

One-dimensional model for the fractional quantum Hall effect

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Abstract. A simple one-dimensional model is proposed, in which N spinless repulsively interacting fermions occupy $M > N$ degenerate states. It is argued that the energy spectrum and the wavefunctions of this system strongly resemble the spectrum and wavefunctions of 2D electrons in the lowest Landau level (the problem of the Fractional Quantum Hall Effect). In particular, Laughlin-type wavefunctions describe ground states at filling factors $\nu = N/M = 1/q$, q odd. Within this model the complementary wavefunction for $\nu = 1 - 1/q$ is found explicitly, and extremely simple ground state wavefunctions for arbitrary odd-denominator filling factors are proposed.

1. Introduction

Thirty years after the discovery of the Fractional Quantum Hall Effect [1] the understanding of this extraordinary phenomenon is still only partial and a consistent theory is absent. After Laughlin's famous article [2], the intriguing concept of composite fermions advanced and developed by Jain [3, 4] became the generally accepted physical description: the fractional QHE is the integer QHE of new particles, moving in a reduced magnetic field. While what exactly a composite fermion is, remains unclear ("electron + $2p$ zeros of the many-electron wavefunction" is not really comprehensible), it was proved by numerical calculations that the proposed ground states for fractional fillings are virtually exact [4], and this justifies the rather bizarre construction of these states involving higher Landau levels which physically are irrelevant.

Currently, we are still in an awkward position: on the one hand many experimental facts support Jain's idea [3] of composite fermions moving in a reduced effective magnetic field, and this is the only physical description available. On the other hand [5], nobody has really shown theoretically, apart from what may be described as wishful thinking, the existence of composite fermions, as (quasi) free particles. So, the nature of the object whose properties are measured experimentally remains a mystery.

While Laughlin's idea [2] certainly gave a clue to understanding the FQHE, some important questions remain unanswered. One of them concerns the $\nu = 2/3$ state (and, generally, all the $\nu = 1 - 1/q$ states with odd q). Because of the electron-hole symmetry, this state, $\Psi_{2/3}$, can be regarded as the $\nu = 1/3$ hole state described by the Laughlin wavefunction, $\Psi_{1/3}$, depending on the coordinates of N holes in a completely filled Landau level. The physical properties should be (and, in fact, are) quite similar to those at $\nu = 1/3$.



Suppose, however, that one wants to have a look at the $\Psi_{2/3}$ function written in terms of $2N$ electron coordinates. To do this, one must (i) write down the Laughlin function $\Psi_{1/3}$ as a superposition of $N \times N$ determinants involving one-particle hole wavefunctions, and (ii) leaving the coefficients in the superposition unchanged, replace each determinant by its complimentary $2N \times 2N$ electron determinant. The resulting unwieldy expression, which nobody knows how to write down explicitly, will represent the $\nu=2/3$ ground state, $\Psi_{2/3}$. It will go to zero at $z_i \rightarrow z_j$ as $(z_i - z_j)$, just like any antisymmetric function, and we will hardly be able to understand why this function should minimize the interaction energy!

This shows the existence of wavefunctions that are as good as the Laughlin function, but which do not have higher order zeros when the electron coordinates coincide. In this sense the $\Psi_{2/3}$ function resembles the wavefunctions for other rational fillings, such as $\Psi_{2/5}$. It remains an open question, what are the relevant properties of these ground state wavefunctions, and this is a clear signal that our understanding is not complete.

2. One-dimensional model for FQHE

It was argued [5, 6] that the energy spectrum responsible for the FQHE arises whenever N spin polarized (or “spinless”) fermions with appropriate repulsive interaction occupy $M > N$ initially degenerate states. Thus it makes sense to look for other, more simple, problems of this kind. In Ref. 6 a one-dimensional model with M degenerate states on a circle was proposed with $\psi_k(\varphi) = (2\pi)^{-1/2} \exp(ik\varphi)$, $k=0..M-1$. Alternatively, another basis of one-particle states $\Phi_n(\varphi)$ localized at $\varphi = 2\pi n/M$ can be used [6].

The Laughlin-like N -particle wavefunction $\Psi_{1/q}$ (q odd) has the familiar form:

$$\Psi_{1/q}(\varphi_1 \dots \varphi_N) = A \prod_{1 \leq i < j \leq N} [\exp(i\varphi_i) - \exp(i\varphi_j)]^q, \quad (1)$$

This can be rewritten in the $\Phi_n(\varphi)$ basis as

$$\Psi_{1/q}(\varphi_1 \dots \varphi_N) = \sum_{(n)} C(n_1 \dots n_N) \Phi_{n_1}(\varphi_1) \dots \Phi_{n_N}(\varphi_N), \quad (2)$$

where the coefficients C (providing just another representation of the same function) are given by:

$$C(n_1 \dots n_N) = A \prod_{1 \leq i < j \leq N} (\omega^{n_i} - \omega^{n_j})^q; \quad \omega = \exp\left(\frac{2\pi i}{M}\right), \quad (3)$$

with a *known* normalization constant A [6] (it was calculated by Dyson [7, 8] a long time ago).

3. The wavefunctions for $\nu=1-1/q$

Rather surprisingly, within my model it can be *proved* that the wavefunctions $\Psi_{1-1/q}$, corresponding to fillings $2/3, 4/5, 6/7 \dots$ have coefficients C given by *exactly the same* expression (3), the powers of ω^n higher than $M-1$ being automatically reduced to the interval $[0, M-1]$.

In summary, starting from the Laughlin-like wavefunction (1) for $\nu=1/q$ we find the following *exact result*. For both cases when either $qN=M$, $\nu=1/q$ (q odd), or $qN=(q-1)M$, $\nu=1-1/q$, the wavefunction in the Φ_n representation (which from now on we denote as Ψ) has the same form:

$$\Psi(n_1 \dots n_N) = \prod_{1 \leq i < j \leq N} (\omega^{n_i} - \omega^{n_j})^q; \quad \omega = \exp(2\pi i / M), \quad (4)$$

the only difference being in the number of variables (n_i).

3. Conjecture

On the basis of this striking result, it is tempting to make the following *conjecture*: in our model *all* the ground states for *arbitrary* rational fillings $\nu=p/q$, or $qN=pM$ (q odd, p and q do not have common divisors) are described by the extremely simple and universal formula (4). As we have seen, this is true for $p=1$ and $p=q-1$. We now *suggest that this is true for all p* .

This conjecture is further supported by the fact that is self-consistent. Indeed, it can be proved that *if* it is true for some filling ν , it is also true for filling $1-\nu$.

Of course, only numerical calculations with small numbers of particles within the proposed model will show whether this conjecture is correct or not.

4. Relation between the model and the true FQHE problem

Such a relation is provided by some fascinating properties of the matrix $\omega^{nm} = \exp(2\pi i n m / M)$. Consider a set of one-particle wavefunctions $\psi_n(m)$ of a discrete variable m , labelled by the index n :

$$\psi_n(m) = \frac{\omega^{nm}}{\sqrt{M}}, \quad n, m = 0 \dots M - 1. \quad (5)$$

This is analogous to plane waves, where both the momentum and the coordinate are discrete. Now introduce the operators X and Y , which shift by 1 the numbers m and n respectively, so that

$$\begin{aligned} X\psi_n(m) &= \omega^n \psi_n(m), \\ Y\psi_n(m) &= \psi_{n+1}(m). \end{aligned} \quad (6)$$

Obviously, $X^M=1$, $Y^M=1$, and $XY=\omega YX$. These are exactly the properties of the elementary *magnetic translations* in the x and y directions for an electron in magnetic field in the (twisted) torus geometry, if M is the number of magnetic fluxes through the surface of the torus (number of degenerate states in a given Landau level).

While Eqs. (5, 6) provide a direct link to the true FQHE problem on a torus, it still remains to be seen whether the true 2D problem can be in some sense reduced to the 1D model considered here.

5. References

- [1] Tsui D C, Stormer H L, and Gossard A C 1982, *Phys. Rev. Lett.* **48**, 1559
- [2] Laughlin R 1983, *Phys. Rev. Lett.* **50** 1395
- [3] Jain J K 1989, *Phys. Rev. Lett.* **63** 199
- [4] Jain J K 2007, *Composite Fermions* (New York, Cambridge University Press)
- [5] Dyakonov M I 2002, Twenty years since the discovery of the fractional Hall effect, *Recent Trends in Theory of Physical Phenomena in High Magnetic Fields*, ed E Schöll Wiley (Preprint arXiv:cond-mat/0209206)
- [6] Dyakonov M I 2002, *J. Phys. IV France* **12**, Pr9-373; (Preprint arXiv:1203.4838)
- [7] Dyson F J 1902, *J. Math. Phys.* **3** 140
- [8] Mehta M L 1991, *Random Matrices* (London, Academic press)