

Future Stability for Reflection Symmetric Solutions of the Einstein-Vlasov System of Bianchi Type VI₀

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Abstract. We present a sketch of the proof of future stability for reflection symmetric solutions of the Einstein-Vlasov system of Bianchi Type VI₀ with an extended introduction to motivate the interest for the Vlasov equation in the context of General Relativity.

1. Introduction

Cosmology describes the dynamics of the Universe as a whole. These dynamics can only be understood in terms of the theory of general relativity which was established by Albert Einstein in 1915. This theory is mathematically complicated enough to have given rise to a research field on its own, the area of Mathematical General Relativity.

A starting point to understand general models are the homogeneous models. In general, the focus has been on the fluid model since it appears (theoretically) relatively natural when dealing with isotropic universes and from observations we also know that the Universe is almost isotropic. However, in order to have a deeper understanding of the dynamics one should go beyond the study of isotropic universes. General statements may vary then depending on the choice of the matter model.

We will deal with the *future asymptotics* of some homogeneous cosmological models within the so called Bianchi class A while the matter is described via an ensemble of free falling particles also called collisionless matter. The late-time behaviour of Bianchi spacetimes with a non-tilted fluid is well understood. In particular, all non-tilted perfect fluid orthogonal Bianchi models except Type IX with a linear equation of state where $0 < \gamma < \frac{2}{3}$, are future asymptotic to the flat Friedmann-Lemître model. Note the restriction on γ here. One cannot expect isotropization for most of the Bianchi models. However, there are two important “conjectures” in the present work:

- (i) The spacetimes considered tend to special (self-similar) solutions
- (ii) For expanding models the dispersion of the velocities of the particles decays

This second “conjecture” means that asymptotically there is a dust-like behaviour for collisionless matter which is the matter model used here. The Einstein-Vlasov system remains a system of partial differential equations (PDE’s) even if one assumes spatial homogeneity. The reason is that although the distribution function written in a suitable frame will not depend on the spatial point, the dependence with respect to the momenta remains. However, sometimes a



reduction to a system of ordinary differential equations is possible due to additional symmetry assumptions. This is no longer possible if one drops some of these additional symmetries. Thus, if one wants to generalize the results obtained until now the theory of finite dimensional dynamical systems is not enough. Most of these results rely on the theory of dynamical systems. Thus, one might be tempted to use techniques from the theory of infinite-dimensional dynamical systems. The first important difficulty would be to choose the suitable (weighted) norm. Another difficulty is that important theorems which have been used for the finite-dimensional case cannot be applied here. All this may work, but this is not the approach taken here.

Here, the main tool used is a bootstrap argument which is often used in non-linear PDE's. For the reflection symmetric Bianchi Type VI₀ we have been able to show that the late-time behaviour remains the same if the LRS condition is dropped assuming small data. We will show that the spacetime will tend to solutions which are even more symmetric. In the case of Bianchi Type VI₀ there cannot be an LRS condition, however it is compatible with an additional discrete symmetry. The analysis of the asymptotics shows that the Bianchi Type VI₀ spacetimes tend to this special class. Note that for Bianchi Type VI₀ there is no corresponding LRS/previous result.

All the results show that the dust model usually assumed in observational cosmology in the 'matter-dominated' Era is robust. Another way of saying the same is that asymptotically collisionless matter is well approximated by the dust system.

2. Relativistic kinetic theory

We will consider collisionless matter as our matter model. Before introducing it more formally we want first to give the motivation for the restriction to the aforesaid kind of matter instead of considering the full Boltzmann equation.

First of all it is of course an enormous simplifying assumption. One assumes by modelling a galaxy as a particle that in a cosmological context the internal structure of the galaxy is irrelevant.

An important physical argument in favour of considering the collisionless model is that collisions between galaxies are not common and even if galaxies fly through each other not so many collisions between stars happen as one might expect. Further, in stellar dynamics collisionless matter is often used since collisions between stars are very unlikely. Actually this led Eddington to state:

“The apparent analogy with the kinetic theory of gases is rejected altogether, and it is taken as a fundamental principle that *the stars describe paths under the general attraction of the stellar system without interfering with one another*”(p. 254 of [1]; italics from Eddington)

Two pages later he continues:

“A regular progression may be traced through rigid dynamics, hydrodynamics, gas-dynamics to stellar dynamics. In the first all the particles move in a connected manner; in the second there is continuity between the motions of contiguous particles ; in the third the adjacent particles act on one another by collision, so that, although there is no mathematical continuity, a kind of physical continuity remains; in the last the adjacent particles are entirely independent.”(p. 256 of [1])

Later, Jeans in the study of stellar dynamics referring to the collisionless Boltzmann equation writes:

“This is the differential equation which must be satisfied by the distribution function f in every problem of stellar dynamics.”(p. 230 of [2])

and on the same page as a footnote:

“The student of the Kinetic Theory will recognise that it is simply Boltzmann’s well-known equation with the collisions left out.”

However, this equation is usually named after Vlasov [3], in particular in the context of mathematical cosmology. Vlasov discovered in the context of plasma physics that pair collision terms do not describe correctly the plasma dynamics and also that these terms are not formally applicable since kinetic terms diverge. His point of view was that only the collective behaviour, i.e. the electromagnetic field created by the charged particles, explains the dynamics of the individual particles.

Maybe the emphasis on the difference of taking one or the other point of view has been the reason that the Vlasov equation is named after him not only in the context of plasma physics. Another reason maybe that although Boltzmann himself assumed that the particles interact only through:

- very long range forces which can be approximated by mean fields
- or very short range forces such as hard core interactions whose effect can be approximated by instantaneous collisions,

in practice the long range forces were often neglected since, for instance, the gravitational force is very weak.

Another fact which has attracted attention to the case of collisionless matter is the discovery that in analyzing the initial singularity of the Einstein-dust equations there arose singularities which are unphysical and related not to gravity but to the chosen matter model. One has seen that these problems do not occur when using collisionless matter as the matter model. Finally, the Vlasov equation is also used in astrophysics to model dark matter, where the particles are now elementary particles.

Of course, at some point it is of interest to know what happens if one includes collisions between the particles. A recent work on the Cauchy problem of the Einstein-Boltzmann system can be found in [4].

2.1. The Einstein-Vlasov system

A cosmological model represents a universe at a certain averaging scale. It is described via a Lorentzian metric $g_{\alpha\beta}$ (we will use the signature $-+++$) on a manifold M and a family of fundamental observers. The interaction between the geometry and the matter is described by the Einstein field equations (we use geometrized units, i.e. the gravitational constant G and the speed of light in vacuum c are set equal to one):

$$G_{\alpha\beta} = 8\pi T_{\alpha\beta},$$

where $G_{\alpha\beta}$ is the Einstein tensor and $T_{\alpha\beta}$ is the energy-momentum tensor. For the matter model we will take the point of view of kinetic theory. The Einstein summation convention to which repeated indices are to be summed over is used. Latin indices run from one to three and Greek ones from zero to three.

Consider a particle with non-zero unit rest mass moving under the influence of the gravitational field. The mean field will be described now by the metric and the components of the metric connection. The worldline x^α of a particle is a timelike curve in spacetime. The unit future-pointing tangent vector to this curve is the 4-velocity v^α and $p^\alpha = mv^\alpha$ is the 4-momentum of the particle. Let T_x be the tangent space at a point x^α in the spacetime M , then we define the *phase space* P_1 for particles of unit mass:

$$P_1 = \{(x^\alpha, p^\alpha) : x^\alpha \in M, p^\alpha \in T_x, p_\alpha p^\alpha = -1, p^0 > 0\},$$

which we will call the mass shell. The collection of particles (galaxies or clusters of galaxies) will be described (statistically) by a non-negative real valued distribution function $f(x^\alpha, p^\alpha)$ on P_1 . This function represents the density of particles at a given spacetime point with given four-momentum. A free particle travels along a geodesic. Consider now a future-directed timelike geodesic parametrized by proper time s . The tangent vector is then at any time future-pointing unit timelike. Thus, the geodesic has a natural lift to a curve on P_1 by taking its position and tangent vector. The equations of motion thus define a flow on P_1 which is generated by a vector field L called geodesic spray or Liouville operator. Using the geodesic equations the restriction of the Liouville operator to the mass shell has the following form:

$$L = p^\alpha \frac{\partial}{\partial x^\alpha} - \Gamma_{\beta\gamma}^a p^\beta p^\gamma \frac{\partial}{\partial p^a}.$$

This operator is sometimes also called geodesic spray. Between collisions the particles follow geodesics. We will consider the collisionless case which is described via the Vlasov equation $L(f) = 0$.

2.2. Energy momentum tensor and characteristics

The unknowns of our system are a 4-manifold M , a Lorentz metric $g_{\alpha\beta}$ on this manifold and the distribution function f on the mass shell P_1 defined by the metric. We have the Vlasov equation defined by the metric for the distribution function and the Einstein field equations. It remains to define the energy-momentum tensor $T_{\alpha\beta}$ in terms of the distribution and the metric. Now considering p^0 as a dependent variable we denote the induced volume of the mass shell considered as a hypersurface in the tangent space at that point by ϖ . Now we define the energy momentum tensor as follows:

$$T_{\alpha\beta} = \int f(x^\alpha, p^a) p_\alpha p_\beta \varpi.$$

One can show that $T_{\alpha\beta}$ is divergence-free and thus it is compatible with the Einstein field equations. For collisionless matter all the energy conditions hold. The Vlasov equation in a fixed spacetime can be solved by the method of characteristics:

$$\frac{dX^a}{ds} = P^a; \quad \frac{dP^a}{ds} = -\Gamma_{\beta\gamma}^a P^\beta P^\gamma.$$

Let $X^a(s, x^\alpha, p^a)$, $P^a(s, x^\alpha, p^a)$ be the unique solution of that equation with initial conditions $X^a(t, x^\alpha, p^a) = x^a$ and $P^a(t, x^\alpha, p^a) = p^a$. Then the solution of the Vlasov equation can be written as:

$$f(x^\alpha, p^a) = f_0(X^a(0, x^\alpha, p^a), P^a(0, x^\alpha, p^a)),$$

where f_0 is the restriction of f to the hypersurface $t = 0$. It follows that if f_0 is bounded the same is true for f . We will assume that f has compact support in momentum space for each fixed t . This property holds if the initial datum f_0 has compact support and if each hypersurface $t = t_0$ is a Cauchy hypersurface. In particular, for the case we will deal with, the spacetime is future complete (theorem 2.1 of [5]).

3. Bianchi spacetimes

Bianchi spacetimes are defined as follows:

Definition 1. *A Bianchi spacetime is defined to be a spatially homogeneous spacetime whose isometry group possesses a three-dimensional subgroup G that acts simply transitively on the spacelike orbits.*

The only Bianchi spacetimes which admit a compact Cauchy hypersurface are Bianchi Types I and IX. In order to be not too restrictive we will consider locally spatially homogeneous spacetimes. Also, we will take the metric approach. If \mathbf{W}^a denote the 1-forms dual to the frame vectors \mathbf{E}_a the metric of a Bianchi spacetime takes the form:

$${}^4g = -dt^2 + g_{ab}(t)\mathbf{W}^a\mathbf{W}^b, \quad (1)$$

where g_{ab} (and all other tensors) on the Lie group G will be described in terms of the frame components of the left invariant frame. A dot above a letter will denote a derivative with respect to the cosmological time t . There are different projections of the energy momentum tensor which are important: $\rho = T^{00}$, $j_a = T_a^0$ and $S_{ab} = T_{ab}$ where ρ is the energy density and j_a is the matter current. Let us also use the notation $S = g^{ab}S_{ab}$.

3.1. Time origin choice and new variables

With the 3+1 formulation our initial data are $(g_{ij}(t_0), k_{ij}(t_0), f(t_0))$, i.e. a Riemannian metric, a second fundamental form and the distribution function of the Vlasov equation, respectively, on a three-dimensional manifold $S(t_0)$. This is the initial data set at $t = t_0$ for the Einstein-Vlasov system. We assume that $k = g^{ab}k_{ab} < 0$ for all time which enables us to set without loss of generality $t_0 = -2/k(t_0)$. The reason for this choice will become clear later and is of a technical nature. We can decompose the second fundamental form introducing σ_{ab} as the trace-free part:

$$k_{ab} = \sigma_{ab} - Hg_{ab}. \quad (2)$$

Using the Hubble parameter $H = -\frac{1}{3}k$ we define:

$$\begin{aligned} \Sigma_a^b &= \frac{\sigma_a^b}{H}; \quad \Sigma_+ = -\frac{1}{2}(\Sigma_2^2 + \Sigma_3^3); \quad \Sigma_- = -\frac{1}{2\sqrt{3}}(\Sigma_2^2 - \Sigma_3^3) \\ \Omega &= 8\pi\rho/3H^2; \quad q = -1 - \frac{\dot{H}}{H^2}; \quad \frac{d\tau}{dt} = H. \end{aligned}$$

The time variable τ is dimensionless. Since we use a left-invariant frame f we can express the Vlasov equation as follows

$$\frac{\partial f}{\partial t} + (p^0)^{-1}C_{ba}^d p^b p_d \frac{\partial f}{\partial p_a} = 0. \quad (3)$$

For the Vlasov equation we obtain by use of (7) for the Bianchi Type VI₀:

$$\frac{\partial f}{\partial t} + (p^0)^{-1}[p_2(p^3 \frac{\partial f}{\partial p_1} - p^1 \frac{\partial f}{\partial p_3}) + p_3(p^2 \frac{\partial f}{\partial p_1} - p^1 \frac{\partial f}{\partial p_2})] = 0.$$

From (3) it is also possible to define the characteristic curve V_a :

$$\frac{dV_a}{dt} = (V^0)^{-1}C_{ba}^d V^b V_d \quad (4)$$

for each $V_i(\bar{t}) = \bar{v}_i$ given \bar{t} . Note that if we define $V = g^{ij}V_i V_j$ due to the antisymmetry of the structure constants we have with (4):

$$\frac{dV}{dt} = \frac{d}{dt}(g^{ij})V_i V_j. \quad (5)$$

4. Bianchi class A spacetimes

Definition 2. A Bianchi class A spacetime is a Bianchi spacetime whose three-dimensional Lie algebra has traceless structure constants, i.e. $C_{ba}^a = 0$.

In this case there is a unique symmetric matrix with components ν^{ij} such that the structure constants can be written as follows:

$$C_{bc}^a = \varepsilon_{bcd} \nu^{da}. \quad (6)$$

Bianchi Type VI₀ is of class A and $\nu = (0, 1, -1)$. We see that the structure constants in the case of Bianchi Type VI₀ are:

$$C_{31}^2 = 1 = -C_{13}^2, \quad C_{21}^3 = 1 = -C_{12}^3. \quad (7)$$

4.1. Reflection symmetry and the equations for diagonal Bianchi Type VI₀

We will restrict ourselves to the diagonal case assuming an additional symmetry, namely the reflection symmetry. In this case we have:

$$f(t, p_1, p_2, p_3) = f(t, -p_1, -p_2, p_3) = f(t, p_1, -p_2, -p_3).$$

One can see that the energy-momentum tensor is then diagonal. Thus, if the metric and the second fundamental form are diagonal initially, they will remain diagonal in the reflection symmetric case. This symmetry implies in particular that there is no matter current, which means that there is no “tilt”. Let us define:

$$n_i = \nu_i \sqrt{\frac{g_{ii}}{g_{jj}g_{kk}}},$$

where (ijk) denotes a cyclic permutation of (123) and the Einstein summation convention is suspended. Define also

$$N_i = \frac{n_i}{H}.$$

The constraint equation is:

$$\Sigma_+^2 + \Sigma_-^2 = 1 - \Omega - \frac{1}{12}(N_2 - N_3)^2$$

and one obtains the evolution equation

$$\partial_t(H^{-1}) = \frac{3}{2} - \frac{1}{24}(N_2 - N_3)^2 + \frac{3}{2}(\Sigma_+^2 + \Sigma_-^2) + \frac{4\pi S}{3H^2}. \quad (8)$$

The evolution equations for the other variables are:

$$\dot{\Sigma}_+ = H[-\frac{1}{6}(N_2 - N_3)^2 - \Sigma_+(3 + \frac{\dot{H}}{H^2}) + \frac{4\pi}{3H^2}(S_2^2 + S_3^3 - 2S_1^1)] \quad (9)$$

$$\dot{\Sigma}_- = H[\frac{N_3^2 - N_2^2}{2\sqrt{3}} - (3 + \frac{\dot{H}}{H^2})\Sigma_- + \frac{4\pi(S_2^2 - S_3^3)}{\sqrt{3}H^2}] \quad (10)$$

$$\dot{N}_2 = -N_2H(-2\Sigma_+ - 2\sqrt{3}\Sigma_- + 1 + \frac{\dot{H}}{H^2}) \quad (11)$$

$$\dot{N}_3 = -N_3H(-2\Sigma_+ + 2\sqrt{3}\Sigma_- + 1 + \frac{\dot{H}}{H^2}). \quad (12)$$

In the case of Bianchi Type VI₀ there is a dust solution with diagonal metric discovered by Ellis and MacCallum:

$$g_{EM} = \text{diag}(t^2, t^1, t^1).$$

Here, the values of the introduced variables are:

$$\Sigma_+ = -\frac{1}{4}; \quad \Sigma_- = 0; \quad N_1 = 0; \quad N_2 = -N_3 = \frac{3}{4}; \quad \Omega = \frac{3}{4}.$$

5. The bootstrap argument

Before coming to the bootstrap argument and in order to have a certain intuition for the bootstrap assumptions, let us consider the linearization in the dust case. For $S = 0$ the linearization gives us the expected estimates and, using the Ellis-MacCallum solution, we arrive at a plausible estimate of P . A first task is to find the suitable bootstrap assumptions. We choose a slightly slower decay for the anisotropy and the curvature variables than in the linearized cases with the hope that, using the central equations, we will be able to obtain the same decay as in the linearized case. For the estimate of P we start with a slower decay than the one obtained in the previous section as well. The assumption of small data here is in the sense that our solutions are not “far away” from our special solution. In general, in order to improve an estimate, the corresponding evolution equation will be integrated. The assumptions made exclude the vacuum case, since the values of Ω due to the constraint equation are near the corresponding values of Ω of the special solution, and thus they are far from being zero.

5.1. Bootstrap assumptions

$$\begin{aligned} |\Sigma_+ + \frac{1}{4}| &\leq A_+(1+t)^{-\frac{3}{8}}, \\ |\Sigma_-| &\leq A_-(1+t)^{-\frac{3}{8}}, \\ |N_2 - \frac{3}{4}| &\leq A_{c1}(1+t)^{-\frac{3}{8}}, \\ |N_3 + \frac{3}{4}| &\leq A_{c2}(1+t)^{-\frac{3}{8}}, \\ P &\leq A_m(1+t)^{-\frac{1}{3}}. \end{aligned}$$

5.2. Estimate of the mean curvature

The first variable we estimate is the trace of the second fundamental form or equivalently the Hubble variable. By integrating (8) and since $t_0 = \frac{2}{3}H^{-1}(t_0)$ (this choice was made in section (3.1)), one obtains from the different bootstrap assumptions for the Hubble variable :

$$\boxed{H = \frac{2}{3}t^{-1}(1 + O(\epsilon_{2/3}t^{-\frac{3}{8}}))}. \quad (13)$$

5.3. Estimate of the metric

Considering the components g^{22} and g^{33} one shows that:

$$\frac{d}{dt}(t^{-\gamma}\bar{g}^{ab}) = t^{-\gamma-1}\bar{g}^{ab}(-\gamma + \frac{p}{q} + \frac{\dot{g}^{ab}}{g^{ab}}t) \leq -\eta t^{-\gamma-1}\bar{g}^{ab},$$

with η positive by use of the bootstrap assumptions and choosing γ in a suitable way which implies that:

$$g^{ab}(t) \leq t_0^{-\gamma+\frac{p}{q}} g^{ab}(t_0) t^{-\frac{p}{q}+\gamma}. \quad (14)$$

We have with (13) for the components g^{22} and g^{33} :

$$\begin{aligned} \eta &= \gamma + \frac{4}{3}(1 + O(\epsilon_{2/3} t^{-\frac{3}{8}}))(1 + \Sigma_+ \pm \sqrt{3}\Sigma_-) - \frac{p}{q}, \\ \frac{4}{3}(1 + \Sigma_+) - \frac{p}{q} &= O(A_+(1+t)^{-\frac{3}{8}}), \end{aligned}$$

which enables us to choose γ in such a way that η is positive. Summarizing, this means that asymptotically up to a positive constant which depends only on t_0 the components (and their inverses) of the metrics g_{VI_0} for Bianchi Type VI₀ have the same decay up to an ϵ as the corresponding components of the Ellis-MacCallum solution respectively.

5.4. Estimate of P

It follows from (4) and using (5):

$$\dot{V} = \dot{g}^{bf} V_b V_f = 2H(\Sigma_a^b - \delta_a^b) g^{af} V_b V_f = 2H(\Sigma_1^1 g^{11} V_1^2 + \Sigma_2^2 g^{22} V_2^2 + \Sigma_3^3 g^{33} V_3^2) - 2HV.$$

The maximum of Σ_1^1 , Σ_2^2 and Σ_3^3 is equal to $\frac{1}{4} + O(t^{-\frac{3}{8}})$. Thus, using now the estimate of H and integrating :

$$V \leq V(t_0)(t/t_0)^{-1+\epsilon}$$

from which it follows choosing $P(t_0) \leq A_m t_0^{\frac{1}{2}-\epsilon}$:

$$P \leq A_m t^{-\frac{1}{2}+\epsilon}$$

which is an improvement of the bootstrap assumption and which has a consequence that:

$$\boxed{\frac{S}{H^2} \leq C t^{-1+\epsilon}}. \quad (15)$$

5.5. Closing the bootstrap argument

What remains are the estimates for Σ_+ , Σ_- , N_2 and N_3 . In terms of the transformed linearization

$$\begin{pmatrix} \hat{\Sigma}_+ \\ \hat{\Sigma}_- \\ \hat{N}_2 \\ \hat{N}_3 \end{pmatrix} = \mathbf{M}_{\mathbf{VI}_0}^{-1} \begin{pmatrix} \tilde{\Sigma}_+ \\ \tilde{\Sigma}_- \\ \tilde{N}_2 \\ \tilde{N}_3 \end{pmatrix}$$

and using the time variable τ we have:

$$\begin{pmatrix} \hat{\Sigma}_+ \\ \hat{\Sigma}_- \\ \hat{N}_2 \\ \hat{N}_3 \end{pmatrix}' = -\frac{3}{4} \begin{pmatrix} 1 & -\sqrt{3} & 0 & 0 \\ \sqrt{3} & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} \hat{\Sigma}_+ \\ \hat{\Sigma}_- \\ \hat{N}_2 \\ \hat{N}_3 \end{pmatrix} + O(A_m^2 e^{-\tau}) \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}.$$

We arrive at

$$\frac{d}{d\tau}[\log(\hat{\Sigma}_+^2 + \hat{\Sigma}_-^2)] \leq -\frac{3}{2} + \epsilon e^{(-\frac{1}{16}-\xi)\tau}$$

and we obtain a similar expression for N_2 and N_3 such that in the end we obtain the desired estimates.

5.6. Results of the bootstrap argument

Let us summarize the results obtained in the following proposition:

Proposition 1. *Consider any C^∞ solution of the Einstein-Vlasov system with reflection Bianchi Type VI_0 symmetry and with C^∞ initial data. Assume that $|\Sigma_+(t_0) + \frac{1}{4}|$, $|\Sigma_-(t_0)|$, $|N_2(t_0) - \frac{3}{4}|$, $|N_3(t_0) + \frac{3}{4}|$ and $P(t_0)$ are sufficiently small. Then at late times the following estimates hold:*

$$\begin{aligned} H(t) &= \frac{2}{3}t^{-1}(1 + O(t^{-\frac{1}{2}+\epsilon})), \\ \Sigma_+ + \frac{1}{4} &= O(t^{-\frac{1}{2}+\epsilon}), \\ \Sigma_- &= O(t^{-\frac{1}{2}+\epsilon}), \\ N_2 - \frac{3}{4} &= O(t^{-\frac{1}{2}+\epsilon}), \\ N_3 + \frac{3}{4} &= O(t^{-\frac{1}{2}+\epsilon}), \\ P(t) &= O(t^{-\frac{1}{2}+\epsilon}). \end{aligned}$$

In the next section we will improve the estimates so that we can get rid of ϵ . However, the results stated in this section represent in fact the core of our results.

6. Main results

6.1. Arzela-Ascoli

We want to use the Arzela-Ascoli theorem. All relevant variables and their derivatives are uniformly bounded. The only variable which is not obviously bounded is the derivative of S . If we can bound the derivative of S then the second derivatives of Σ_- , Σ_+ , N_2 , N_3 and H are also bounded. The only term of the time derivative of S which could cause problems is the time derivative of the distribution function, but it can be handled with the Vlasov equation and through integration by parts. We obtain a term which can be bounded by S . Let $\{t_n\}$ be a sequence tending to infinity and let $(\Sigma_-)_n(t) = \Sigma_-(t + t_n)$, $(\Sigma_+)_n(t) = \Sigma_+(t + t_n)$, $(N_2)_n(t) = N_2(t + t_n)$, $(N_3)_n(t) = N_3(t + t_n)$, $H_n(t) = H(t + t_n)$ and $S_n(t) = S(t + t_n)$. Using the bounds, the Arzela-Ascoli theorem can be applied. This implies that, after passing to a subsequence, $(\Sigma_-)_n$, $(\Sigma_+)_n$, $(N_2)_n$, $(N_3)_n$, H_n and S_n converge uniformly on compact sets to a limit $(\Sigma_-)_\infty$, $(\Sigma_+)_\infty$, $(N_2)_\infty$, $(N_3)_\infty$, H_∞ and S_∞ respectively. The first derivative of these variables converges to the corresponding derivative of the limits since we have been able to bound the derivative of S in the last section. Going to this limit it is easy to see that the variable D_∞ is zero and consequently $H_\infty = \frac{2}{3}t^{-1}$. Thus, we obtain the optimal decay rates for the metric and for its derivative. This implies that we obtain the optimal decay rates for P . Since S/H^2 is zero asymptotically, we obtain the same estimates for Σ_- , Σ_+ , N_2 and N_3 as in the Einstein-dust case. Introducing these estimates in (8), we also obtain the optimal estimate for H .

6.2. Optimal estimates

We can summarize the results:

Theorem 1. *Consider any C^∞ solution of the Einstein-Vlasov system with reflection Bianchi Type VI₀ symmetry and with C^∞ initial data. Assume that $|\Sigma_+(t_0) + \frac{1}{4}|$, $|\Sigma_-(t_0)|$, $|N_2(t_0) - \frac{3}{4}|$, $|N_3(t_0) + \frac{3}{4}|$ and $P(t_0)$ are sufficiently small. Then at late times the following estimates hold:*

$$\begin{aligned} H(t) &= \frac{2}{3}t^{-1}(1 + O(t^{-\frac{1}{2}})), \\ \Sigma_+ + \frac{1}{4} &= O(t^{-\frac{1}{2}}), \\ \Sigma_- &= O(t^{-\frac{1}{2}}), \\ N_2 - \frac{3}{4} &= O(t^{-\frac{1}{2}}), \\ N_3 + \frac{3}{4} &= O(t^{-\frac{1}{2}}), \\ P(t) &= O(t^{-\frac{1}{2}}). \end{aligned}$$

From this theorem we can obtain some other results. Let λ_i be the eigenvalues of k_{ij} with respect to g_{ij} :

$$\det(k_j^i - \lambda \delta_j^i) = 0. \quad (16)$$

We define $p_i = \frac{\lambda_i}{k}$ as the *generalized Kasner exponents*. From (2) we see that the eigenvalues (16) of the second fundamental form with respect to the induced metric are also the solutions of:

$$\det(\sigma_j^i - [\lambda - \frac{1}{3}k]\delta_j^i) = 0.$$

Let us define the eigenvalues of σ_{ij} with respect to g_{ij} by $\hat{\lambda}_i$:

$$\hat{\lambda}_i = \lambda_i - \frac{1}{3}k.$$

Using the optimal estimates for Σ_+ , Σ_- and H and the fact that the sum of the generalized Kasner exponents is equal to one, we finally arrive at the generalized Kasner exponents for Bianchi Type VI₀ $(\frac{1}{2}, \frac{1}{4}, \frac{1}{4})$ up to an error of order $O(t^{-\frac{1}{2}})$. V_2 and V_3 become constant asymptotically. Now $f(t_0, p)$ has compact support on p and denoting by \hat{p} the momenta in an orthonormal frame since $f(t, \hat{p})$ is constant along the characteristics, we have $|f(t, \hat{p})| \leq \|f_0\| = \sup\{|f(t_0, \hat{p})|\}$. Let us summarize this in the following corollary:

Corollary 1. *Consider the same assumptions as in Theorem 1. Then*

$$g_{VI_0} = t \operatorname{diag}(tK_1, K_2, K_3),$$

with $K_n = C_n + O(t^{-\frac{1}{2}})$ and where C_1 - C_3 are independent of time. The corresponding result for the inverse metric also holds.

$$\begin{aligned} p_{VI_0} &= p_{EM} + O(t^{-\frac{1}{2}}) \\ \rho &= \rho_{EM}(1 + O(t^{-\frac{1}{2}})) \\ S_{ij} &\leq C|f_0|t^{-3} \end{aligned}$$

The fact that this quotient vanishes asymptotically means that matter behaves asymptotically as dust as it is expected.

7. Conclusions and Outlook

For Bianchi Type VI_0 even for the reflection symmetric case there is no analogous previous result. The reason is that it is not compatible with the LRS-symmetry. Thus, our result on Bianchi Type VI_0 clearly shows that the methods developed are powerful in the sense that one can obtain results which have been out of reach with the techniques developed until now. An important question is whether it is possible to remove the small data assumptions. We will work on this question using techniques developed in [6]. For details of the presented results we refer to [7].

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