

Empirical study of the GARCH model with rational errors

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Abstract. We use the GARCH model with a fat-tailed error distribution described by a rational function and apply it to stock price data on the Tokyo Stock Exchange. To determine the model parameters we perform Bayesian inference to the model. Bayesian inference is implemented by the Metropolis-Hastings algorithm with an adaptive multi-dimensional Student's t-proposal density. In order to compare our model with the GARCH model with the standard normal errors, we calculate the information criteria AIC and DIC, and find that both criteria favor the GARCH model with a rational error distribution. We also calculate the accuracy of the volatility by using the realized volatility and find that a good accuracy is obtained for the GARCH model with a rational error distribution. Thus we conclude that the GARCH model with a rational error distribution is superior to the GARCH model with the normal errors and it can be used as an alternative GARCH model to those with other fat-tailed distributions.

1. Introduction

In finance volatility plays a central role for risk management such as derivative price estimation and portfolio allocation for which it is important to measure a reliable volatility from the data observed in the financial markets. Since volatility is not a direct observable in the financial markets we have to rely on a certain estimation technique. Parametric volatility models designed to capture asset return and volatility properties are often used in empirical finance. The most popular and successful model is the GARCH model[1], which is a generalized version of the ARCH model[2].

In the GARCH model asset returns r_t at time t are expressed as $r_t = \sigma_t \epsilon_t$ where σ_t^2 is the time-changing volatility which is given by a function of past returns and past volatilities. In the original GARCH model the standard normal distribution, i.e. $N(0, 1)$ was used for ϵ_t errors. It is known that the GARCH model well captures relevant properties of asset returns such as the fat-tailed behavior of the return distribution and the volatility clustering that are classified as stylized facts[3]. On the other hand in empirical studies it is often observed that the GARCH model does not sufficiently account for the leptokurtosis of the return distributions. To circumvent this it is advocated to apply a different distribution having a fatter tail than that of the normal distribution for the ϵ_t error. Several distributional forms having a fatter tail than the normal distribution such as Student's t-distribution[4] and the generalized error distribution



(GED)[5] are applied for the ϵ_t error term. By using the Student's t-distributions or GED for ϵ_t errors, usually one gets a better goodness-of-fit to the financial return data. However the Student's t-distributions and GED are not necessarily the optimal solution for the ϵ_t error term of the GARCH model and one could also choose other fat-tailed distributions.

In this study we apply Padé approximants described by a rational function for the ϵ_t error term. The Padé approximants are flexible to approximate a function in a certain domain. In finance Padé approximants are used to describe the interest rate return distributions[6, 7], where the parameters of rational functions are obtained by fitting to the interest rate return distributions. Here we apply a rational function for the ϵ_t error of the GARCH model. In [8] the GARCH model with rational errors was investigated by using USD/JPN exchange rate returns and the goodness-of-fit by the Akaike information criterion (AIC)[9] and deviance information criterion (DIC)[10] showed that the GARCH model with rational errors is superior to the GARCH model with normal errors.

We further investigate the effectiveness of the GARCH model with rational errors by using stock return data on the Tokyo Stock Exchange. In this study in order to clarify the model-effectiveness, in addition to AIC and DIC, we utilize realized volatility which is a model-free estimate of the integrated volatility. Using realized volatility as a proxy of the true volatility we calculate the accuracy of the volatility by a loss function for both models and using the loss function we determine which model is more effective.

2. GARCH model with normal error distribution

Bollerslev introduced the GARCH(p,q) model[1] which is a generalized version of the ARCH model[2]. The GARCH(p,q) model is expressed as

$$y_t = \sigma_t \epsilon_t, \quad (1)$$

and

$$\sigma_t^2 = \omega + \sum_{i=1}^q \alpha_i y_{t-i}^2 + \sum_{i=1}^p \beta_i \sigma_{t-i}^2, \quad (2)$$

where α_i , β_i and ω are parameters of the GARCH model. These parameters are determined so that the model matches the return data. Since the volatility σ_t^2 should be positive the GARCH parameters are restricted to $\omega > 0$, $\alpha_i > 0$ and $\beta_i > 0$ to ensure a positive volatility. ϵ_t is an independent normal error following $N(0, 1)$ and the return time series is given by y_t . In this study we focus on the GARCH(1,1) model, i.e. $p = 1$ and $q = 1$, where the volatility process is given by

$$\sigma_t^2 = \omega + \alpha y_{t-1}^2 + \beta \sigma_{t-1}^2, \quad (3)$$

and hereafter the GARCH model simply denotes the GARCH(1,1) model. Moreover for the GARCH model with normal errors we call it the GARCH-N model.

3. GARCH model with rational error distribution

In general a rational function of the Padé approximants is expressed by two polynomial functions $T_M(x)$ and $B_N(x)$ as

$$P_{M,N}(x) = \frac{T_M(x)}{B_N(x)}, \quad (4)$$

where M and N stand for the degrees of the polynomial $T_M(x)$ and $B_N(x)$ respectively. In order to consider $P_{M,N}(x)$ as a probability distribution we have to impose conditions that it must be positive and normalized to 1. Furthermore similar to the normal distribution we assume that $P_{M,N}(x)$ takes a maximum value at the origin and it is symmetric to the $x = 0$ axis, i.e. $P_{M,N}(x) = P_{M,N}(-x)$. In [6] possible normalizable distributions with finite variances are derived

to approximate the interest rate distributions. The simplest normalized probability distribution with tunable parameters a_1 and a_2 is given by

$$P_{0,4}(x) = \frac{a_1}{\pi(1 + (a_1^2 + 2a_2)x^2 + a_2^2x^4)}. \quad (5)$$

The variance of this probability distribution is calculated to be $-1/a_2$. Since usually the variance of the GARCH error distribution is set to 1 we also set the variance of $P_{0,4}(x)$ to 1, i.e. $a_2 = -1$. Finally we obtain our rational error distribution for the GARCH model as

$$P(x) = \frac{a}{\pi(1 + (a^2 - 2)x^2 + x^4)}. \quad (6)$$

When we use the rational error distribution of (6) for ϵ_t of the GARCH model we call it the GARCH model with rational errors (GARCH-RE model)[8].

4. Realized volatility

Recent availability of high frequency financial data enables us to calculate the realized volatility constructed as a sum of squared intraday returns[11, 12, 13, 14], see also e.g.[15]. Let us assume that the logarithmic price process $\ln p(s)$ follows a continuous time stochastic diffusion,

$$d \ln p(s) = \tilde{\sigma}(s) dW(s), \quad (7)$$

where $W(s)$ stands for a standard Brownian motion and $\tilde{\sigma}(s)$ is a spot volatility at time s . Under this assumption the integrated volatility defined by

$$\sigma_h^2(t) = \int_t^{t+h} \tilde{\sigma}(s)^2 ds, \quad (8)$$

where h stands for the interval to be integrated. In empirical finance "daily volatility" is of primary importance and for the daily volatility h takes one day. Since $\tilde{\sigma}(s)$ is latent and not observed in the financial markets, (8) can not be evaluated analytically.

Let us define a sampling period Δ by $\Delta = h/n$, i.e. we sample n returns in the time interval of h . Then the $i - th$ intraday return on the day t with Δ sampling period is given by a log-price difference as

$$r_{t+i\Delta} = \ln P_{t+i\Delta} - \ln P_{t+(i-1)\Delta}, \quad (9)$$

where P_t is an asset price at time t . Using these intraday returns the realized volatility RV_t on the day t is given by a sum of squared intraday returns as

$$RV_t = \sum_{i=1}^n r_{t+i\Delta}^2. \quad (10)$$

Under ideal circumstance RV_t is proved to converge to the integrated volatility of (8) in the limit of $n \rightarrow \infty$. However in the real financial markets there exist several types of bias such as microstructure noise[16], and thus in the presence of the bias the convergence of RV_t to the integrated volatility is not guaranteed. Let us assume that the log-price observed in financial markets is contaminated with independent noise[17], i.e.

$$\ln P_t^* = \ln P_t + \xi_t, \quad (11)$$

where $\ln P_t^*$ is the observed log-price in the markets which consists of the true log-price $\ln P_t$ and noise ξ_t with mean 0 and variance ρ^2 . Under this assumption the observed return r_t^* is given by

$$r_t^* = r_t + \eta_t, \quad (12)$$

where $\eta_t = \xi_t - \xi_{t-\Delta}$. Thus RV_t^* actually observed from the market data is obtained as a sum of the squared returns r_t^* ,

$$RV_t^* = \sum_{i=1}^n (r_{t+i\Delta}^*)^2, \quad (13)$$

$$= RV_t + 2 \sum_{i=1}^n r_{t+i\Delta} \eta_{t+i\Delta} + \sum_{i=1}^n \eta_{t+i\Delta}^2. \quad (14)$$

With these independent noises the bias appears as $\sum_{i=1}^n \eta_{t+i\Delta}^2$ which corresponds to $\sim 2n\rho^2$. Thus due to the bias the RV_t^* diverges as $n \rightarrow \infty$.

Practically in order to avoid the distortion from microstructure noise one needs to choose a good sampling period which reduces the microstructure noise bias and at the same time to maintain the accuracy of the realized volatility. The optimal sampling period is suggested to be around 5min[18]. One could also use kernel-based estimations which are designed to reduce the microstructure noise[17, 19, 20].

Another type of bias is due to "non-trading hours". Since stock markets are not open 24 hours the high-frequency data are only available for a part of 24 hours. At the Tokyo stock exchange market domestic stocks are traded in the two trading sessions: (a) morning trading session 9:00-11:00. (b) afternoon trading session 12:30-15:00. The daily realized volatility calculated without including intraday returns during the non-trading periods can be underestimated. When we consider volatility only in each trading session[21, 22] this bias problem does not arise. Otherwise we need to deal with this bias appropriately.

Hansen and Lunde[23] advocated an idea to circumvent the problem by introducing an adjustment factor which modifies the realized volatility so that the average of the realized volatility matches the variance of the daily returns. Let (R_1, \dots, R_N) be N daily returns constructed by close-close daily log-price difference. The adjustment factor c (HL adjustment factor) is given by

$$c = \frac{\sum_{t=1}^N (R_t - \bar{R})^2}{\sum_{t=1}^N RV_t}, \quad (15)$$

where \bar{R} denotes the average of R_t . Then using this factor the daily realized volatility is modified to cRV_t . Although originally the HL adjustment factor is introduced to correct the bias of the non-trading hours it can also correct the microstructure noise bias effects to some extent.

5. Empirical results

In this study we analyze the stock price data of Panasonic Co. traded on the Tokyo Stock Exchange. This stock is listed in the Topix core 30 index which includes the 30 most liquid

Table 1. Results of the Bayesian inference for GARCH-RE and GARCH-N models. The values marked by * show the autocorrelation time τ_{int} defined by $\tau_{int} = 1 + 2 \sum_{t=1}^{\infty} ACF(t)$, where $ACF(t)$ stands for the autocorrelation function.

| | GARCH-RE | | GARCH-N | |
|----------|---------------------------|----------|-------------------------|---------|
| α | 0.132(38) | 5.4(10)* | 0.148(31) | 2.7(3)* |
| β | 0.858(41) | 5.8(12)* | 0.836(33) | 2.8(2)* |
| ω | $2.8(1.2) \times 10^{-5}$ | 6.5(16)* | $1.3(5) \times 10^{-5}$ | 3.3(4)* |
| a | 1.57(9) | 4.2(6)* | — | — |
| AIC | -4151.29 | | -4148.35 | |
| DIC | -4156.30 | | -4151.98 | |

and highly market capitalized stocks. Our data set begins June 3, 2006 and ends December 30, 2009. Figure 1 shows the daily return time series of Panasonic Co. We apply the GARCH-RE and GARCH-N models for the daily returns shown in Figure 1 and estimate the daily volatilities corresponding to those daily returns. The parameter estimation of the GARCH-RE and GARCH-N models is conducted by the Bayesian inference. A popular approach to perform the the Bayesian inference is the Markov Chain Monte Carlo (MCMC) methods. Since there exist a variety of MCMC methods we need to choose an adequate method for the Bayesian inference of the GARCH model. We perform the Bayesian inference by the Metropolis-Hastings algorithm[24, 25] with an adaptive multi-dimensional Student's t-proposal density (MHAS algorithm)[26, 27, 28, 29]. In the MH algorithm we need to specify the proposal density. In [30] the proposal densities constructed from an auxiliary process are used for the MH algorithm. References [31, 32] use a multi-dimensional Student's t-proposal density for which density parameters are determined by the maximum likelihood method. Here we use MHAS algorithm where density parameters of a multi-dimensional Student's t-proposal density are determined adaptively during the Monte Carlo simulations so that the multi-dimensional Student's t-proposal density matches the posterior distributions of the model. The MHAS algorithm has been shown to be very efficient for the Bayesian inference of the GARCH models[26, 27, 28, 29]. The implementation of the MHAS algorithm was done as follows. We discarded the first 6000 Monte Carlo updates by the MHAS algorithm. Then we accumulated 50000 Monte Carlo samples for analysis. Table 1 shows the values of the parameters averaged over the Monte Carlo samples. The values marked by * show the autocorrelation time of the Monte Carlo data generated by the MHAS algorithm. We find that the values of the autocorrelation time are small which indicates that the MHAS algorithm generates effectively uncorrelated Monte Carlo samples.

In order to compare the goodness-of-fit of the models we utilize two information criteria: AIC[9] and DIC[10]. The AIC is defined by $AIC = -\ln L(\bar{\theta}) - 2k$ where k is the number of the parameters of the model and $L(\bar{\theta})$ is the likelihood function of the model at $\bar{\theta}$. θ stands for $\theta = (\alpha, \beta, \omega, a)$ for the GARCH-RE model and $\theta = (\alpha, \beta, \omega)$ for the GARCH-N model.

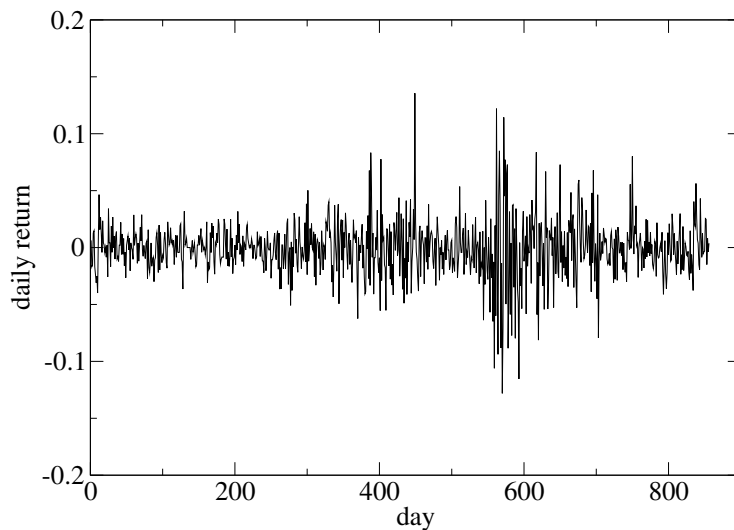


Figure 1. Daily return time series of Panasonic Co.

$\bar{\theta}$ stands for the parameters averaged over the Monte Carlo samples. The DIC is defined by $2[\ln L(\bar{\theta}) - 2E(\ln L(\theta))]$ where $E(\ln L(\theta))$ is the Monte Carlo average of $\ln L(\theta)$. For both AIC and DIC the model with the smallest value is chosen as the one which would best predict the time series. As seen in Table 1 both of AIC and DIC give smaller values for the GARCH-RE model. Thus we find that the GARCH-RE model is superior to the GARCH-N model.

Next we compare the GARCH-RE and GARCH-N models with the accuracy of the volatility. To do that we measure the difference between the volatility from the models and the true volatility. Since we do not know the value of the true volatility we use the realized volatility as a proxy of the true volatility. The realized volatility is constructed by a sum of squared intraday returns as (10). Figure 2 shows the average realized volatility as a function of sampling period, i.e. the realized volatility is averaged at each sampling period. Such plot is call "volatility signature plot" advocated in [33] to visualize the microstructure noise bias on the realized volatility. As expected in (14) we find that the realized volatility diverges at small sampling period (or at high sampling frequency). In this study we correct this bias with the HL adjustment factor which adjusts the average of the realized volatility to the variance of the daily return.

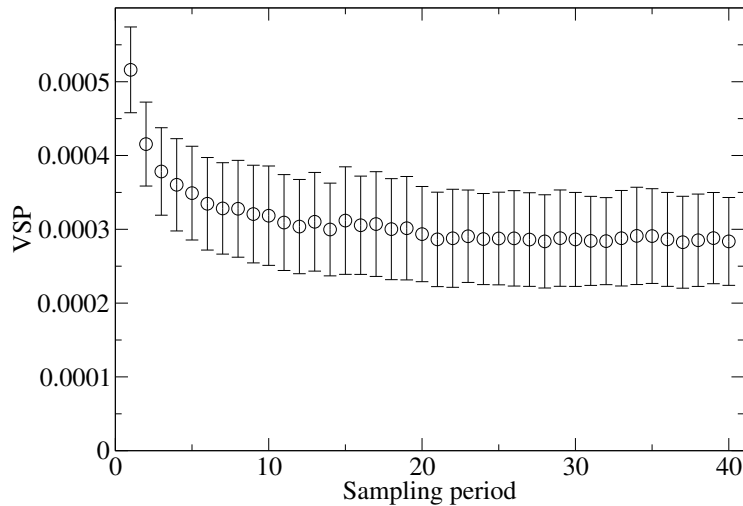


Figure 2. Volatility signature plot: Average realized volatility at each sampling period.

Figure 3 shows the HL adjustment factor as a function of sampling period. The HL adjustment factor decreases as the sampling period decreases. This decrease is explained by the microstructure noise bias which inflates the realized volatility at small sampling periods. As the sampling period increases the HL adjustment factor reaches a plateau around 2 where the microstructure noise bias effects are expected to be small. This factor of 2 means that the original realized volatility is underestimated due to non-trading hours and the size of the volatility during no-trading hours is about the same size as that during the trading hours. On the Tokyo Stock Exchange there are two non-trading periods: lunch break and night break. Since during the lunch break the size of the volatility is observed to be small[21] the dominant contribution to the factor of 2 comes from the night break.

Figure 4 compares volatilities from the GARCH-RE and GARCH-N models, and the realized volatility at 1-min sampling period. To quantify the accuracy of the volatility we measure a loss

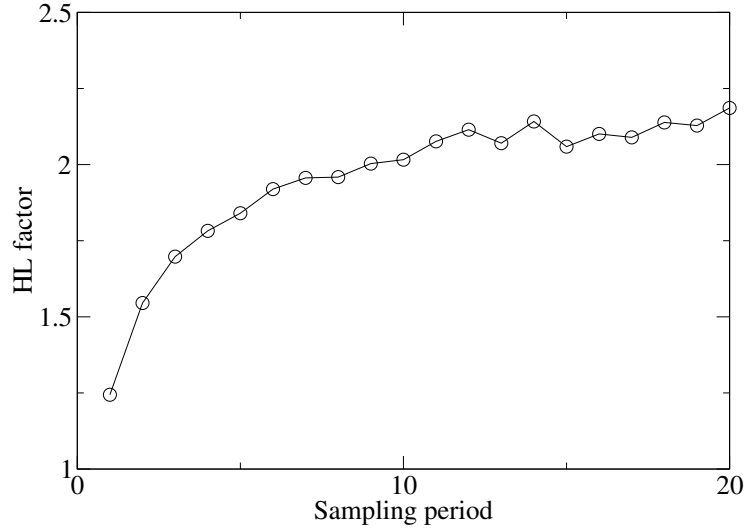


Figure 3. HL adjustment factor as a function of sampling period.

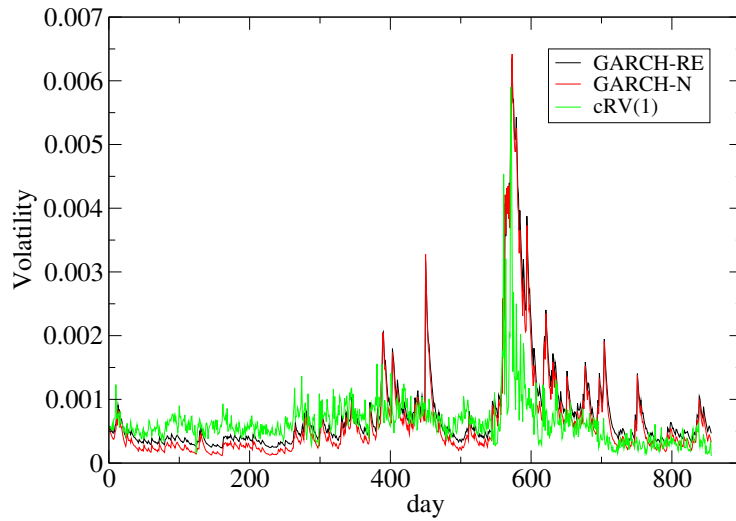


Figure 4. Volatility from GARCH-RE, GARCH-N models and the realized volatility at 1-min sampling period. The realized volatility is adjusted by the HL adjustment factor.

function of the root mean square percentage error (RMSPE) defined by

$$RMSPE = \left(\sum_{t=1}^N \left(\frac{\bar{\sigma}_t^2 - cRV_t}{cRV_t} \right)^2 \right)^{1/2}, \quad (16)$$

where $\bar{\sigma}_t^2$ is the volatility estimated from the Bayesian inference of the GARCH-RE or GARCH-N models. $\bar{\sigma}_t^2$ is also adjusted so that the average of $\bar{\sigma}_t^2$, i.e. $\sum_{t=1}^N \bar{\sigma}_t^2 / N$ matches the variance

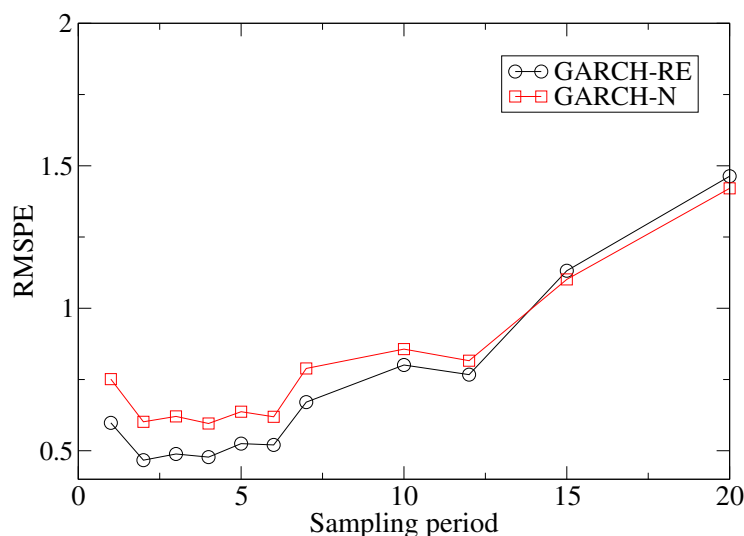


Figure 5. RMSPE of GARCH-RE and GARCH-N models as a function of sampling period.

of the daily returns. Figure 5 shows RMSPE of the GARCH-RE and GARCH-N models. We find that RMSPE takes a minimum around 1 to 6-min sampling periods where the GARCH-RE model gives smaller values. It is also noted that the sampling frequencies which take the minimum of the RMSPE are very similar to the optimum sampling frequencies obtained from the mean squared error of the realized volatility[18]. Our result of the RMSPE also indicates that the GARCH-RE model is more effective than the GARCH-N model.

6. Conclusions

We performed Bayesian inference of the GARCH-RE model and the GARCH-N model for the stock price data of Panasonic Co. on the Tokyo Stock Exchange. The Bayesian inference is implemented by the MHAS algorithm. In order to compare models we calculated the information criteria AIC and DIC, and find that both criteria favor the GARCH-RE model. We also calculated the accuracy of the volatility by the RMSPE and found that the smaller RMSPE is obtained for the GARCH-RE model. Thus we conclude that the GARCH-RE model is superior to the GARCH-N model and it can be used as an alternative GARCH model to those with other fat-tailed distributions.

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