G Weingart, “Ternutator Identities” J Phys A **42** (2009) 475209, [arXiv:0908.1738 [hep-th]].
- [8] G Dito and M Flato, “Generalized Abelian Deformations: Application to Nambu Mechanics” Lett Math Phys **39** (1997) 107–125 [arXiv:hep-th/9609114].
- [9] V T Filippov, “n-Lie Algebras” Sib Math Journal **26** (1986) 879–891; “On n-Lie Algebra of Jacobians” Sib Math Journal **39** (1998) 573–581.
- [10] P Gautheron, “Some Remarks Concerning Nambu Mechanics” Lett Math Phys **37** (1996) 103–116.
- [11] P Hanlon and M Wachs, “On Lie k-Algebras” Adv Math **113** (1995) 206–236.

- [12] P Higgins, "Groups with multiple operators" *Proc London Math Soc* **6** (1956) 366-416.
- [13] A G Kurosh, "Multioperator rings and algebras" *Russian Math Surveys* **24** (1969) 1-13.
- [14] T Lada and J Stasheff, "Introduction to SH Lie algebras for physicists" *Int J Theor Phys* **32** (1993) 1087-1103.
- [15] Y Nambu, "Generalized Hamiltonian Dynamics" *Phys Rev* **D7** (1973) 2405-2412.
- [16] A P Pojidaev, "Enveloping Algebras of Filippov Algebras" *Comm Algebra* **31** (2003) 883-900.
- [17] M Schlesinger and J D Stasheff, "The Lie algebra structure of tangent cohomology and deformation theory" *J Pure Appl Algebra* **38** (1985) 313-322.
- [18] L Takhtajan, "On foundation of the generalized Nambu mechanics" *Comm Math Phys* **160** (1994) 295-315 [arXiv:hep-th/9301111].
- [19] L Vainerman and R Kerner, "On special classes of n-algebras" *J Math Physics* **37** (1996) 2553-2565.
- [20] I Vaisman, "A survey on Nambu-Poisson brackets" *Acta Math Univ Comenianae* **LXVIII**, 2 (1999) 213-241 [arXiv:math/9901047].