

An effective model of QCD thermodynamics¹

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Abstract. A combined effective model reproducing the equation of state of hadronic matter as obtained in recent lattice QCD simulations is presented. The model reproduces basic physical characteristics encountered in dense hadronic matter in the quark-gluon plasma (QGP) phase and the lower temperature hadron resonance gas phase. The hadronic phase is described by means of an extended Mott-Hagedorn resonance gas while the QGP phase is described by the extended PNJL model. The dissociation of hadrons is obtained by including the state dependent hadron resonance width.

1. Introduction

Simulations of lattice QCD (LQCD) are in practice the only reliable approach to QCD thermodynamics which covers the broad region of strongly interacting matter properties from the hadron gas at low temperatures to a deconfined quark gluon plasma phase at high temperatures. Recently, finite temperature LQCD simulations have overcome the difficulties of reaching the low physical light quark masses and approaching the continuum limit which makes this theoretical laboratory now a benchmark for modeling QCD under extreme conditions.

We are going to present a combined effective model reproducing the equation of state of hadronic matter as obtained in recent lattice QCD simulations [1, 2]. The model should reproduce basic physical characteristics of processes encountered in dense hadronic matter, from the hot QCD phase through the critical temperature region till the lower temperature hadron resonance gas phase. In medium properties of hadrons are different from those in the vacuum. The very notion of the mass shell should be modified, as was postulated quite long ago [3]. The interaction becomes effectively nonlocal due to the Mott effect and hadrons eventually gradually dissolve into quarks and gluons in the high temperature phase. Then, with the increasing temperature, quark masses are less and less important although the massless Stefan-Boltzmann limit would be eventually reached only at extremely high temperature.

It has been shown that the equation of state derived from that time QCD lattice calculation [4] can be reproduced by a simple hadron gas resonance model. The rapid rise of the number of degrees of freedom in lattice QCD data around the critical temperature $T_c \sim 150 - 170$ MeV, can be explained quantitatively by a resonance gas below the critical temperature T_c [5, 6].

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For higher temperatures the model is modified by introducing finite widths of heavy hadrons [7, 8] with a heuristic ansatz for the spectral function which reflects medium modifications of hadrons. This fits nicely the lattice data, also above T_c as is shown in Fig. 1.

This Mott-Hagedorn type model [9] has been constructed to fit nicely the lattice data, also above T_c where it does so because it leaves light hadrons below a mass threshold of $m_0 = 1$ GeV unaffected. The description of the lattice data at high temperatures is accidental because the effective number of those degrees of freedom approximately coincides with that of quarks and gluons. The QGP presence in this region is formally simulated here by the smart choice of the mass cut-off parameter such that cut-off defined stable light hadrons provide the same number of degrees of freedom as partonic components of the QGP.

The considered model is here gradually refined to take into account those physical processes present in the full QCD treatment. The uniform treatment of all hadronic resonances, without artificial stability island, is reached by a state-dependent hadron resonance width $\Gamma_i(T)$ given by the inverse collision time scale in a resonance gas [10].

In order to remove this unphysical aspect of the otherwise appealing model one has to extend the spectral broadening also to the light hadrons and thus describe their disappearance due to the Mott effect while simultaneously the quark and gluon degrees of freedom appear at high temperatures due to chiral symmetry restoration and deconfinement.

In the present contribution we will report results obtained by introducing a unified treatment of all hadronic resonances with a state-dependent width $\Gamma_i(T)$ in accordance with the inverse hadronic collision time scale from a recent model for chemical freeze-out in a resonance gas [10]. The appearance of quark and gluon degrees of freedom is introduced by the Polyakov-loop improved Nambu–Jona-Lasinio (PNJL) model [11, 12]. The model is further refined by adding perturbative corrections to $\mathcal{O}(\alpha_s)$ for the high-momentum region above the three-momentum cutoff inherent in the PNJL model. One obtains eventually a good agreement with lattice QCD data, comparable with all important physical characteristics taken into account.

2. Extended Mott-Hagedorn resonance gas

2.1. No quarks and gluons; hadronic spectral function with state-independent ansatz

We introduce the width Γ of a resonance in the statistical model through the spectral function

$$A(M, m) = N_M \frac{\Gamma \cdot m}{(M^2 - m^2)^2 + \Gamma^2 \cdot m^2} , \quad (1)$$

a Breit-Wigner distribution of virtual masses with a maximum at $M = m$ and the normalization factor

$$\begin{aligned} N_M &= \left[\int_{m_0^2}^{\infty} d(M^2) \frac{\Gamma \cdot m}{(M^2 - m^2)^2 + \Gamma^2 \cdot m^2} \right]^{-1} \\ &= \frac{1}{\frac{\pi}{2} + \arctan\left(\frac{m^2 - m_0^2}{\Gamma \cdot m}\right)} . \end{aligned} \quad (2)$$

And the model ansatz for the resonance width Γ is given by [7]

$$\Gamma(T) = C_\Gamma \left(\frac{m}{T_H} \right)^{N_m} \left(\frac{T}{T_H} \right)^{N_T} \exp\left(\frac{m}{T_H} \right) , \quad (3)$$

where $C_\Gamma = 10^{-4}$ MeV, $N_m = 2.5$, $N_T = 6.5$ and the Hagedorn temperature $T_H = 165$ MeV.

The energy density of this model with zero resonance proper volume for given temperature T and chemical potentials: μ_B for baryon number and μ_S for strangeness, can be cast in the form

$$\begin{aligned} \varepsilon(T, \mu_B, \mu_S) &= \sum_{i: m_i < m_0} g_i \varepsilon_i(T, \mu_i; m_i) \\ &+ \sum_{i: m_i \geq m_0} g_i \int_{m_0^2}^{\infty} d(M^2) A(M, m_i) \varepsilon_i(T, \mu_i; M), \end{aligned} \quad (4)$$

where $m_0 = 1$ GeV and the energy density per degree of freedom with a mass M is

$$\varepsilon_i(T, \mu_i; M) = \int \frac{d^3k}{(2\pi)^3} \frac{\sqrt{k^2 + M^2}}{\exp\left(\frac{\sqrt{k^2 + M^2} - \mu_i}{T}\right) + \delta_i}, \quad (5)$$

with the degeneracy g_i and the chemical potential $\mu_i = B_i \cdot \mu_B + S_i \cdot \mu_S$ of hadron i . For mesons, $\delta_i = -1$ and for baryons $\delta_i = 1$. According to Eq. (1) the energy density of hadrons consists of the contribution of light hadrons for $m_i < m_0$ and the contribution of heavier hadrons smeared with the spectral function for $m_i \geq m_0$.

For simplicity, we assume $n_S = 0$ for the strangeness number density and $n_B = 0$ for the baryon number density. Then $\mu_B = 0$ and $\mu_S = 0$ always, so the temperature is the only significant statistical parameter here. In such a case and for a fixed volume we have

$$\varepsilon + P = T \cdot s = T \cdot \frac{\partial P}{\partial T}, \quad (6)$$

where $P = P(T)$ and $s = s(T)$ are the pressure and entropy density, respectively. This is the first order ordinary differential equation for the pressure and the general solution reads

$$P(T) = \frac{T}{T_0} \cdot P_0 + T \int_{T_0}^T dT' \frac{\varepsilon(T')}{T'^2}, \quad (7)$$

where $P_0 = P(T_0)$. To have well-defined solution for the initial temperature $T_0 = 0$ one has to assume that $\lim_{T_0 \rightarrow 0} P(T_0)/T_0 = s_0 < \infty$. Then

$$P(T) = s_0 \cdot T + T \int_0^T dT' \frac{\varepsilon(T')}{T'^2}. \quad (8)$$

And the entropy density reads:

$$s(T) = \frac{\partial P}{\partial T} = s_0 + \int_0^T dT' \frac{\varepsilon(T')}{T'^2} + \frac{\varepsilon(T)}{T}, \quad (9)$$

where $s(0) = s_0$. We put $s_0 = 0$ as suggested by the Nernst postulate. The sound velocity squared is given by

$$c_s^2 = \frac{\partial P}{\partial \varepsilon}. \quad (10)$$

In Fig. 1 we show the results for the thermodynamic quantities (pressure, energy density and squared speed of sound) of the MHRG model at this stage. The nice correspondence with results from lattice QCD is not accidental for the temperature region $T \sim T_c \sim 200$ MeV, where has

been shown in [1] that a hadron resonance gas perfectly describes the lattice QCD data. For $T > T_c$ the broadening of the spectral function (1) which affects at this stage of the model only the hadronic resonances with $m > m_0$ leads to the vanishing of their contribution at about $2T_c$ while the light hadrons with masses $m < m_0$ are not affected and gradually reach the Stefan-Boltzmann (SB) limit determined by their number of degrees of freedom. As has been noted in [13], this number ($\sum_{i=\pi,K,\eta,f_0,\rho,\omega,K^*,\eta',f_0,a_0} g_i = 3 + 4 + 1 + 1 + 9 + 3 + 6 + 1 + 1 + 3 = 32$) accidentally (or by duality arguments) coincides with that of the quarks and gluons ($\sum_{i=q,g} g_i = 7/8 * N_c * N_f * N_s * 2 + (N_c^2 - 1) * 2 = 31.5$) for $N_c = N_f = 3$. Therefore, imposing that all mesons lighter than $m_0 = 1$ GeV are stable provides us with a SB limit at high temperatures which fakes that of quarks and gluons in the case for three flavors.

Although providing us with an excellent fit of the lattice data, the high-temperature phase of this model is unphysical since it ignores the Mott effect for light hadrons. Due to the chiral phase transition at T_c , the quarks loose their mass and therefore the threshold of the continuum of quark-antiquark scattering states is lowered. At the same time the light meson masses, however, remain almost unaffected by the increase in the temperature of the system. Consequently, they merge the continuum and become unbound - their spectral function changes from a delta-function (on-shell bound states) to a Breit-Wigner-type (off-shell, resonant scattering states). This phenomenon is the hadronic analogue [14] of the Mott-Anderson transition for electrons in solid state physics (insulator-conductor transition).

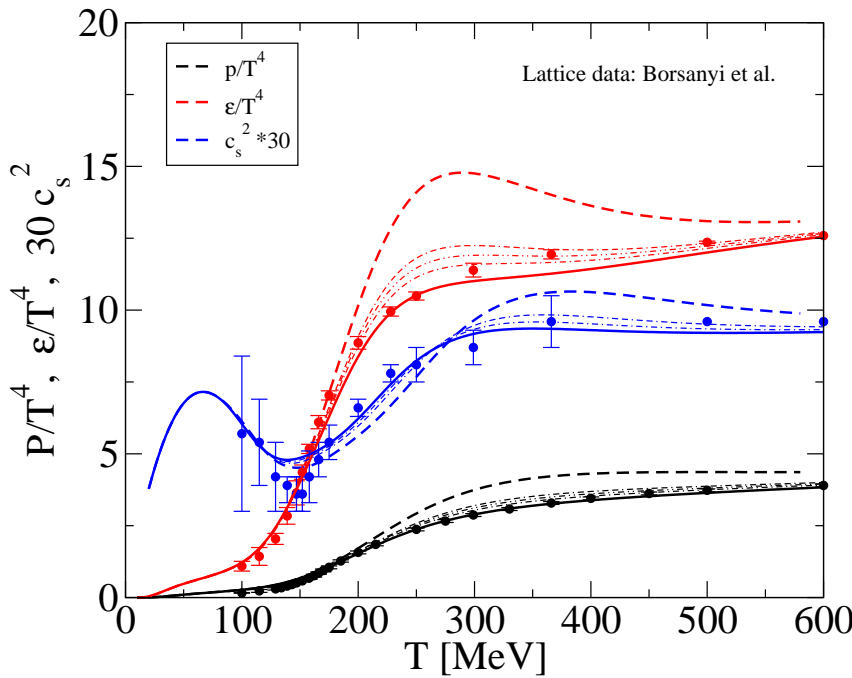


Figure 1. (Color online) Thermodynamic quantities for the old Mott-Hagedorn Resonance Gas model [7]. Different line styles correspond to different values for the parameter N_m in the range from $N_m = 2.5$ (dashed line) to $N_m = 3.0$ (solid line). Lattice QCD data are from Ref. [1].

It has been first introduced for the hadronic-to-quark-matter transition in [15]. Later, within the NJL model, a microscopic approach to the thermodynamics of the Mott dissociation of mesons in quark matter has been given in the form of a generalized Beth-Uhlenbeck equation of state [16], see also [17].

2.2. Hadronic spectral function with state-dependent ansatz

As a microscopic treatment of the Mott effect for all resonances is presently out of reach, we introduce an ansatz for a state-dependent hadron resonance width $\Gamma_i(T)$ given by the inverse collision time scale recently suggested within an approach to the chemical freeze-out and chiral condensate in a resonance gas [10]

$$\Gamma_i(T) = \tau_{\text{coll},i}^{-1}(T) = \sum_j \lambda \langle r_i^2 \rangle_T \langle r_j^2 \rangle_T n_j(T) , \quad (11)$$

which is based on a binary collision approximation and relaxation time ansatz using for the in-medium hadron-hadron cross sections the geometrical Povh-Hüfner law [18]. In Eq. (11) the coefficient λ is a free parameter, $n_j(T)$ is the partial density of the hadron j and the mean squared radii of hadrons $\langle r_i^2 \rangle_T$ obtain in the medium a temperature dependence which is governed by the (partial) restoration of chiral symmetry. For the pion this was quantitatively studied within the NJL model [19] and it was shown that close to the Mott transition the pion radius is well approximated by

$$r_\pi^2(T) = \frac{3}{4\pi^2} f_\pi^{-2}(T) = \frac{3M_\pi^2}{4\pi^2 m_q} |\langle \bar{q}q \rangle_T|^{-1} . \quad (12)$$

Here the Gell-Mann–Oakes–Renner relation has been used and the pion mass shall be assumed chirally protected and thus temperature independent.

For the nucleon, we shall assume the radius to consist of two components, a medium independent hard core radius r_0 and a pion cloud contribution $r_N^2(T) = r_0^2 + r_\pi^2(T)$, where from the vacuum values $r_\pi = 0.59$ fm and $r_N = 0.74$ fm one gets $r_0 = 0.45$ fm. A key point of our approach is that the temperature dependent hadronic radii shall diverge when hadron dissociation (Mott effect) sets in, driven basically by the restoration of chiral symmetry. As a consequence, in the vicinity of the chiral restoration temperature all meson radii shall behave like that of the pion and all baryon radii like that of the nucleon.

The resulting energy density behaviour is shown in Fig. 2. This part of the model we call Mott-Hagedorn-Resonance-Gas (MHRG). When all hadrons are gone at $T \sim 250$ MeV, we are clearly missing degrees of freedom!

3. Quarks, gluons and hadron resonances

We improve the PNJL model over its standard versions [11, 12] by adding perturbative corrections in $\mathcal{O}(\alpha_s)$ for the high-momentum region above the three-momentum cutoff Λ . In the second step, the MHRG part is replaced by its final form, using the state-dependent spectral function for the description of the Mott dissociation of all hadron resonances above the chiral transition. The total pressure obtains the form

$$P(T) = P_{\text{MHRG}}(T) + P_{\text{PNJL,MF}}(T) + P_2(T) . \quad (13)$$

where $P_{\text{MHRG}}(T)$ stands for the pressure of the MHRG model, accounting for the dissociation of hadrons in hot dense matter.

The $\mathcal{O}(\alpha_s)$ corrections can be split in quark and gluon contributions

$$P_2(T) = P_2^{\text{quark}}(T) + P_2^{\text{gluon}}(T) , \quad (14)$$

where P_2^{quark} stands for the quark contribution and P_2^{gluon} contains the ghost and gluon contributions. The total perturbative QCD correction to $\mathcal{O}(\alpha_s)$ is

$$P_2 = -\frac{8}{\pi} \alpha_s T^4 (I_\Lambda^+ + \frac{3}{\pi^2} ((I_\Lambda^+)^2 + (I_\Lambda^-)^2)) , \quad (15)$$

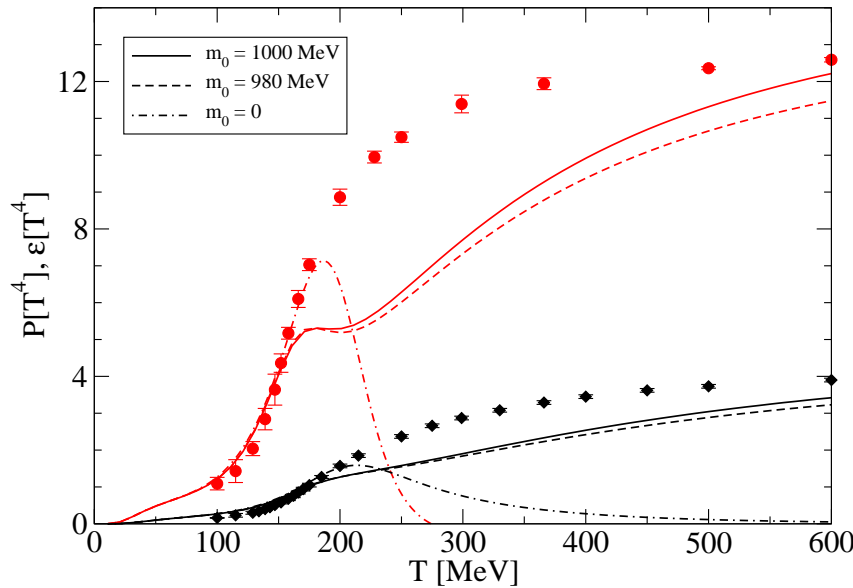


Figure 2. (Color online) Energy density (red lines and symbols) and pressure (black lines and symbols) for the state-dependent width model of Eq. (11) and three values of the mass threshold m_0 : 1 GeV (solid lines), 980 MeV (dashed lines), 0 (dash-dotted lines). Lattice QCD data are from Ref. [1].

where $I_{\Lambda}^{\pm} = \int_{\Lambda/T}^{\infty} \frac{dx}{e^x \pm 1}$. The corresponding contribution to the energy density is given in standard way by Eq. (6).

We will now include an effective description of the dissociation of hadrons due to the Mott effect into the hadron resonance gas model by including the state dependent hadron resonance width (11) into the definition of the HRG pressure

$$P_{\text{MHRG}}(T) = \sum_i \delta_i d_i \int \frac{d^3p}{(2\pi)^3} dM A_i(M) T \ln \left(1 + \delta_i e^{-\sqrt{p^2 + M^2}/T} \right). \quad (16)$$

From the pressure as a thermodynamic potential all relevant thermodynamical functions can be obtained. Combining the α_s corrected meanfield PNJL model for the quark-gluon subsystem with the MHRG description of the hadronic resonances we obtain the results shown in the right panel of Fig. 1 where the resulting partial contributions in comparison with lattice QCD data from Ref. [1] are shown.

We see that the lattice QCD thermodynamics is in full accordance with a hadron resonance gas up to a temperature of ~ 170 MeV which corresponds to the pseudocritical temperature of the chiral phase transition. The lattice data saturate below the Stefan-Boltzmann limit of an ideal quark-gluon gas at high temperatures. The PNJL model, however, attains this limit by construction. The deviation is to good accuracy described by perturbative corrections to $\mathcal{O}(\alpha_s)$ which vanish at low temperatures due to an infrared cutoff procedure. The transition region $170 \leq T[\text{MeV}] \leq 250$ is described by the MHRG model, resulting in a decreasing HRG pressure which vanishes at $T \sim 250$ MeV.

We have presented two stages of an effective model description of QCD thermodynamics at finite temperatures which properly accounts for the fact that in the QCD transition region it is dominated by a tower of hadronic resonances. To this end we have further developed a generalization of the Hagedorn resonance gas thermodynamics which includes the finite lifetime

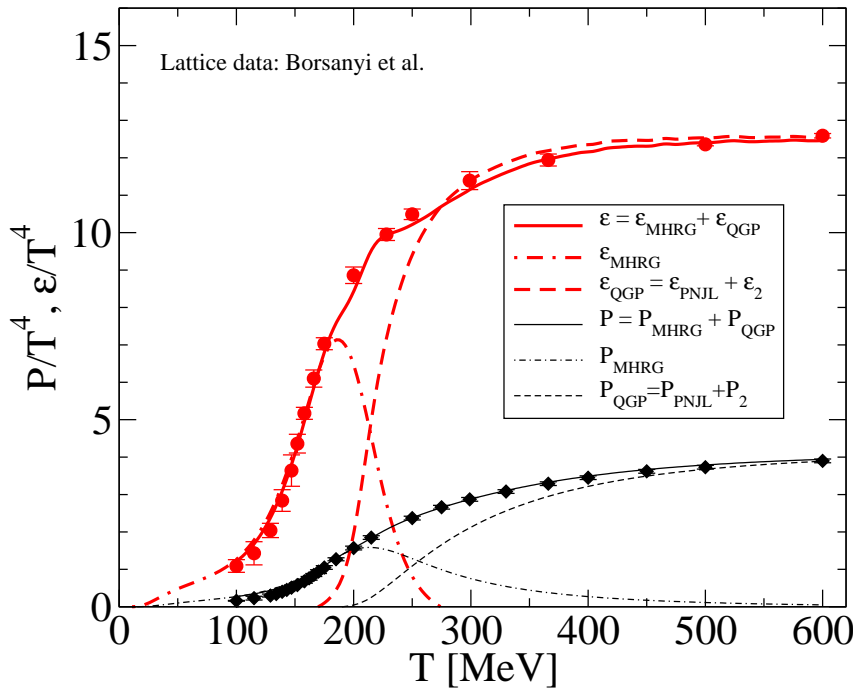


Figure 3. (Color online) Thermodynamic quantities for the new Mott-Hagedorn Resonance Gas where quark-gluon plasma contributions are described within the PNJL model including α_s corrections (dashed lines). Hadronic resonances are described within the resonance gas with finite width, as an implementation of the Mott effect (dash-dotted line). The sum of both contributions (solid lines) is shown for the energy density (thick lines) and pressure (thin lines) in comparison with the lattice data from [1].

of hadronic resonances in a hot and dense medium by a model ansatz for a temperature- and mass dependent spectral function.

4. Conclusion and outlook

After having presented the MHRG-PNJL model with the state-dependent spectral function approach we show the summary of the thermodynamic quantities in Fig. 4. We have presented two stages of an effective model description of QCD thermodynamics at finite temperatures which properly accounts for the fact that in the QCD transition region it is dominated by a tower of hadronic resonances. In the first of the two stages of the developments we presented here, we have used the fact that the number of low-lying mesonic degrees of freedom with masses below ~ 1 GeV approximately equals that of the thermodynamic degrees of freedom of a gas of quark and gluons. In the second one we have further developed a generalization of the Hagedorn resonance gas thermodynamics which includes the finite lifetime of heavy resonances in a hot and dense medium by a model ansatz for a temperature- and mass dependent spectral function which is motivated by a model for the collision time successfully applied in the kinetic description of chemical freeze-out from a hadron resonance gas. A next step should take into account also the effects of continuum correlations in hadronic scattering channels in accordance with the Levinson theorem [20] as discussed recently for the example of pion dissociation within the PNJL model [21]. Our account for $\mathcal{O}(\alpha_s)$ corrections from quark and gluon scattering in the plasma may be seen from this perspective.

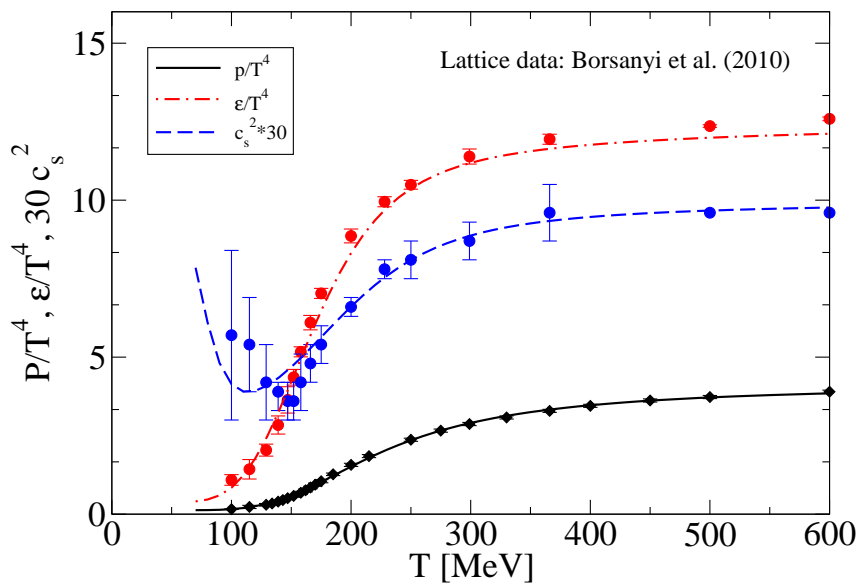


Figure 4. (Color online) Thermodynamic quantities as in Fig. 1 for the MHRG-PNJL model compared to lattice data from Ref. [1].

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