New scale classification within the representational theory of measurement

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Abstract. Within the representational theory of measurement, Stevens presented a classification of scales based on the set of empirical operations preserved by each type of scale. In this classification, the group structure of admissible transformations of each scale type were exposed. In this paper, we present recent studies that focus the scale classification on the group structure of admissible transformations. Then the classification is extended to other types of scales.

1. Introduction

Introduced by Stevens in [1] , the representational theory of measurement formalizes the concept of scale. The scales are presented as structures that define the representation process of manifestations of empirical quantities by symbols defined on a representational abstract space. During this process, called measurement, an empirical relational structure is preserved and transposed into the representational abstract space. In his seminal paper, Stevens proposed to classify scales into 4 types: nominal, ordinal, interval and ratio scales depending on the set of relations preserved during the measurement process. In parallel, Stevens remarks that for each type of scale, the set of all possible admissible transformations is a group under the composition law. In recent studies, we shown that the classification can be driven by such groups and by the subgroup relation [2]. This paper specifically develops this approach in comparison with other classifications and will propose the inclusion of new scale types.

2. Presentation of the classification

The representational theory of measurement has inspired several studies, especially for ordinal scales that represent a central interest in psychophysics [3]. The link between scale types and group structures remained unexplored until 1987, when Luce and Narens studied measurement scales on continuous spaces [4]. More recently, Narens showed the importance of an approach based on group theory to analyse the representational theory of measurement [5]. We will see that the group theory is the central pivot of the scale classification. Within the representational theory of measurement, a scale $< X, S, M, R_X, R_S >$ is defined by an empirical relational system $< X, R_X >$, a representational relational system $< S, R_S >$, and an homomorphism M from X to S that preserves the relational structure. The representation theorem induces for each type of scale the existence of a set of admissible transformations [6]. An admissible transformation is a mapping f into the representation set S that transforms a scale $< X, S, M, R_X, R_S >$ into

Table 1. Scale classification given by Stevens.

Scale	Basis empirical operation	mathematical group structure	admissible transfromation
nominal	determination of equality	permutation group	$y = f(x)$, f is a bijection
ordinal	+ determination of greater or less	isotonic group	$y = f(x)$, f is a monotonic increasing function
interval	+ determination of equality of intervals	general linear group	$y = ax + b, a > 0$
ratio	+ determination of equality of ratio	similarity group	$y = ax, a > 0$

another scale $< X, S, f \circ M, R_X, R_S >$ of same type such that:

$$\forall r_s \in R_s, n = arity(r_s), \forall x_1, \ldots, x_n \in X, r_s(M(x_1), \ldots, M(x_n)) \Leftrightarrow r_s(f(M(x_1)), \ldots, f(M(x_n))) \quad (1)$$

According that $S = M(X)$, the preceding definition is defined on the representation set.

$$\forall r_s \in R_s, n = arity(r_s), \forall s_1, \ldots, s_n \in S, r_s(s_1, \ldots, s_n) \Leftrightarrow r_s(f(s_1), \ldots, f(s_n)) \quad (2)$$

2.1. Nominal scales

In the case of a nominal scale, the representational relational structure includes only the equality relation. $R_S = \{=\}$. The admissible transformations are then mappings that respect the following constraint.

$$\forall s_1, s_2 \in S, s_1 = s_2 \Leftrightarrow f(s_1) = f(s_2) \quad (3)$$

The set of all admissible transformations of a nominal scale is then the set of bijective mappings on S. Such set defines, with the operator of function composition, the group of all permutations on S also called the Symmetric group and denoted $Sym(S)$.

2.2. Ordinal scales

With the ordinal scales the relational structure preserved during the measurement process includes also an ordering relation. Then the set of admissible transformations are relations that preserve a relational system $< S, \{=, <\} >$.

$$\forall s_1, s_2 \in S, s_1 < s_2 \Leftrightarrow f(s_1) < f(s_2) \quad (4)$$

The isotonic group cited in the seminal paper is also called order-preserving group and is the group of monotonic increasing functions on S. This group is a subgroup of $Sym(S)$ and will be denoted $OA(S)$ for order-preserving automorphisms on S.

2.3. Interval scales

Stevens explicitly defined the interval scales by their admissible transformations:

$$\forall x \in \Re, f(x) = a.x + b, a > 0 \quad (5)$$

In the seminal paper, the group of the admissible transformations of interval scales is given as the General linear group but is actually a subgroup of the affine group $GA(1,\Re)$ on $\Re$. Indeed, If the preceding constraint is compared with the definition, given in (6), of a member of $GA(1,\Re)$, it is deduced that any admissible transformation of interval scales is an affine increasing mapping. The group of the admissible transformations of interval scales is $GA(1,\Re)\cap OA(\Re)$. By definition this group is a subgroup of $OA(\Re)$.

$$\forall x \in \Re, f \in GA(1,\Re) \Leftrightarrow f(x) = a.x + b, a \neq 0 \tag{6}$$

The relations preserved by the admissible transformations are given by: (3), (4) and (7).

$$\forall s_1, s_2, s_3, s_4 \in \Re, s_1 - s_2 = s_3 - s_4 \Leftrightarrow f(s_1) - f(s_2) = f(s_3) - f(s_4) \tag{7}$$

2.4. Ratio scales

Finally, the similarity group associated in the original classification with the ratio scales is the Linear group $GL(1,\Re)$. Actually, and for the same reason than for interval scales, the set of admissible transformations of ratio scales (see (8)) is the set $GL(1,\Re)\cap OA(\Re)$. It's a group, subgroup of $GA(1,\Re)\cap OA(\Re)$. The relations preserved by the admissible transformations are: (3), (4) and (9).

$$\forall x \in \Re, f(x) = a.x, a > 0 \tag{8}$$

$$\forall s_1, s_2 \neq 0, s_3, s_4 \neq 0 \in \Re, \frac{s_1}{s_2} = \frac{s_3}{s_4} \Leftrightarrow \frac{f(s_1)}{f(s_2)} = \frac{f(s_3)}{f(s_4)} \tag{9}$$

Table 2. Scale classification updated with recent group theory.

Scale	Basis empirical operation	mathematical group structure	admissible transformation
nominal	determination of equality	Symmetric group $Sym(S)$	$y = f(x)$, f is a bijection
ordinal	+ determination of greater or less	$OA(S)$	$y = f(x)$, f is a monotonic increasing function
interval	+ determination of equality of intervals	$GA(1,\Re)\cap OA(\Re)$	$y = ax + b, a > 0$
ratio	+ determination of equality of ratio	$GL(1,\Re)\cap OA(\Re)$	$y = ax, a > 0$

3. New scale types

According to this approach, other scales introduced in the literature can be added to the classification.

The first trivial example is the scale type which set of admissible transformations is the group $T(\Re)$ of translations on $\Re$: $f(x) = x+b$. Actually, by the composition with exponential function, the group of translations is isomorphic to the group of ratios. We can conclude that this scale type is identical to a ratio scale.

3.1. Metrical scales
In [7] Coombs introduces the metrical scales as scales that preserve a distance from the empirical space of manifestations to the representational space. In this case, the admissible transformations are bijective mapping that maintain distances: isometries. The preserved relational structure is then $< S, \{=, r_d\} >$ such that r_d is the quaternary relation:

$$\forall s_1, s_2, s_3, s_4 \in S, (s_1, s_2) r_d (s_3, s_4) \Leftrightarrow d(s_1, s_2) = d(s_3, s_4) \tag{10}$$

The admissible transformations are then bijective mappings that respect:

$$\forall s_1, s_2, s_3, s_4 \in S, d(s_1, s_2) = d(s_3, s_4) \Leftrightarrow d(f(s_1), f(s_2)) = d(f(s_3), f(s_4)) \tag{11}$$

Both isometries and scaling functions respect this constraint. A subgroup including only isometries can be defined after reinforcement of the last constraint.

$$\forall s_1, s_2 \in S, d(s_1, s_2) = \alpha \Leftrightarrow d(f(s_1), f(s_2)) = \alpha \tag{12}$$

The set of admissible transformations is then reduced to a group of isometries. As an example, a metrical scale must be used for the measurement of angle over the full circle. Indeed, in this case nor ordinal or interval or ratio scales can be used due to the lack of ordering relation. But a distance can be defined on the circle. In recent studies we proposed the concept of fuzzy scales that are actually varieties of metrical scales.

3.2. Multidimensional scales
If we consider the affine group and the linear group in higher dimensions, we obtain new multidimensional scales that are not the simple combination of several ratio scales or interval scales. Let first have a look on the linear group $GL(n, \Re)$ and on the affine group $GA(n, \Re)$.

$$f \in GL(n, \Re) \Leftrightarrow \forall x \in \Re^n, f(x) = A.x, A \in \Re^{nxn}, detA \neq 0 \tag{13}$$

$$f \in GA(n, \Re) \Leftrightarrow \forall x \in \Re^n, f(x) = A.x + B, A \in \Re^{nxn}, B \in \Re^{nxn}, detA \neq 0 \tag{14}$$

These 2 groups can be used to define the set of admissible transformation of the extension to higher dimensions of interval scales and ratio scales. In this case the set of invariant properties define a large field of mathematics that we won't open in this paper. The study of multi-dimensional scale is directly linked with the analysis of the invariant properties of the admissible transformations.

4. Conclusion
Considering the groups of admissible transformations and the *subgroup of* relation between them to classify the scales is a way to change the viewpoint on scales and to analyse new types of scales. In the new representation of scales, appear the historical branch including the order preserving scales, the branch of multi-dimensional scales, and the branch of metrical scales. These last branches are not really new but must be studied within the group theory approach. The advantage of considering scales with the group theory is to benefit from the last advances of this field. In particular the analysis of invariant properties of each scale gives the tools for the computation of data derived from measurements.

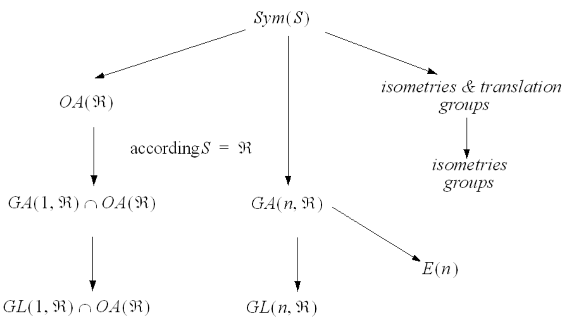

Figure 1. Classification of groups of admissible transformations.

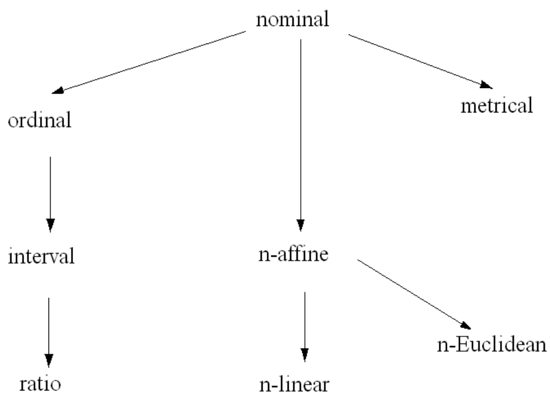

Figure 2. Proposal for a general classification of scales.

References

[1] S.S. Stevens, On the theory of the scales of measurement, *Science*, **103** (1946), 677680.
[2] E. Benoit, The role of fuzzy scales in measurement theory, *Measurement* (2013) 10.1016/j.measurement.2013.04.043.
[3] L. Finkelstein, M. S. Leaning, A review of the fundamental concepts of measurement, *Measurement* **2**(1) (1984) 25-34.
[4] R.D. Luce, L. Narens, Measurement scales on the continuum, *Science* **236** (1987) 1527-1532.
[5] L. Narens, A meaningful justification for the representational theory of measurement, *Journal of Mathematical Psychology* 46 (2002) 1527-1532.
[6] L. Finkelstein, Representation by symbol systems as an extension of the concept of measurement, *Kybernetes* **4** (1975), 215-223.
[7] C. H. Coombs, A theory of psychological scaling, Engineering Research Institute, University of Michigan, 1952.