Journal of Physics: Conference Series **453** (2013) 012009 doi:10.1088/1742-6596/453/1/012009

Using Noether symmetries to specify f(R) gravity

Andronikos Paliathanasis

Department of Physics, Section of Astronomy, Astrophysics and Mechanics, University of Athens, Panepistemiopolis, Athens 157 83, Greece

E-mail: anpaliat@phys.uoa.gr

Abstract. A detailed study of the modified gravity, f(R) models is performed, using the fact that the Noether point symmetries of these models are geometric symmetries of the mini superspace of the theory. It is shown that the requirement that the field equations admit Noether point symmetries selects definite models in a self-consistent way. As an application in Cosmology we consider the Friedman -Robertson-Walker spacetime and show that the only cosmological model which is integrable via Noether point symmetries is the $\left(R^b-2\Lambda\right)^c$ model, which generalizes the Lambda Cosmology. Furthermore using the corresponding Noether integrals we compute the analytic form of the main cosmological functions.

Keywords: General Relativity, Modified Gravity, Noether symmetries
Pacs - numbers:98.80.-k,95.35.+d,95.36.+x

1. Introduction

The recent cosmological data indicate that the universe is spatially flat and has suffered two acceleration phases. An early acceleration phase (inflation), which occurred prior to the radiation dominated era and a recently initiated accelerated expansion.

An easy way to explain this expansion is to consider an additional fluid with negative equation of state parameter, usually called dark energy, that dominates the universe at late times. In spite of that, the absence of a fundamental physical theory, regarding the mechanism inducing the cosmic acceleration, has given rise to a plethora of alternative cosmological scenarios. Most of them are based either on the existence of new fields in nature (dark energy) or in some modification of Einstein's general relativity (GR), with the present accelerating stage appearing as a sort of geometric effect ("geometrical" dark energy).

The simplest dark energy probe is the cosmological constant Λ (vacuum) leading to the ΛCDM cosmology [1–3]. However, it has been shown that ΛCDM cosmology suffers from two major drawbacks known as the fine tuning problem and the coincidence problem [4]. Besides ΛCDM cosmology, many other candidates have been proposed in the literature, such as time-varying $\Lambda(t)$ cosmologies, quintessence, $k-$essence, tachyons, modifications of gravity, Chaplygin gas and others [5–10].

There are other possibilities to explain the present accelerating stage. For instance, one may consider that the dynamical effects attributed to dark energy can be resembled by the effects of a nonstandard gravity theory. In other words, the present accelerating stage of the universe can be driven only by cold dark matter, under a modification of the nature of gravity. Such a reduction of the so-called dark sector is naturally obtained in the $f(R)$ gravity theories [11]. In the original nonstandard gravity models, one modifies the Einstein-Hilbert action with a general function


Content from this work may be used under the terms of the Creative Commons Attribution 3.0 licence. Any further distribution of this work must maintain attribution to the author(s) and the title of the work, journal citation and DOI.
Published under licence by IOP Publishing Ltd

$f(R)$ of the Ricci scalar R. The $f(R)$ approach is a relatively simple but still a fundamental tool used to explain the accelerated expansion of the universe. A pioneering fundamental approach was proposed long ago with $f(R) = R + mR^2$ [12]. Later on, the $f(R)$ models were further explored from different points of view in [13–15] and indeed a large number of functional forms of $f(R)$ gravity is currently available in the literature [16–21].

The aim of the present work is to investigate which $f(R)$ models admit extra Noether point symmetries and use the first integrals of these models to determine analytic solutions of their field equations. The idea to use Noether symmetries in cosmological studies is not new and indeed a lot of attention has been paid in the literature (see [22–37]).

The main reasons for the consideration of this hypothesis is that (a) the Noether point symmetries provide integrals, which assist the integrability of the system, (b) is a geometric criterion because the Noether symmetries are associated with the geometry of the field equations. A fundamental approach to derive the Noether point symmetries of a given dynamical system moving in a Riemannian space has been proposed in [38]. A similar analysis can be found in [39, 40].

The structure of the paper is as follows. The basic theoretical elements of the problem are presented in section 2, where we also introduce the basic FRW cosmological equations in the framework of $f(R)$ models. The geometrical Noether point symmetries and their connections to the $f(R)$ models are discussed in sections 5. In section 6 we provide analytical solutions for those $f(R)$ models which are Liouville integrable via Noether point symmetries. In section 7 we study the Noether symmetries in spatially non flat $f(R)$ cosmological models. Finally, we draw our main conclusions in section 8.

2. Cosmology with a modified gravity

Consider the modified Einstein-Hilbert action:

$$S = \int d^4x\sqrt{-g}\left[\frac{1}{2k^2}f\left(R\right) + \mathcal{L}_m\right] \tag{1}$$

where $\mathcal{L}_m$ is the Lagrangian of dust-like ($p_m = 0$) matter and $k^2 = 8\pi G$. Varying the action with respect to the metric[1] we arrive at

$$(1 + f')G^{\mu}_{\nu} - g^{\mu\alpha}f_{R,\,\alpha\,;\,\nu} + \left[\frac{2\Box f' - (f - Rf')}{2}\right]\delta^{\mu}_{\nu} = k^2\,T^{\mu}_{\nu} \tag{2}$$

where the prime denotes derivative with respect to R, G^{μ}_{ν} is the Einstein tensor and T^{μ}_{ν} is the ordinary energy-momentum tensor of matter. Based on the matter era we treat the expanding universe as a perfect fluid which includes only cold dark matter with comoving observers $U^{\mu} = \delta^{\mu}_{0}$. Thus the energy momentum tensor becomes $T_{\mu\nu} = \rho_m U_{\mu}U_{\nu}$, where ρ_m is the energy density of the cosmic fluid.

Now, in the context of a flat FRW model the metric is

$$ds^2 = -dt^2 + a^2(t)(dx^2 + dy^2 + dz^2). \tag{3}$$

The components of the Einstein tensor are computed to be:

$$G^0_0 = -3H^2,\; G^a_b = -\delta^a_b\left(2\dot{H} + 3H^2\right)\;. \tag{4}$$

[1] We use the metric i.e. the Hilbert variational approach.

Inserting (4) into the modified Einstein's field equations (2), for comoving observers, we derive the modified Friedman's equation

$$3f^{'}H^2 = k^2\rho_m + \frac{f^{'}R - f}{2} - 3Hf^{''}\dot{R} \tag{5}$$

$$2f^{'}\dot{H} + 3f^{'}H^2 = -2Hf^{''}\dot{R} - \left(f^{'''}\dot{R}^2 + f^{''}\ddot{R}\right) - \frac{f - Rf^{'}}{2}. \tag{6}$$

The contraction of the Ricci tensor provides the Ricci scalar

$$R = g^{\mu\nu}R_{\mu\nu} = 6\left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2}\right) = 6(2H^2 + \dot{H})\ . \tag{7}$$

The Bianchi identity $\bigtriangledown^{\mu}T_{\mu\nu} = 0$ leads to the matter conservation law:

$$\dot{\rho}_m + 3H\rho_m = 0 \tag{8}$$

whose solution is

$$\rho_m = \rho_{m0}a^{-3}. \tag{9}$$

Note that the over-dot denotes derivative with respect to the cosmic time t and $H \equiv \dot{a}/a$ is the Hubble parameter.

If we consider $f(R) = R$ then the field equations (2) boil down to the Einstein's equations a solution of which is the Einstein de Sitter model. On the other hand, the concordance Λ cosmology is fully recovered for $f(R) = R - 2\Lambda$.

From the current analysis it becomes clear that unlike the standard Friedman equations in Einstein's GR the modified equations of motion (5) and (6) are complicated and thus it is difficult to solve them analytically.

We would like to stress here that within the context of the metric formalism the above $f(R)$ cosmological models must obey simultaneously some strong conditions [42]. These are: (i) $f^{'} > 0$ for $R \geq R_0 > 0$, where R_0 is the Ricci scalar at the present time. If the final attractor is a de Sitter point we need to have $f^{'} > 0$ for $R \geq R_1 > 0$, where R_1 is the Ricci scalar at the de Sitter point, (ii) $f^{''} > 0$ for $R \geq R_0 > 0$, (iii) $f(R) \approx R - 2\Lambda$ for $R \gg R_0$ and finally (iv) $0 < \frac{Rf^{''}}{f^{'}}(r) < 1$ at $r = -\frac{Rf^{'}}{f} = -2$

3. Modified gravity versus symmetries

In the last decade a large number of experiments have been proposed in order to constrain dark energy and study its evolution. Naturally, in order to establish the evolution of the dark energy ("geometrical" in the current work) equation of state parameter a realistic form of $H(a)$ is required while the included free parameters must be constrained through a combination of independent DE probes (for example SNIa, BAOs, CMB etc). However, a weak point here is the fact that the majority of the $f(R)$ models appeared in the literature are plagued with no clear physical basis and/or many free parameters. Due to the large number of free parameters many such models could fit the data. The proposed additional criterion of Noether point symmetry requirement is a physically meaning-full geometric ansatz.

According to the theory of general relativity, the space-time symmetries (Killing and homothetic vectors) via the Einstein's field equations, are also symmetries of the energy momentum tensor. Due to the fact that the $f(R)$ models provide a natural generalization of GR one would expect that the theories of modified gravity must inherit the symmetries of the space-time as the usual gravity (GR) does.

Furthermore, besides the geometric symmetries we have to consider the dynamical symmetries, which are the symmetries of the field equations (Lie symmetries). If the field equations are derived from a Lagrangian then there is a special class of Lie symmetries, the Noether symmetries, which lead to conserved currents or, equivalently, to first integrals of the equations of motion. The Noether integrals are used to reduce the order of the field equations or even to solve them. Therefore a sound requirement, which is possible to be made in Lagrangian theories, is that they admit extra Noether symmetries. This assumption is model independent, because it is imposed after the field equations have been derived, therefore it does not lead to conflict with the geometric symmetries while, at the same time, serves the original purpose of a selection rule. Of course, it is possible that a different method could be assumed and select another subset of viable models. However, symmetry has always played a dominant role in Physics and this gives an aesthetic and a physical priority to our proposal.

In the Lagrangian context, the main field equations (5) and (6), described in section 2, can be produced by the following Lagrangian:

$$L\left(a,\dot{a},R,\dot{R}\right)=6af^{'}\ \dot{a}^{2}+6a^{2}f^{''}\ \dot{a}\dot{R}+a^{3}\left(f^{'}R-f\right) \tag{10}$$

in the space of the variables $\{a,R\}$. Using eq.(10) we obtain the Hamiltonian of the current dynamical system

$$E=6af^{'}\ \dot{a}^{2}+6a^{2}f^{''}\ \dot{a}\dot{R}-a^{3}\left(f^{'}R-f\right) \tag{11}$$

or

$$E=6a^{3}\left[f^{'}H^{2}-\frac{1}{6f^{'}}\left(\left(f^{'}R-f\right)-6\dot{R}Hf^{''}\right)\right]\ . \tag{12}$$

Combining the first equation of motion (5) with eq.(12) we find

$$\rho_{m}=\frac{E}{2k^{2}}\ a^{-3}\ . \tag{13}$$

The latter equation together with $\rho_m=\rho_{m0}a^{-3}$ implies that

$$\rho_{m0}=\frac{E}{2k^{2}}\Rightarrow\Omega_{m}\rho_{cr,0}=\frac{E}{2k^{2}}\Rightarrow E=6\Omega_{m}H_{0}^{2} \tag{14}$$

where $\Omega_m=\rho_{m0}/\rho_{cr,0}$, $\rho_{cr,0}=3H_0^2/k^2$ is the critical density at the present time and H_0 is the Hubble constant.

We note that the current Lagrangian eq.(10) is time independent implying that the dynamical system is autonomous hence the Hamiltonian E is conserved ($\frac{dE}{dt}=0$).

4. Noether symmetries

Before we proceed we review briefly the basic definitions concerning Lie and Noether point symmetries of systems of second order ordinary differential equations (ODEs)

$$\ddot{x}^{i}=\omega^{i}\left(t,x^{j},\dot{x}^{j}\right). \tag{15}$$

The one point parameter transformation

$$\bar{t}\ =\ t+\varepsilon\xi\left(t,x^{i}\right) \tag{16}$$

$$\bar{x}^{i}\ =\ x^{i}+\varepsilon\eta^{i}\left(t,x^{i}\right) \tag{17}$$

with generator $X = \xi\left(t, x^j\right)\partial_t + \eta^i\left(t, x\right)\partial_i$ is a Lie point symmetry of the system of ODEs (15) if the following condition is satisfied [39, 40]

$$X^{[2]}\left(\ddot{x}^i - \omega\left(t, x^j, \dot{x}^j\right)\right) = 0 \tag{18}$$

where $X^{[2]}$ is the second prolongation of X defined by the formula

$$X^{[2]} = \xi\partial_t + \eta^i\partial_i + \left(\dot{\eta}^i - \dot{x}^i\dot{\xi}\right)\partial_{\dot{x}^i} + \left(\ddot{\eta}^i - \dot{x}^i\ddot{\xi} - 2\ddot{x}^i\dot{\xi}\right)\partial_{\ddot{x}^i}. \tag{19}$$

Condition (18) is equivalent to the relation

$$\left[X^{[1]}, A\right] = \lambda\left(x^a\right)A \tag{20}$$

where $X^{[1]}$ is the first prolongation of X and A is the Hamiltonian vector field

$$A = \partial_t + \dot{x}\partial_x + \omega^i\left(t, x^j, \dot{x}^j\right)\partial_{\dot{x}^i}. \tag{21}$$

If the system of ODEs results from a first order Lagrangian $L = L\left(t, x^j, \dot{x}^j\right)$, then a Lie symmetry X of the system (15) is a Noether symmetry of the Lagrangian if the additional condition is satisfied

$$X^{[1]}L + L\frac{d\xi}{dt} = \frac{dg}{dt} \tag{22}$$

where $g = g\left(t, x^j\right)$ is the gauge function. To every Noether symmetry there corresponds a first integral (a Noether integral) of the system of equations (15) which is given by the formula:

$$I = \xi E_H - \frac{\partial L}{\partial\dot{x}^i}\eta^i + g \tag{23}$$

where E_H is the Hamiltonian of the Lagrangian

$$E_H = \dot{x}^i\frac{\partial L}{\partial x^i} - L \tag{24}$$

The vector field X in the augmented space $\{t, a, R\}$ is

$$X = \xi\left(t, a, R\right)\partial_t + \eta^{(1)}\left(t, a, R\right)\partial_a + \eta^{(2)}\left(t, a, R\right)\partial_R \tag{25}$$

and the first prolongation

$$X^{[1]} = \xi\partial_t + \eta^{(1)}\partial_a + \eta^{(2)}\partial_R + \left(\eta^{(1)} - \dot{a}\dot{\xi}\right)\partial_{\dot{a}} + \left(\dot{\eta}^{(2)} - \dot{R}\dot{\xi}\right)\partial_R. \tag{26}$$

Having given the basic formula for the Noether symmetries we look for analytic solutions of the dynamical system with Lagrangian (10) with the use of Noether Integrals.

5. Noether symmetries of $f\left(R\right)$ gravity

The Noether condition (22) for the Lagrangian (10) is equivalent with the following system of eight equations

$$\xi_{,a} = 0 \tag{27}$$
$$\xi_{,R} = 0 \tag{28}$$

$$a^2 f'' \eta^{(1)}_{,R} = 0 \tag{29}$$

$$f' \eta^{(1)} + a f'' \eta^{(2)} + 2a f' \eta^{(1)}_{,a} + a^2 f'' \eta^{(2)}_{,a} - \frac{1}{2} a f' \xi_{,t} = 0 \tag{30}$$

$$2a f'' \eta^{(1)} + a^2 f''' \eta^{(2)} + a^2 f'' \eta^{(1)}_{,a} + 2a f' \eta^{(1)}_{,R} + a^2 f'' \eta^{(2)}_{,R} - \frac{1}{2} a^2 f'' \xi_{,t} = 0 \tag{31}$$

$$-3a^2 R f' \eta^{(1)} + 3a^2 f \eta^{(1)} - a^3 R f'' \eta^{(2)} + a^3 \left(f - f^{'} R \right) \xi_{,t} + g_{,t} = 0 \tag{32}$$

$$12a f' \eta^{(1)}_{,t} + 6a^2 f'' \eta^{(2)}_{,t} + a^3 \left(f^{'} R - f \right) \xi_{,a} - g_{,a} = 0 \tag{33}$$

$$6a^2 f'' \eta^{(1)}_{,t} + a^3 \left(f^{'} R - f \right) \xi_{,R} - g_{,R} = 0 \tag{34}$$

The solution of the system (27)-(34) will determine the Noether symmetries.

Since the Lagrangian (10) is in the form $L = T\left(a, \dot{a}, R, \dot{R}\right) - V\left(a, R\right)$, the results of [38] can be used [2]. The kinematic term defines a two dimensional metric in the space of $\{a, R\}$ with line element

$$d\hat{s}^2 = 12 a f' da^2 + 12 a^2 f'' da \; dR \tag{35}$$

while the "potential" is

$$V(a, R) = -a^3 (f^{'} R - f) \; . \tag{36}$$

The Ricci scalar of the two dimensional metric (35) is computed to be $\hat{R} = 0$, therefore the space is a flat space[3] with a maximum homothetic algebra. The homothetic algebra of the metric (35) consists of the vectors

$$\mathbf{K}^1 = a\partial_a - 3\frac{f'}{f''}\partial_R \; , \; \mathbf{K}^2 = \frac{1}{a}\partial_a - \frac{1}{a^2}\frac{f'}{f''}\partial_R$$
$$\mathbf{K}^3 = \frac{1}{a}\frac{1}{f''}\partial_R \; , \; \mathbf{H} = \frac{a}{2} \; \partial_a + \frac{1}{2}\frac{f'}{f''}\partial_R$$

where $\mathbf{K}$ are Killing vectors ($\mathbf{K}^{2,3}$ are gradients) and $\mathbf{H}$ is a gradient Homothetic vector.

Therefore applying theorem 2 of [38] we have the following cases:

Case 1: If $f\left(R\right)$ is arbitrary the dynamical system admits as Noether symmetry the $X^1 = \partial_t$ with Noether integral the Hamiltonian E.

Case 2: If $f\left(R\right) = R^{\frac{3}{2}}$ the dynamical system admits the extra Noether symmetries

$$X^2 = \mathbf{K}^2, \; X^3 = t\mathbf{K}^2 \tag{37}$$

$$X^4 = 2t\partial_t + \mathbf{H} + \frac{5}{6}\mathbf{K}^1. \tag{38}$$

with corresponding Noether Integrals

$$I_2 = \frac{d}{dt}\left(a\sqrt{R}\right) \tag{39}$$

$$I_3 = t\frac{d}{dt}\left(a\sqrt{R}\right) - a\sqrt{R} \tag{40}$$

[2] Where T is the "kinetic" term and V is the "potential

[3] All two dimensional Riemannian spaces are Einstein spaces implying that if $\hat{R} = const$ the space is maximally symmetric [41] and if $\hat{R} = 0$, the space admit gradient Homothetic vector, i.e. is flat.

$$I_4 = 2tE - 6a^2\dot{a}\sqrt{R} - 6\frac{a^3}{\sqrt{R}}\dot{R}. \tag{41}$$

the non vanishing commutators of the Noether algebra are

$$\left[X^1, X^3\right] = X^2 \qquad \left[X^1, X^4\right] = 2X^1$$

$$\left[X^2, X^4\right] = \frac{8}{3}X^2 \qquad \left[X^3, X^4\right] = \frac{2}{3}X^3$$

Case 3: If $f(R) = R^{\frac{7}{8}}$ the dynamical system admits the extra Noether symmetries

$$X^5 = 2t\partial_t + \mathbf{H}\ ,\ X^6 = t^2\partial_t + t\mathbf{H} \tag{42}$$

with corresponding Noether Integrals

$$I_5 = 2tE - \frac{21}{8}\frac{d}{dt}\left(a^3R^{-\frac{1}{8}}\right) \tag{43}$$

$$I_6 = t^2E - \frac{21}{8}t\frac{d}{dt}\left(a^3R^{-\frac{1}{8}}\right) + \frac{21}{8}a^3R^{-\frac{1}{8}}. \tag{44}$$

and the non vanishing commutators of the Noether algebra are

$$\left[X^1, X^5\right] = 2X^1 \qquad \left[X^1, X^6\right] = X^5 \qquad \left[X^5, X^6\right] = 2X^6$$

From the time dependent integrals (43),(44) and the Hamiltonian we construct the Ermakov-Lewis invariant [43, 44]

$$\Sigma = 4I_6E - I_5^2 \tag{45}$$

Case 4: If $f(R) = (R - 2\Lambda)^{\frac{3}{2}}$ the dynamical system admits the extra Noether symmetries

$$\bar{X}^2 = e^{\sqrt{m}t}\mathbf{K}^2\ ,\ \bar{X}^3 = e^{-\sqrt{m}t}\mathbf{K}^2 \tag{46}$$

with corresponding Noether Integrals

$$\bar{I}_2 = e^{\sqrt{m}t}\left(\frac{d}{dt}\left(a\sqrt{R-2L}\right) - 9\sqrt{m}a\sqrt{R-2\Lambda}\right) \tag{47}$$

$$\bar{I}_3 = e^{-\sqrt{m}t}\left(\frac{d}{dt}\left(a\sqrt{R-2L}\right) + 9\sqrt{m}a\sqrt{R-2\Lambda}\right) \tag{48}$$

where $m = \frac{2}{3}\Lambda$. The non vanishing commutators of the Noether algebra are

$$\left[X^1, \bar{X}^2\right] = \sqrt{m}\bar{X}^2 \qquad \left[\bar{X}^3, X^1\right] = \sqrt{m}\bar{X}^3$$

From the time dependent integrals (47),(48) we construct the time independent integral $\bar{I}_{23} = \bar{I}_2\bar{I}_3$.

Case 5: If $f(R) = (R - 2\Lambda)^{\frac{7}{8}}$ the dynamical system admits the extra Noether symmetries

$$\bar{X}^5 = \frac{1}{\sqrt{m}}e^{2\sqrt{m}t}\partial_t + e^{2\sqrt{m}t}\ \mathbf{H} \tag{49}$$

$$\bar{X}^6 = -\frac{1}{\sqrt{m}}e^{-2\sqrt{m}t}\partial_t + e^{-2\sqrt{m}t}\ \mathbf{H} \tag{50}$$

with corresponding Noether Integrals

$$\bar{I}_5 = e^{2\sqrt{m}t}\left[\frac{1}{\sqrt{m}}E - \frac{21}{8}\frac{d}{dt}\left(a^3\left(R-2\Lambda\right)^{-\frac{1}{8}}\right) + \frac{21}{4}\sqrt{m}a^3\left(R-2\Lambda\right)^{-\frac{1}{8}}\right] \tag{51}$$

$$\bar{I}_6 = e^{-2\sqrt{m}t}\left[\frac{1}{\sqrt{m}}E + \frac{21}{8}\frac{d}{dt}\left(a^3\left(R-2\Lambda\right)^{-\frac{1}{8}}\right) + \frac{21}{4}\sqrt{m}a^3\left(R-2\Lambda\right)^{-\frac{1}{8}}\right] \tag{52}$$

and the non vanishing commutators of the Noether algebra are

$$\left[X^1,\bar{X}^5\right] = 2\sqrt{m}\bar{X}^5 \qquad \left[\bar{X}^6,X^1\right] = 2\sqrt{m}\bar{X}^6$$

$$\left[\bar{X}^5,\bar{X}^6\right] = \frac{4}{\sqrt{m}}X^1$$

From the time dependent integrals (43),(44) and the Hamiltonian we construct the Ermakov-Lewis invariant [44]

$$\phi = E^2 - \bar{I}_5\bar{I}_6 \tag{53}$$

Case 6: If $f\left(R\right) = R^n$ (with $n \neq 0, 1, \frac{3}{2}, \frac{7}{8}$) the dynamical system admits the extra Noether symmetry

$$X^7 = 2t\partial_t + \mathbf{H} + \left(\frac{4n}{3} - \frac{7}{6}\right)\mathbf{K}^1 \tag{54}$$

with corresponding Noether Integral

$$I_7 = 2tE - 8na^2R^{n-1}\dot{a}\left(2-n\right) - 4na^3R^{n-2}\dot{R}\left(2n-1\right)\left(n-1\right). \tag{55}$$

and the commutator of the Noether algebra is $\left[X^1,X^7\right] = 2X^1$.

We note that the Noether subalgebra of case 2, $\left\{X^1,X^2,X^3\right\}$ and the algebra of case 4 $\left\{X^1,\bar{X}^2,\bar{X}^3\right\}$ is the same Lie algebra but not in the same representation. The same observation applies to the subalgebra of case 3 $\left\{X^1,X^5,X^6\right\}$ and the algebra of case 5 $\left\{X^1,\bar{X}^5,\bar{X}^6\right\}$. This connection between the Lie groups is useful because it reveals common features in the dynamic systems, as is the common transformation to the normal coordinates of the systems.

For the cosmological viability of the models see [21, 45, 46]

6. Analytic Solutions

Using the Noether symmetries and the associated Noether integrals we solve analytically the differential eqs.(5), (6) and (7) for the cases where the dynamical system is Liouville integrable, that is for cases 2-5. Case 6 (i.e. $f\left(R\right) = R^n$) is not Liouville integrable via Noether point symmetries, since the Noether integral (55) is time dependent[4].

6.1. Power law model R^μ with $\mu = \frac{3}{2}$

In this case the Lagrangian eq.(10) of the $f(R) = R^{\frac{3}{2}}$ model is written as

$$L = 9a\sqrt{R}\dot{a}^2 + \frac{9a^2}{2\sqrt{R}}\dot{a}\dot{R} + \frac{a^3}{2}R^{\frac{3}{2}} \tag{56}$$

Changing the variables from (a, R) to (z, w) via the relations:

$$a = \left(\frac{9}{2}\right)^{-\frac{1}{3}}\sqrt{z} \quad R = \frac{w^2}{z} \tag{57}$$

[4] In the appendix Appendix A we present special solutions for the $f\left(R\right) = R^n$ model, using the zero order invariants.

the Lagrangian (56) and the Hamiltonian (11) become

$$L = \dot{z}\dot{w} + V_0 w^3 \tag{58}$$

$$E = \dot{z}\dot{w} - V_0 w^3 \tag{59}$$

where $V_0 = \frac{1}{9}$. The equations of motion in the new coordinate system are

$$\ddot{w} = 0 \tag{60}$$

$$\ddot{z} - 3V_0 w^2 = 0 \tag{61}$$

The Noether integrals (39),(40) in the coordinate system $\{z, y\}$ are

$$I_1^{'} = \dot{w} \quad , \quad I_2^{'} = t\dot{w} - w \tag{62}$$

The general solution of the system is:

$$y\left(t\right) = I_1' t - I_2^{'} \tag{63}$$

$$z\left(t\right) = \frac{1}{36\left(I_1^{'}\right)^2}\left(I_1^{'}t - I_2^{'}\right)^4 + z_1 t + z_0 \tag{64}$$

The Hamiltonian constrain gives $E = z_1 I_1^{'}$ where $z_{0,1}$ are constants and the singularity condition results in the constrain

$$\frac{1}{36\left(I_1^{'}\right)^2}\left(I_2^{'}\right)^4 + z_0 = 0. \tag{65}$$

6.2. Power law model R^{μ} with $\mu = \frac{7}{8}$
In this case the Lagrangian eq.(10) is written as

$$L = \frac{21a}{4R^{\frac{1}{8}}}\dot{a}^2 - \frac{21}{16}\frac{a^2}{R^{\frac{9}{8}}}\dot{a}\dot{R} - \frac{1}{8}a^3 R^{\frac{7}{8}}. \tag{66}$$

Changing now the variables from (a, R) to (ρ, σ) via the relations:

$$a = \left(\frac{21}{4}\right)^{-\frac{1}{3}}\sqrt{\rho e^{\sigma}} \quad R = \frac{e^{12\sigma}}{\rho^4}. \tag{67}$$

The Lagrangian (94) and the Hamiltonian (11) become

$$L = \frac{1}{2}\dot{\rho}^2 - \frac{1}{2}\rho^2\dot{\sigma}^2 + V_0\frac{e^{12\sigma}}{\rho^2} \tag{68}$$

$$E = \frac{1}{2}\dot{\rho}^2 - \frac{1}{2}\rho^2\dot{\sigma}^2 - V_0\frac{e^{12\sigma}}{\rho^2}. \tag{69}$$

where $V_0 = -\frac{1}{42}$. The Euler-Lagrange equations provide the following equations of motion:

$$\ddot{\rho} + \rho\dot{\sigma}^2 + 2V_0\frac{e^{12\sigma}}{\rho^3} = 0 \tag{70}$$

$$\ddot{\sigma} + \frac{2}{\rho}\dot{\sigma}\dot{\rho} + 12V_0\frac{e^{12\sigma}}{\rho^2} = 0. \tag{71}$$

The Noether integrals (43), (44) and the Ermakov-Lewis invariant 45 in the coordinate system $\{u, v\}$ are

$$I_5^{'} = 2tE - \rho\dot{\rho} \tag{72}$$

$$I_6^{'} = t^2E - t\rho\dot{\rho} + \frac{1}{2}\rho^2. \tag{73}$$

$$\Sigma = \rho^4\dot{\sigma}^2 + 4V_0e^{12\sigma}. \tag{74}$$

Using the Ermakov-Lewis Invariant, the Hamiltonian (68) and equation (70) are written:

$$\frac{1}{2}\dot{\rho}^2 - \frac{1}{2}\frac{\Sigma}{\rho^2} = E \tag{75}$$

$$\ddot{\rho} + \frac{\Sigma}{\rho^3} = 0. \tag{76}$$

And the analytical solution of the system is

$$\rho(t) = \left(\rho_2 t^2 + \rho_1 t + \frac{\left((\rho_1)^2 - 4\Sigma\right)}{4\rho_2}\right)^{\frac{1}{2}} \tag{77}$$

$$\exp(\sigma(t)) = \left\{\frac{21}{2}\Sigma\left[\left(\tanh\left[\sigma_0\rho_2\sqrt{\Sigma} - 6\arctan h\left(\frac{2\rho_2 t + \rho_1}{2\sqrt{\Sigma}}\right)\right]\right)^2 - 1\right]\right\}^{\frac{1}{12}} \tag{78}$$

where $B(t) = \left(\frac{1}{2}\frac{2\rho_2 t + \rho_1}{\sqrt{\Sigma}}\right)$ and $\rho_{1,2}$, σ_0 are constants with Hamiltonian constrain $E = \frac{1}{2}\rho_2$. The singularity constrain gives $(\rho_1)^2 = 4\Sigma$

In the case $\Sigma = 0$ the analytical solution is

$$\rho(t) = \left(\rho_2 t^2 + \rho_1 t + \frac{1}{2}\frac{(\rho_1)^2}{\rho_2}\right)^{\frac{1}{2}} \tag{79}$$

$$\exp\sigma(t) = \left[\frac{1}{24\sqrt{V_0}}\frac{(2\rho_2 t + \rho_1)}{\left(4\sigma_0\rho_2^2 t + 2\sigma_0\rho_2\rho_1 - 1\right)}\right]^{\frac{1}{6}} \tag{80}$$

The singularity constrain gives $\rho_1 = 0$, then the solution is

$$a(t) = \frac{a_0 t^{\frac{7}{6}}}{(a_2 t - 1)^{\frac{1}{6}}} \tag{81}$$

In contrast with the claim of [47] this model is analytically solvable and there exists models which admit Noether integrals with time dependent gauge functions.

6.3. Λ_{bc}CDM model with $(b, c) = (1, \frac{3}{2})$
Inserting $f(R) = (R - 2\Lambda)^{3/2}$ into eq.(10) we obtain

$$L = 9a\sqrt{R - 2\Lambda}\dot{a}^2 + \frac{9a^2}{2\sqrt{R - 2\Lambda}}\dot{a}\dot{R} + \frac{a^3}{2}(R + 4\Lambda)\sqrt{R - 2\Lambda} \tag{82}$$

Changing now the variables from (a, R) to (x, y) via the relations:

$$a = \left(\frac{9}{2}\right)^{-\frac{1}{3}} \sqrt{x} \quad R = 2\Lambda + \frac{y^2}{x} \tag{83}$$

the Lagrangian (82) and the Hamiltonian (11) become

$$L = \dot{x}\dot{y} + V_0 \left(y^3 + \bar{m}xy\right) \tag{84}$$

$$E = \dot{x}\dot{y} - V_0 \left(y^3 + \bar{m}xy\right) \tag{85}$$

where $V_0 = \frac{1}{9}$ and $\bar{m} = 6\Lambda$.
The equations of motion, using the Euler-Lagrange equations, in the new coordinate system are

$$\ddot{x} - 3V_0 y^2 - \bar{m}V_0 x = 0 \tag{86}$$

$$\ddot{y} - \bar{m}V_0 y = 0. \tag{87}$$

The Noether integrals (47),(48) in the coordinate system $\{x, y\}$ are

$$\bar{I}_1' = e^{\omega t}\dot{y} - \omega e^{\omega t} y \tag{88}$$

$$\bar{I}_2' = e^{-\omega t}\dot{y} + \omega e^{-\omega t} y. \tag{89}$$

where $\omega = \sqrt{2\Lambda/3}$. From these we construct the time independent first integral

$$\Phi = I_1 I_2 = \dot{y}^2 - \omega^2 y^2. \tag{90}$$

The constants of integration are further constrained by the condition that at the singularity $(t = 0)$, the scale factor has to be exactly zero, that is, $x(0) = 0$.
The general solution of the system (86)-(87) is:

$$y\left(t\right) = \frac{I_2}{2\omega} e^{\omega t} - \frac{I_1}{2\omega} e^{-\omega t} \tag{91}$$

$$x\left(t\right) = x_{1G} e^{\omega t} + x_{2G} e^{-\omega t} + \frac{1}{4\bar{m}\omega^2} \left(I_2 e^{\omega t} + I_1 e^{-\omega t}\right)^2 + \frac{\Phi}{\bar{m}\omega^2}. \tag{92}$$

The Hamiltonian constrain gives $E = \omega \left(x_{1G} I_1 - x_{2G} I_2\right)$ where $x_{1G,2G}$ are constants and the singularity condition results in the constrain

$$x_{1G} + x_{2G} + \frac{1}{4\bar{m}\omega^2} \left(I_1 + I_2\right)^2 + \frac{\Phi}{\bar{m}\omega^2} = 0. \tag{93}$$

At late enough times the solution becomes $a^2(t) \propto e^{2\omega t}$

6.4. Λ_{bc}CDM model with $(b, c) = (1, \frac{7}{8})$
In this case the Lagrangian eq.(10) of the $f(R) = (R - 2\Lambda)^{7/8}$ model is written as

$$L = \frac{21a}{4\left(R - 2\Lambda\right)^{\frac{1}{8}}} \dot{a}^2 - \frac{21}{16} \frac{a^2}{\left(R - 2\Lambda\right)^{\frac{9}{8}}} \dot{a}\dot{R} - \frac{1}{8} a^3 \frac{\left(R - 16\Lambda\right)}{\left(R - 2\Lambda\right)^{\frac{1}{8}}}. \tag{94}$$

Changing now the variables from (a, R) to (u, v) via the relations:

$$a = \left(\frac{21}{4}\right)^{-\frac{1}{3}} \sqrt{ue^v} \quad R = 2\Lambda + \frac{e^{12v}}{u^4}. \tag{95}$$

The Lagrangian (94) and the Hamiltonian (11) become

$$L = \frac{1}{2}\dot{u}^2 - \frac{1}{2}u^2\dot{v}^2 + V_0\frac{\bar{m}}{4}u^2 + V_0\frac{e^{12v}}{u^2} \tag{96}$$

$$E = \frac{1}{2}\dot{u}^2 - \frac{1}{2}u^2\dot{v}^2 - V_0\frac{\bar{m}}{4}u^2 - V_0\frac{e^{12v}}{u^2}. \tag{97}$$

where $\bar{m} = -28\Lambda$, $V_0 = -\frac{1}{42}$.
The Euler-Lagrange equations provide the following equations of motion:

$$\ddot{u} + u\dot{v}^2 - \frac{V_0\bar{m}}{2}u + 2V_0\frac{e^{12v}}{u^3} = 0 \tag{98}$$

$$\ddot{v} + \frac{2}{u}\dot{u}\dot{v} + 12V_0\frac{e^{12v}}{u^4} = 0. \tag{99}$$

The Noether integrals (51),(52) and the Ermakov-Lewis invariant (53) in the coordinate system $\{u, v\}$ are

$$I_+ = \frac{1}{\lambda}e^{2\lambda t}E - e^{2\lambda t}u\dot{u} + \lambda e^{2\lambda t}u^2 \tag{100}$$

$$I_- = \frac{1}{\lambda}e^{-2\lambda t}E - e^{-2\lambda t}u\dot{u} + \lambda e^{-2\lambda t}u^2. \tag{101}$$

$$\phi = u^4\dot{v}^2 + 4V_0e^{12v}. \tag{102}$$

where $\lambda = \frac{1}{2}\sqrt{\frac{2}{3}\Lambda}$.

Using the Ermakov-Lewis Invariant (102), the Hamiltonian (97) and equation (98) are written:

$$\frac{1}{2}\dot{u}^2 - V_0\frac{m}{8}u^2 - \frac{1}{2}\frac{\phi}{u^2} = E \tag{103}$$

$$\ddot{u} - \frac{V_0m}{4}u + \frac{\phi}{u^3} = 0. \tag{104}$$

The solution of (104) has been given by Pinney [49] and it is the following:

$$u(t) = \left(u_1e^{2\lambda t} + u_2e^{-2\lambda t} + 2u_3\right)^{\frac{1}{2}} \tag{105}$$

where u_{1-3}. From the Hamiltonian constrain (103) and the Noether Integrals (100),(101) we find

$$E = -2\lambda u_3 \ , \ I_+ = 2\lambda u_2 \ , \ I_- = 2\lambda u_1.$$

Replacing (105) in the Ermakov-Lewis Invariant (102) and assuming $\phi \neq 0$ we find:

$$\exp(v(t)) = 2^{\frac{1}{6}}\phi^{\frac{1}{12}}e^{-A(t)}\left(4V_0 + e^{-12A(t)}\right)^{-\frac{1}{6}} \tag{106}$$

where

$$A(t) = \arctan\left[\frac{2\lambda}{\sqrt{\phi}}\left(u_1e^{2\lambda t} + u_3\right)\right] + 4\lambda^2u_1\sqrt{\phi}. \tag{107}$$

Then the solution is

$$a^2(t) = 2^{-\frac{1}{3}}\phi^{\frac{1}{12}}e^{-A(t)}\left(4V_0 + e^{-12A(t)}\right)^{-\frac{1}{6}}\left(u_1e^{2\lambda t} + u_2e^{-2\lambda t} + 2u_3\right)^{\frac{1}{2}} \tag{108}$$

where from the singularity condition $x(0) = 0$ we have the constrain $u_1 + u_2 + 2u_3 = 0$, or

$$2E - (I_+ + I_-) = 0. \tag{109}$$

At late enough time we find $A(t) \simeq A_0$, which implies $a^2(t) \propto e^{\lambda t}$.

In the case where $\phi = 0$ equations (103),(104) describe the hyperbolic oscillator and the solution is

$$u(t) = \sinh \lambda t \ , \ 2E = \lambda^2. \tag{110}$$

From the Ermakov-Lewis Invariant we have

$$\exp(v(t)) = \left(\frac{\lambda \sinh \lambda t}{\lambda v_1 \sinh \lambda t - 12\sqrt{|V_0|}e^{-2\lambda t}} \right)^{\frac{1}{6}} \tag{111}$$

where v_1 is a constant. The analytical solution is

$$a^2(t) = \left(\frac{\lambda \sinh^7 \lambda t}{\lambda v_1 \sinh \lambda t - 12\sqrt{|V_0|}e^{-2\lambda t}} \right)^{\frac{1}{6}} \tag{112}$$

7. Noether symmetries in spatially non flat $f(R)$ models

In this section we study further the Noether symmetries in non flat $f(R)$ cosmological models. In the context of a FRW spacetime the Lagrangian of the overall dynamical problem and the Ricci scalar are

$$L = 6f'a\dot{a}^2 + 6f''\dot{R}a^2\dot{a} + a^3\left(f'R - f\right) - 6Kaf' \tag{113}$$

$$R = 6\left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2 + K}{a^2}\right) \tag{114}$$

where K is the spatial curvature. Note that the two dimensional metric is given by eq.(35) while the "potential" in the Lagrangian takes the form

$$V_K(a, R) = -a^3(f'R - f) + Kaf'. \tag{115}$$

Based on the above equations and using the theoretical formulation presented in section 5, we find that the $f(R)$ models which admit non trivial Noether symmetries are the $f(R) = (R - 2\Lambda)^{3/2}$, $f(R) = R^{3/2}$ and $f(R) = R^2$. The Noether symmetries can be found in section 5.

In particular, inserting $f(R) = (R - 2\Lambda)^{3/2}$ into the Lagrangian (113) and changing the variables from (a, R) to (x, y) [see section 6.3] we find

$$L = \dot{x}\dot{y} + V_0\left(y^3 + \bar{m}xy\right) - \bar{K}y \tag{116}$$

$$E = \dot{x}\dot{y} - V_0\left(y^3 + \bar{m}xy\right) + \bar{K}y \tag{117}$$

where $\bar{K} = 3(6^{1/3}K)$. Therefore, the equations of motion are

$$\begin{aligned} \ddot{x} - 3V_0y^2 - \bar{m}V_0x + \bar{K} &= 0 \\ \ddot{y} - \bar{m}V_0y &= 0 \ . \end{aligned}$$

The constant term $\bar{K}$ appearing into the first equation of motion is not expected to affect the Noether symmetries (or the integrals of motion). Indeed we find that the corresponding Noether

symmetries coincide with those of the spatially flat $f(R) = (R - 2\Lambda)^{3/2}$ model. However, in the case of $K \neq 0$ (or $\bar{K} \neq 0$) the analytical solution for the x-variable is written as

$$x_K(t) \equiv x(t) + \frac{\bar{K}}{\omega^2} \tag{118}$$

where $x(t)$ is the solution of the flat model $K = 0$ (see section 6.3). Note that the solution of the y-variable remains unaltered.

Similarly, for the case of the $f(R) = R^{3/2}$ model the analytical solution is

$$z_K(t) = z(t) + \bar{K} \tag{119}$$

where $z(t)$ is the solution of the spatially flat model (see section 6.1).

8. Conclusion

In the literature the functional forms of $f(R)$ of the modified $f(R)$ gravity models are mainly defined on a phenomenological basis. In this article we use the Noether symmetry approach to constrain these models with the aim to utilize the existence of non-trivial Noether symmetries as a selection criterion that can distinguish the $f(R)$ models on a more fundamental level. Furthermore the resulting Noether integrals can be used to provide analytic solutions.

In the context of $f(R)$ models, the system of the modified field equations is equivalent to a two dimensional dynamical system moving in M^2 (mini superspace) under the constraint $\bar{E}$ =constant. Following the general methodology of [31,38], we require that the two dimensional system admits extra Noether symmetries. This requirement fixes the $f(R)$ function and the analytical solutions are computed. It is interesting that two well known dynamical systems appear: the anharmonic oscillator and the Ermakov-Pinney system. We recall that the field equations of the $\Lambda-$cosmology is equivalent with that of the hyperbolic oscillator.

Acknowledgments

This research was partially funded by the University of Athens Special Account of Research Grants no 10812.

Appendix A. Special solutions for the Power law model R^n

The case $f(R) = R^n$ is not Liouville integrable via Noether point symmetries. The zero order invariant will be used in order to find special solutions. Inserting $f(R) = R^n$, $\left(n \neq 0, 1, \frac{3}{2}, \frac{7}{8}\right)$ into eq.(10) we obtain

$$L\left(a, \dot{a}, R, \dot{R}\right) = 6naR^{n-1}\dot{a}^2 + 6n(n-1)a^2R^{n-2}\dot{a}\dot{R} + (n-1)a^3R^n \tag{A.1}$$

and the modified field equations are

$$\ddot{a} + \frac{1}{a}\dot{a}^2 - \frac{1}{6}aR = 0 \tag{A.2}$$

$$\ddot{R} + \frac{n-2}{R}\dot{R}^2 - \frac{1}{n-1}\frac{R}{a^2}\dot{a}^2 + \frac{2}{a}\dot{a}\dot{R} - \frac{(n-3)}{6n(n-1)}R^2 = 0 \tag{A.3}$$

$$E = 6naR^{n-1}\dot{a}^2 + 6n(n-1)a^2R^{n-2}\dot{a}\dot{R} - (n-1)a^3R^n. \tag{A.4}$$

The Noether symmetry (54) is also and a Lie symmetry, hence we have the zero order invariants

$$a_0 = at^{-N} \ , \ R_0 = Rt^{-2}. \tag{A.5}$$

Applying the zero order invariants in the field equations (A.2)-(A.4) and in the Noether integral (55) we have the following results.

The dynamical system admit a special solution of the form

$$a = a_0 t^N \ , \ R = 6N\left(2N-1\right)t^{-2} \tag{A.6}$$

where the constants N, E and I_7 are

$$N = \frac{1}{2} \ , \ E = 0 \ , \ I_7 = 0$$

or

$$N = -\frac{\left(2n-1\right)\left(n-1\right)}{n-2} \ , \ E = 0 \ , \ I_7 = 0$$

or

$$N = \frac{2}{3}n \ , \ E = \left(\frac{12n}{9}\right)^n \left(4n-3\right)^{n-1}\left(13n-8n^2-3\right)a_0^3 \ , \ I_7 = 0.$$

Another special solution is the deSitter solution for $n = 2$

$$a = a_0 e^{H_0 t} \ , \ R = 12H_0^2 \tag{A.7}$$

where $I_7 = 0$ and the spacetime is empty i.e. $E = 0$.

References

[1] Weinberg S 1989 Rev Mod Phys **61** 1
[2] Peebles P J and Ratra B 2003 Rev. Mod. Phys. **75** 559
[3] Padmanabhan T 2003 Phys. Rept. **380** 235
[4] Perivolaropoulos L. 2008 (Preprint arxXiv.0811.4684)
[5] Ratra B and Peebles P J E 1988 Phys. Rev D. **37** 3406
[6] Overduin J M and Cooperstock F I 1998 Phys. Rev. D **58** 043506
[7] Basilakos S Plionis M and Solà S 2009 Phys. Rev. D **80** 3511
[8] Linder E V 2004 Phys. Rev. Lett. **70** 023511
[9] Lima J A S Silva F E and Santos R C 2008 Class. Quantum Grav. **25** 205006
[10] Brookfield A W et.al 2006 Phys. Rev. Lett. **96** 061301
[11] Sotiriou T P and Faraoni V 2010 Rev. Mod. Phys. **82** 451
[12] Starobinsky A A 1980 Phys. Lett. B **91** 99
[13] Carrol S Duvvuri M V Trodden M and Turner M S 2004 Phys. Rev. D. **70** 043528
[14] Amendola L Polarski D and Tsujikawa S 2007 Phys. Rev. Lett. **98** 131302
[15] Amendola L Gannouji R Polarski D and Tsujikawa S 2007 Phys. Rev. D **75** 083504
[16] Capozzielo S 2002 Int. J. Mod. Phys. D. **11** 483
[17] Nojiri S and Odintsov S D 2006 Phys. Rev. D. **73** 124038
[18] Hu W and Sawicki I 2007 Phys. Rev. D. **76** 064004
[19] Starobinsky A A 2007 JETP Lett. **86** 157
[20] Tsujikawa S 2008 Phys. Rev. D. **77** 023507
[21] Amendola L and Tsujikawa S 2008 Phys. Lett. B. **660** 125
[22] Capozziello S de Ritis R and Marino A A 1997 Class. Quantum Gravity **14** 3259
[23] Cotsakis S Leach P G L and Pantazi H 1998 Grav. Cosm. **4** 314
[24] Rubano C and Scudellaro P 2002 Gen. Relat. Grav. **34** 307
[25] Sanyal A K Modak B Rubano C and Piedipalumbo E 2005 Gen. Relat. Grav. **37** 407
[26] Szydlowski M et al. 2006 Gen. Rel. Grav. **38** 795
[27] Capozziello S Stabile A and Troisi A 2007 Class. Quant. Grav. **24** 2153
[28] Capozziello S Nesseris S and Perivolaropoulos L 2007 JCAP **0712** 009
[29] Bonanno A Esposito G Rubano C and Scudellaro P 2007 Gen. Rel. Grav. **39** 189
[30] Vakili B Phys. Lett. B. 2008 **664** 16
[31] Tsamparlis M and Paliathanasis A 2011Gen. Rel. Grav. 43 1861
[32] Basilakos S Tsamparlis M and Paliathanasis A 2011 Phys. Rev. D **83** 103512 (Preprint arXiv:1104.2980)
[33] Vakili B and Khazaie F 2012 Class. Quantum Grav. **29** 035015 (Preprint arxiv:1109.3352)

[34] Jamil M Mahomed F M and Momenti D 2011 Phys. Lett. B **702** 315 (Preprint arxiv:1105.2610)
[35] Christodoulakis T et al 2012 (Preprint arXiv:1208.0462)
[36] Capozziello S De Laurentis M and Odintsov S D 2012 Eur. Phys. J. **C72** 2086 (Preprint arxiv:1206.4842)
[37] Wei H Guo X J and Wang L F 2012 Phys. Lett. B **707** 298
[38] Tsamparlis M and Paliathanasis A 2011 J. Phys. A: Math. and Theor. **44** 175202 (Preprint arXiv:1101.5771)
[39] Olver P J 1986 Applications of Lie Groups to Differential Equations Springer - Verlag N.Y.
[40] Stephani H 1989 Differential Equations: Their Solutions using Symmetry Cambridge University Press N.Y.
[41] Barnes A 1993 Class. Quantum Grav. **10** 1139
[42] Amendola L and Tsujikawa S 2012 Dark Energy: Theory and Observations Cambridge University Press N.Y.
[43] Moyo S and Leach P G L 2002 J. Phys. A: Math. and Theor. **35** 5333
[44] Tsamparlis M and Paliathanasis A 2012 J. Phys. A: Math. and Theor. **45** 275202 (Preprint arXiv:1205.4114)
[45] Amendola L Polarski D and Tsujikawa S 2007 Int. J. Mod. Phys. D **16** 1555 (Preprint astro-ph/0605384)
[46] Paliathanasis A Tsamparlis M and Basilakos S 2011 Phys. Rev. D. **84** 123514 (Preprint arXiv:1111.4547)
[47] Nayem Sk and Sanyal A K 2012 Astrophys. Space Sci. **342** 549 (Preprint arXiv:1208.2306)
[48] Lewis H R 1967 Phys. Rev. Lett. **18** 510
[49] Pinney E 1950 Proc. Am. Math. Soc. **1** 681