

# Methods for measuring risk-aversion: problems and solutions

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**Abstract.** Risk-aversion is a fundamental parameter determining how humans act when required to operate in situations of risk. Its general applicability has been discussed in a companion presentation, and this paper examines methods that have been used in the past to measure it and their attendant problems. It needs to be borne in mind that risk-aversion varies with the size of the possible loss, growing strongly as the possible loss becomes comparable with the decision maker's assets. Hence measuring risk-aversion when the potential loss or gain is small will produce values close to the risk-neutral value of zero, irrespective of who the decision maker is. It will also be shown how the generally accepted practice of basing a measurement on the results of a three-term Taylor series will estimate a limiting value, minimum or maximum, rather than the value utilised in the decision. A solution is to match the correct utility function to the results instead.

## 1. Introduction

Risk-aversion is a well established parameter in the field of economics and decision science, and the fact that it can be defined precisely in mathematical terms means that it offers the potential for accurate measurement. The paper will examine how far risk-aversion may be deduced from the individual's spending either (i) on insurance or physical protection to avoid an already-imposed risk or (ii) on a ticket for a lottery. A derivation of the Taylor series method of estimation will be presented first, followed by the more generally valid process of matching the correct utility function to the observed data in order to measure risk-aversion.

## 2. When the individual seeks to avoid an already-imposed risk: the protection/insurance method

Let us consider an individual of starting wealth,  $w_0$ , facing a loss,  $z$ , with probability,  $p$ , in a given period and use the maximum insurance premium,  $t$ , that he is prepared to pay to avoid any possibility of this loss to estimate his dimensionless risk-aversion,  $\varepsilon$ . (Alternatively he may be prepared to spend the same sum,  $t$ , on some other form of physical protection.)

If he does not buy insurance, his expected utility at the end of the period will be:

$$E[U_1] = pu(w_0 - z) + (1 - p)u(w_0) \quad (1)$$

whereas if he buys the insurance at a premium,  $t$ , his utility at the end of the period will be  $u(w_0 - t)$ , with certainty. The individual will be content to pay the premium,  $t$ , provided:

$$u(w_0 - t) \geq pu(w_0 - z) + (1 - p)u(w_0) \quad (2)$$

The term,  $u(w_0 - t)$ , may be expanded using the first three terms of a Taylor series as



$$u(w_0 - t) \approx u(w_0) - tu'(w_0) + \frac{t^2 u''(w_0)}{2} \quad (3)$$

under the assumption that  $t \ll w_0$ . The term,  $u(w_0 - z)$  may be expanded in a similar fashion, provided it is possible to assume that  $z \ll w_0$ :

$$u(w_0 - z) \approx u(w_0) - zu'(w_0) + \frac{z^2 u''(w_0)}{2} \quad (4)$$

The necessity of assuming that  $z \ll w_0$  means that this analysis can apply to the insurance of, for example a washing machine costing £250, with an annual insurance premium of tens of pounds, but cannot be used to analyse the insurance of the average person's house, with a value of £150,000, for example. This point will be returned to later.

Substituting from equations (3) and (4) into inequality (2) gives:

$$u(w_0) - tu'(w_0) + \frac{1}{2}t^2 u''(w_0) \geq pu(w_0) - pz u'(w_0) + \frac{1}{2}pz^2 u''(w_0) + u(w_0) - pu(w_0) \quad (5)$$

Hence

$$u''(w_0) \left( \left( \frac{t}{z} \right)^2 - p \right) \geq \frac{2}{z} \left( \frac{t}{z} - p \right) u'(w_0) \quad (6)$$

Both the probability,  $p$ , of loss and the ratio of the premium to the potential loss,  $t/z$ , will be strictly fractional, which implies that

$$(t/z)^2 < t/z \text{ and } p < \sqrt{p} \quad (7)$$

Now the expected loss is given by

$$E(Z) = pz + (1-p)0 = pz \quad (8)$$

(c.f. equation (1)). For the case where  $t$  is set less than or equal to the expected loss,  $t \leq pz$  or  $t/z \leq p$  so that, from equation (7),  $(t/z)^2 - p < 0$ . The same condition holds when an individual is prepared to set his premium,  $t$ , above the expected loss but in the range:  $p < t/z < \sqrt{p}$ . Since utility rises with wealth, it follows that  $u'(w_0) > 0$ . Hence dividing inequality (6) by first  $u'(w_0)$  and then  $(t/z)^2 - p$  gives

$$\frac{u''(w_0)}{u'(w_0)} \leq \frac{2}{z} \frac{(t/z) - p}{(t/z)^2 - p} = \frac{2}{z} \frac{p - (t/z)}{p - (t/z)^2} \text{ for } t/z < \sqrt{p} \quad (9)$$

Thus

$$-\frac{u''(w_0)}{u'(w_0)} \geq \frac{2}{z} \frac{p - t/z}{p - (t/z)^2} \text{ for } t/z < \sqrt{p} \quad (10)$$

In other words:

$$-\frac{u''(w_0)}{u'(w_0)} \geq \frac{2(pz - t)}{(t^2 - pz^2)} \text{ for } t/z < \sqrt{p} \quad (11)$$

Using the definition of risk-aversion,  $\varepsilon_{\min} = -w_0 u''(w_0)/u'(w_0)$  we may now calculate the minimum value for risk-aversion at which a premium,  $t$ , will be sanctioned (which is not the same thing as the actual value of risk-aversion at which the decision would be made):

$$\varepsilon_{\min} = \frac{2w_0(pz - t)}{(t^2 - pz^2)} \quad (12)$$

The fact that equation gives a minimum value becomes plain when one considers a more risk averse individual, with  $\varepsilon > \varepsilon_{\min}$ , with the same wealth,  $w_0$ , and facing the same potential loss,  $z$ : it is intuitively obvious that he will be even keener to buy insurance at this premium,  $t$ .

The validity of equation (12) is restricted to cases where

- the possible loss,  $z$ , is much less than the individual's starting wealth,  $w_0$ :  $z \ll w_0$
- the ratio of the premium,  $t$ , to the possible loss,  $z$ , is less than the square root of the probability of loss,  $p$ :  $t/z < \sqrt{p}$

### 3. Where the individual is invited to take a risk: a lottery

Consider now a lottery offering a prize,  $z$ , with probability of winning,  $p$ . The maximum amount,  $t$ , that an individual of starting wealth,  $w_0$ , is prepared to pay for a ticket may be used to deduce information about the individual's risk-aversion. A Taylor series analysis similar to that given in Section 2 may be used to find the maximum risk-aversion consistent with buying a lottery ticket at price,  $t$ :

$$\varepsilon_{\max} = \frac{2w_0(pz - t)}{pz^2 - 2pzt + t^2} \quad (13)$$

Again only a limit is derived.  $\varepsilon_{\max}$  is not the value of risk-aversion at which the decision takes place, but an upper bound on that value. This fact is not always made plain [1], but becomes intuitively obvious when one considers a more risk confident individual, with  $\varepsilon < \varepsilon_{\max}$ , and the same wealth,  $w_0$ : he will be even keener to take the risk of betting on the lottery.

The conditions for equation (13) to be valid are that  $t \ll w_0$  and  $z - t \ll w_0$ , implying that the possible prize,  $z$ , must be very much smaller than the person's starting assets,  $w_0$ . This excludes many lotteries in common use, such as national lotteries where the prize can be orders of magnitude greater than the starting wealth of the person buying the lottery ticket.

### 4. Difficulties in measuring risk-aversion by the insurance and lottery methods

It is clear from equations (12) and (13) that the limiting values of a person's risk aversion depends on his starting wealth,  $w_0$ . This dependence occurs not only explicitly but also via his choice of maximum insurance premium or maximum ticket price,  $t$ , which it is reasonable to assume will be affected by how much money the person has:  $t = t(w_0)$ . While a casual reading of equations (12) and (13) might suggest that risk-aversion will rise with wealth, in fact the reverse is more likely to be true, a fact that is explicable in the two cases considered above in terms of the variability of  $t$  with starting wealth. For example, a poor person might decide to insure a washing machine for more than the expected loss, indicating a positive risk-aversion, while a rich person might decide that he can stand the loss, thus setting his maximum acceptable premium at zero. Putting  $t = 0$  into equation (12) shows the rich person's minimum risk-aversion is negative in this case. His greater wealth allows him to be risk confident in this case, and his risk-aversion is less than that of the poor person considered.

The individual may consider several insurance premia,  $t$ , and carry out a comparison of utilities in each case. Considering the terms on the left-hand side of equations (12) and (13) namely,  $t$ ,  $z$ ,  $p$  and  $w_0$ , it is clear that  $z$ ,  $p$  and  $w_0$  will remain constant for all such comparisons. The price or premium,  $t$ , will change before each successive comparison of utilities, but it will stay constant during the comparison process itself. Thus, if the actual risk-aversion used at the point of decision is governed by the same factors as its limiting values, then risk-aversion will be constant during the process of each utility comparison. This adds to the arguments set out in [2] for risk-aversion being constant during the process of any single decision. Such is, in any case, the universal assumption made in practice when utility functions are used to examine a financial decision under uncertainty.

But although a person's risk-aversion will not change during the decision process, it is likely to vary between individuals with different starting wealths and may also vary between individuals possessing the same starting wealth. Moreover, it has been shown elsewhere, [3], [4], [5], that it is rational for a person to experience a different risk-aversion when he is considering different investment decisions. It is rational for those decisions having only a small effect on his economic state either way to incur a lower risk-aversion than those where a poor outcome could threaten his economic survival. Thus any exercise to measure risk-aversion needs to take account of both variations in starting wealth between individuals and the financial importance of the decision to individuals, if the results are to be interpreted correctly.

The approach based on a Taylor series expansion requires the excursions from the starting wealth to be small, so that, for example, the possible win from a lottery must not be comparable with the starting wealth and therefore of low importance to all the individuals in the sample. This makes it rational for them all to employ a low risk-aversion in their assessment of insurance against the loss, rendering any study trying to pick out differences between groups problematic if the Taylor series is the chosen method of interpretation. Thus it is not surprising that Cramer et al. [1] reported difficulties in differentiating between the (maximum) risk-aversions of entrepreneurs and those in paid employment when they considered how the maximum acceptable price for a ticket in a lottery with a 1000 guilder prize (about €450).

Measurement of minimum risk-aversion from the maximum acceptable insurance premium suffers from a similar limitation. It can be used only for potential losses that are relatively unthreatening to the individuals' wealth, in which circumstance it is rational for the individuals all to use a low risk-aversion. The situation will be significantly different for most people faced with the problem of insuring their houses against fire: the potential loss will in many cases be almost all their wealth, and they would be expected to display high risk-aversions in considering house insurance as a consequence.

To measure risk-aversion in the general case it is necessary first to derive a general utility function with risk-aversion as its parameter and then to estimate risk-aversion by matching the function to the characteristics of the decision through adjusting this parameter.

## 5. Measuring risk-aversion from the maximum acceptable insurance premium using a utility function

A reason for the popularity of estimating risk-aversion via Taylor series expansions of utility rather than through a utility function is that the former avoids the need to specify a form of the latter. While Pratt [6] reduced the number of feasible utility functions, the number can be reduced much further to just one family. It has been shown [6] that, if an individual's risk-aversion stays constant, not over all time and decisions, but merely during the course of each decision, then only one general family of utility functions can be representative of human decision making. These are the Power utility functions, governed by risk-aversion as their sole parameter.

Using the Atkinson utility function [7], [2] (one of the Power family), it is possible to define the reluctance to invest,  $R_{120A}$ , as the difference in utility before and after investment, normalised to the starting utility relative to the utility of one unit of money [8], which is dependent on the reluctance to invest based on the Power utility,  $R_{120P}$ :

$$R_{120P} = 1 - p + p(1-c)^{1-\varepsilon} - (1-b)^{1-\varepsilon} \quad (14)$$

where  $b$  is the normalised premium:  $b = t/w_0$ , while  $c$  is the normalised potential loss:  $c = z/w_0$ .

For any given premium that a respondent is prepared to pay, it is required to find the value of risk-aversion,  $\varepsilon$ , that minimises the reluctance to invest, equivalent to maximising the desire to invest. The curve of  $R_{120A}$  versus  $\varepsilon$  will be smooth, and, if care is taken near  $\varepsilon = 1.0$ , it is possible to use calculus to find the valley minimum of  $R_{120A}$ :

$$(w_0^{1-\varepsilon} - 1)R'_{120P} - R_{120P} \log w_0 = 0 \quad (15)$$

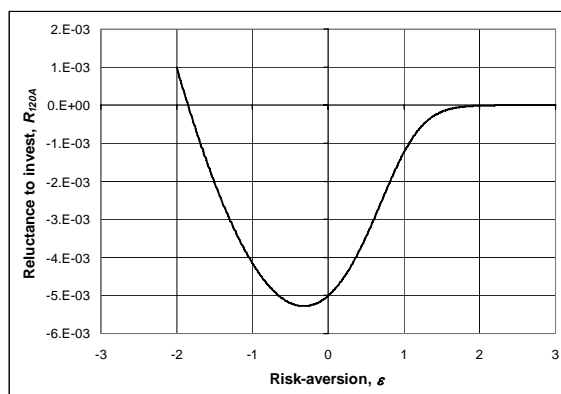
where  $R'_{120P} = dR_{120P}/d\varepsilon$  is given by:

$$R'_{120P} = p(1-c)^{1-\varepsilon} \log(1-c) - (1-b)^{1-\varepsilon} \log(1-b) \quad (16)$$

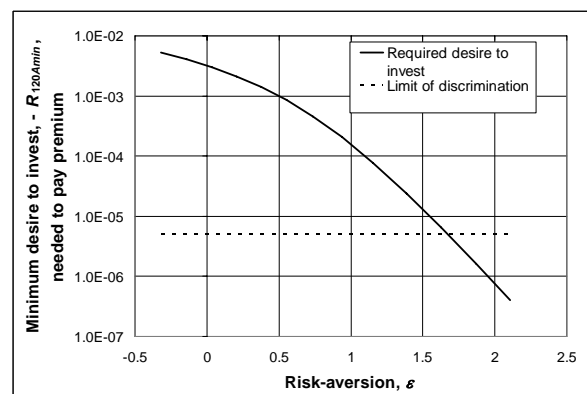
Solving equations (15) and (16) gives the risk-aversion associated with the minimum of  $R_{120A}$ , the value of risk-aversion at which the decision to invest is made, known as "the permission point" [4].

Not only is the utility function method not dependent on small-perturbation analysis but it has the advantage that it can identify the unique value of risk-aversion associated with the decision to invest in addition to the minimum value that could allow that decision to stand, which occurs when  $R_{120A} = 0$ .

Fig. 1 shows the behaviour of the reluctance to invest for the case of washing machine insurance, where the individual has wealth of £1,000 and the probability of complete loss of £250 is 0.1. The individual's maximum acceptable premium is assumed here to be £20. The reluctance to invest is zero, viz.  $R_{120A} = 0$ , meaning that the utility difference is zero at a risk-aversion of  $-1.85$ , but the individual will be keener to buy the insurance at higher risk-aversions, with his minimum reluctance to invest,  $-5.28 \times 10^{-3}$ , occurring at a risk-aversion of  $-0.32$ . The criterion for accepting the insurance premium is that the person paying it must experience at least a minimum desire to invest, a variable that depends on his risk-aversion,  $-R_{120Amin} = -R_{120Amin}(\varepsilon)$ . This minimum level of desire to invest will fall as risk-aversion increases (Fig.2). It is suggested that it is this process, which is rather more complex than simply finding the point where the utility difference is zero (viz.  $R_{120A} = 0$ ), that provides the true explanation how higher premia will be sanctioned at higher levels of risk-aversion.

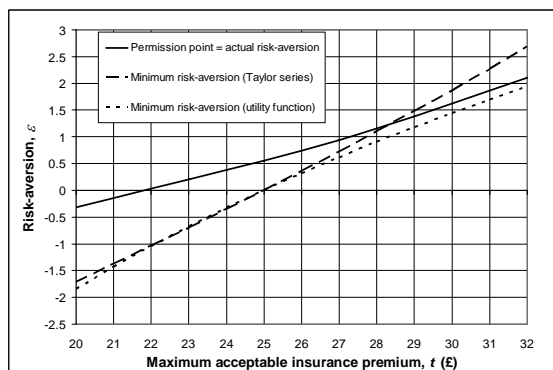


**Figure 1** Reluctance to invest vs. risk-aversion



**Figure 2** Minimum desire to invest needed to pay premium vs. risk-aversion

Fig. 3 shows a comparison between the methods for a range of insurance premia. The Taylor series expansion method matches the utility-function calculation of risk-aversion well near  $\varepsilon = 0$ , but it



**Figure 3.** Risk-aversion versus premium

begins to fail as the risk-aversion moves away from zero. Only the utility function method is able to calculate the actual risk-aversion used by the subject when deciding to accept the insurance premium. While the actual risk-aversion and the minimum risk-aversion begin to converge for high, positive values, it be noted that  $|R_{120A}| < 5 \times 10^{-6}$  for a premium of £31 and above, meaning that it would not be sensible to authorise such a premium, as it lies beyond the point of indeterminate decision [3]. This limit, only 20% up on the expected loss of £25 is thus

very tight in this example where rather small sums of money are at stake. This reveals a further problem arising from the requirement of the Taylor series approach that experimental studies are restricted to risks that are small compared with the individual's assets.

The capability of the utility function method to find the risk-aversion in general, insurance situations, including where the potential loss is large compared with the starting wealth allows a larger range of risk-aversions to be analysed. This improves the accuracy of risk-aversion observation.

## 6. Conclusions

Risk-aversion is a key parameter characterising human decision making in the necessarily uncertain environment in which we exist. The value of risk-aversion is likely to differ among different decisions for the same person, and among different people facing the same decision [9], [10]. It is possible to calculate a rational risk-aversion for a given insurance/protection situation, but it is likely that there may be differences between individuals.

Methods of estimating risk-aversion based on a Taylor series expansion suffer from two major weaknesses. First, they estimate only a limiting value, maximum or a minimum, of risk-aversion that could be compatible with the subject's economic choice, not the actual value used by the subject in making his decision. Secondly, their validity is restricted to situations where no outcome of the decision is going to make a great difference to the subject's economic well-being. By contrast, methods based on matching a utility function to the observed economic behaviour allow the estimation of not only the minimum or maximum compatible risk-aversion but also the actual value of risk-aversion used by the subject in deciding which economic course to take when a lot is at stake.

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