Mathematical problems of interaction of different dimensional physical fields

G Chkadua

King's College London, Department of Mathematics, Strand, London WC2R 2LS, UK

E-mail: george.chkadua@kcl.ac.uk

Abstract. In the paper we consider mathematical problem of interaction of different dimensional physical fields for complex composite structures. We investigate the mixed transmission problem arising in the model of fluid-solid acoustic interaction when a piezo-ceramic elastic body (Ω^+) is embedded in an unbounded fluid (Ω^-). The corresponding physical process is described by boundary-transmission problems for second order partial differential equations. In particular, in the bounded domain Ω^+ we have 4×4 dimensional matrix strongly elliptic second order partial differential equation, while in the unbounded complement domain Ω^- we have a scalar Helmholtz equation describing an acoustic wave propagation. The physical kinematic and dynamic relations mathematically are described by appropriate boundary and transmission conditions. With the help of the potential method and theory of pseudodifferential equations the uniqueness and existence theorems are proved in Sobolev-Slobodetski spaces.

1. Introduction

In applications we rather often encounter multi-component composed bodies, where in different components we have different dimensional physical fields. In such cases, we actually have interaction problems for physical fields with different dimensions in the adjacent domains. The corresponding mathematical models for composed bodies are described by boundary, boundary-contact and transmission problems for systems of partial differential equations (PDE). The matrix operators generated by these PDEs have different orders in different domains. Furthermore, the situation becomes complicated because of the fact that on the interfaces between the adjacent domains of the composed body one needs to find appropriate interface conditions relating the different dimensional physical fields.

Due to the rapidly increasing use of composite materials in modern industrial and technological processes, on the one hand, and in biology and medicine, on the other hand, the mathematical modeling related to complex composite structures and their mathematical analysis became very important form the theoretical and practical points of views in recent years.

We consider the model of fluid-solid acoustic interaction when a piezo-ceramic elastic body (Ω^+) is embedded in an unbounded fluid (Ω^-). In this case in the domain Ω^+ we have four-dimensional piezoelectric field (a displacement vector with three components and an electric potential) while in the unbounded domain Ω^- we have a scalar acoustic wave field.

One such example of interacting of different dimensional fields is a piezoelectric transducer. Which can be used either to transform an electric current to an acoustic pressure field or, the opposite, to produce an electric current from an acoustic field. These devices are generally useful for applications that require the generation of sound in air and liquids. Examples of such

applications include phased array microphones, ultrasound equipment, inkjet droplet actuators, drug discovery, sonar transducers, bioimaging, and acousto-biotherapeutics (see [6], [19–22]).
Only few papers are devoted to the strict mathematical analysis of the above named problems. In the following papers mainly are obtained numerical results (see [13], [15], [16]).
In the present paper for boundary-transmission problems generated by the above mentioned mathematical model, we prove the existence and uniqueness theorems in Sobolev-Slobodetski spaces.
Same type problems for the classical model of elasticity (elastic body is embedded in fluid) are considered and studied in the papers: [1–5], [7–9], [11], [12], [14], [17], [18]. In these papers in the bounded domain Ω^+ is three dimensional elastic field (displacement vector with three components) and scalar wave field in unbounded domain Ω^-.
In our case addition of electric potentials complicates investigation and needs special analyses. We investigate the above mentioned problems with the use of the potential method and the theory of pseudodifferential equations on manifolds.

2. Piezoelectric field

Let Ω^+ be a bounded 3-dimensional domain in $\mathbb{R}^3$ with a compact, C^∞-smooth boundary $S = \partial\Omega^+$ and $\Omega^- := \mathbb{R}^3\backslash\overline{\Omega^+}$. Assume that the domain Ω^+ is filled with an anisotropic homogeneous piezoelectric material.
The basic equation of steady state oscillation of the piezoelectricity for anisotropic homogeneous media is written as follows:

$$A(\partial,\omega)U + F = 0 \qquad \text{in } \Omega^+,$$

where $U = (u,\varphi)^\top$, $u = (u_1,u_2,u_3)$ is a displacement vector, $\varphi = u_4$ is an electric potential and $F = (F_1,F_2,F_3,F_4)^\top$. The three-dimensional vector (F_1,F_2,F_3) is mass force density, while $-F_4$ is charge density. $A(\partial,\omega)$ is the matrix differential operator,

$$A(\partial,\omega) = [A_{jk}(\partial,\omega)]_{4\times 4},$$

$$\begin{aligned} A_{jk}(\partial,\omega) &= c_{ijkl}\partial_i\partial_l + \rho_1\omega^2\delta_{jk}, \qquad A_{j4}(\partial,\omega) = e_{lij}\partial_l\partial_i \\ A_{4k}(\partial,\omega) &= -e_{ikl}\partial_i\partial_l \qquad\qquad\quad A_{44}(\partial,\omega) = \varepsilon_{il}\partial_i\partial_l \qquad j,k=1,2,3, \end{aligned} \tag{1}$$

where $\omega > 0$ is the oscillation (frequency) parameter, ρ_1 is the density of piezoelectric material, $c_{ijlk}, e_{ikl}, \varepsilon_{il}$ are elastic, piezoelectric and dielectric constants respectively, δ_{jk} is the Kronecker symbol and summation is over repeated indices. These constants satisfy the following symmetry conditions:

$$c_{ijkl} = c_{jikl} = c_{klij}, \;\; e_{ijk} = e_{ikj}, \;\; \varepsilon_{ij} = \varepsilon_{ji}, \qquad i,j,k,l = 1,2,3.$$

Moreover from physical considerations it is assumed that the quadratic forms $c_{ijkl}\xi_{ij}\xi_{kl}$ and $\varepsilon_{ij}\eta_i\eta_j$ are positive definite, which provides positiveness of the internal energy:

$$c_{ijkl}\xi_{ij}\xi_{kl} \;\geq\; c_0\,\xi_{ij}\xi_{ij} \qquad \forall\xi_{ij} = \xi_{ji} \in \mathbb{R}, \tag{2}$$

$$\varepsilon_{ij}\eta_i\eta_j \;\geq\; c_1\,|\eta|^2 \qquad \forall\eta = (\eta_1,\eta_2,\eta_3) \in \mathbb{R}^3, \tag{3}$$

where c_0 and c_1 are positive constants.
$A(\partial,\omega)$ is a strongly elliptic nonselfadjoint operator.

In the theory of piezoelasticity the components of the three-dimensional mechanical stress vector acting on a surface element with a normal $n = (n_1, n_2, n_3)$ have the form

$$\sigma_{ij} n_i = c_{ijlk} n_i \partial_l u_k + e_{lij} n_i \partial_l \varphi \qquad j = 1, 2, 3,$$

while the normal components of the electric displacement vector $D = (D_1, D_2, D_3)^\top$ read as

$$-D_i n_i = -e_{ikl} n_i \partial_l u_k + \varepsilon_{il} n_i \partial_l \varphi.$$

Let us introduce the following matrix differential operator

$$T(\partial, n) = [T_{jk}(\partial, n)]_{4\times 4},$$

$$\begin{aligned} T_{jk}(\partial, n) &= c_{ijlk} n_i \partial_l, \qquad & T_{j4}(\partial, n) = e_{lij} n_i \partial_l, \\ T_{4k}(\partial, n) &= -e_{ikl} n_i \partial_l, \qquad & T_{44}(\partial, n) = \varepsilon_{il} n_i \partial_l, \qquad j, k = 1, 2, 3. \end{aligned}$$

For a vector $U = (u, \varphi)^\top$ we have

$$T(\partial, n)U = (\sigma_{1j} n_j,\ \sigma_{2j} n_j,\ \sigma_{3j} n_j,\ -D_i n_i)^\top. \tag{4}$$

The components of the vector TU given by (4) have the physical sense: the first three components correspond to the mechanical stress vector in the theory of electroelasticity, while the forth one is the normal component of the electric displacement vector.
In Green's formulae, there appears also the following boundary operator associated with the adjoint differential operator $A^*(\partial, \omega) = A^\top(-\partial, \omega)$,

$$\widetilde{T}(\partial, n) = [\widetilde{T}_{jk}(\partial, n)]_{4\times 4},$$

where

$$\begin{aligned} \widetilde{T}_{jk}(\partial, n) &= T_{jk}(\partial, n), \qquad & \widetilde{T}_{j4}(\partial, n) = -T_{j4}(\partial, n), \\ \widetilde{T}_{4k}(\partial, n) &= -T_{4k}(\partial, n), \qquad & \widetilde{T}_{44}(\partial, n) = T_{44}(\partial, n), \qquad j, k = 1, 2, 3. \end{aligned}$$

3. Scalar field

We assume that the exterior domain Ω^- is filled by a homogeneous isotropic (fluid) medium with the constant density ρ_2. Further, let some physical process (the propagation of acoustic wave) in Ω^- be described by a complex-valued scalar function (scalar field) w being a solution of the homogeneous wave equation (Helmholtz equation)

$$\Delta \mathrm{w} + \rho_2 \omega^2 \mathrm{w} = 0 \qquad \text{in } \Omega^-, \tag{5}$$

where $\Delta = \sum\limits_{j=1}^{3} \frac{\partial^2}{\partial x_j^2}$ is the Laplace operator.
We say that a solution w to the Helmholtz equation (5) satisfies the classical Sommerfeld radiation condition if the following relation holds

$$\frac{\partial \mathrm{w}(x)}{\partial |x|} - i\sqrt{\rho_2}\,\omega \mathrm{w}(x) = O(|x|^{-2}), \qquad |x| \to \infty. \tag{6}$$

Note that if solutions of Helmholtz equation (5) in Ω^- satisfy Sommerfeld radiation condition then (see [23])

$$\mathrm{w}(x) = O(|x|^{-1}), \qquad |x| \to \infty.$$

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Denote by $Som(\Omega^-)$ a class of solutions of the Helmholtz equation (5) satisfying the Sommerfeld radiation condition.

Let Ω be a domain in $\mathbb{R}^3$ with a compact simply connected boundary $\partial\Omega \in C^\infty$.

By $H^s(\Omega)$ $(H^s_{loc}(\Omega))$ and $H^s(\partial\Omega)$, $s \in \mathbb{R}$, we denote the L_2 based Sobolev-Slobodetski spaces on a domain Ω and on a closed manifold $\partial\Omega$.

If $\mathcal{M}$ is a smooth proper submanifold of a manifold $\partial\Omega$, then we denote by $\widetilde{H}^s(\mathcal{M})$ the subspace of $H^s(\partial\Omega)$,

$$\widetilde{H}^s(\mathcal{M}) := \{g : g \in H^s(\partial\Omega), \operatorname{supp} g \subset \overline{\mathcal{M}}\},$$

while $H^s(\mathcal{M})$ denotes the space of restrictions on $\mathcal{M}$ of functions from $H^s(\partial\Omega)$,

$$H^s(\mathcal{M}) := \{r_{\mathcal{M}} g : g \in H^s(\partial\Omega)\},$$

where $r_{\mathcal{M}}$ is the restriction operator onto $\mathcal{M}$.

4. Formulation of mixed type interaction problem for steady state oscillation equations

Let us consider interaction problem of fluid and piezoelectric body. We assume that $S = \partial\Omega^+ = \partial\Omega^- \in C^\infty$.

Let the boundary $S = \partial\Omega^+ = \partial\Omega^-$ be divided into two disjoint parts S_D and S_N, i.e. $S = \overline{S_D} \cup \overline{S_N}$, $S_D \cap S_N = \varnothing$ and $l_m := \partial S_D = \partial S_N \in C^\infty$.

Mixed type problem (M_ω): Find a vector-function $U = (u, \varphi)^\top \in [H^1(\Omega^+)]^4$ and scalar function $\mathrm{w} \in H^1_{loc}(\Omega^-) \bigcap Som(\Omega^-)$ satisfying the following condition

$$\begin{aligned}
A(\partial, \omega)U &= 0 && \text{in } \Omega^+, && (7)\\
\Delta \mathrm{w} + \rho_2\omega^2 \mathrm{w} &= 0 && \text{in } \Omega^-, && (8)\\
\{u \cdot n\}^+ &= b_1\{\partial_n \mathrm{w}\}^- + f_0 && \text{on } S, && (9)\\
\{[T(\partial, n)U]_j\}^+ &= b_2\{\mathrm{w}\}^- n_j + f_j && \text{on } S, \quad j = 1, 2, 3, && (10)\\
\{\varphi\}^+ &= f^{(D)} && \text{on } S_D, && (11)\\
\{[T(\partial, n)U]_4\}^+ &= f^{(N)} && \text{on } S_N, && (12)
\end{aligned}$$

where $f_0 \in H^{-1/2}(S)$, $f_j \in H^{-1/2}(S)$, $j = 1, 2, 3$, $f^{(D)} \in H^{1/2}(S_D)$, $f^{(N)} \in H^{-1/2}(S_N)$. $b_1 b_2 \neq 0$, $\mathrm{Im}[\bar{b}_1 b_2] = 0$ and the symbols $\{\cdot\}^\pm$ denote the traces on S from $\Omega^\pm$.

Note that the transmission conditions (9)-(10) are kinematic and dynamic conditions. For interaction problem of fluid and piezoelectric body $\mathrm{w}(x) = P^{sc}(x)$ is pressure of scattered acoustic wave,

$$b_1 = [\rho_2\omega^2]^{-1}, \quad b_2 = -1, \quad f_0(x) = f_0^{inc}(x) = [\rho_2\omega^2]^{-1}\{\partial_n P^{inc}(x)\}^-,$$

$$(f_1, f_2, f_3) = b_2 n(x)\{P^{inc}(x)\}^-,$$

where P^{inc} is plane incident wave,

$$P^{inc}(x) = e^{id\cdot x}, \quad d = \omega\sqrt{\rho_2}\eta, \quad |\eta| = 1,$$

here $d = (d_1, d_2, d_3)$ denotes the direction of propagation of the plain incident wave.

5. Uniqueness of solution of the problem (M_ω)

We denote by $J_M(\Omega^+)$ the set of values of the frequency parameter $\omega > 0$ for which the following boundary value problem

$$
\begin{aligned}
A(\partial,\omega)U &= 0 && \text{in } \Omega^+, && (13)\\
\{u\cdot n\}^+ &= 0 && \text{on } S, && (14)\\
\{[T(\partial,n)U]_j\}^+ &= 0 && \text{on } S, \quad j=1,2,3, && (15)\\
\{\varphi\}^+ &= 0 && \text{on } S_D, && (16)\\
\{[T(\partial,n)U]_4\}^+ &= 0 && \text{on } S_N, && (17)
\end{aligned}
$$

has a nontrivial solution $U = (u,\varphi)^\top \in [H^1(\Omega^+)]^4$, where homogeneous mixed boundary conditions are given (see. [10]).

Solutions of the problem (13)-(17) are called Jones modes, while the corresponding values of ω are called Jones eigenfrequencies. The spaces of Jones modes corresponding to ω we denote by $X_{M,\omega}(\Omega^+)$. The spaces of Jones modes for the differential operator $A^*(\partial,\omega)$ denote by $X^*_{M,\omega}(\Omega^+)$.

Note that $J_M(\Omega^+)$ are at most enumerable, and for each ω corresponding space of Jones modes $X_{M,\omega}(\Omega^+)$ are of finite dimension.

THEOREM 5.1 *Let a pair (U,w) be a solution of the homogeneous problem (M_ω). Then $U \in X_{M,\omega}(\Omega^+)$ and $\mathrm{w} = 0$ in Ω^-.*

COROLLARY 5.2 *Let $\omega \notin J_M(\Omega^+)$. Then the homogeneous problem (M_ω) has only the trivial solution.*

6. Existence results for the problem (M_ω)

THEOREM 6.1 *If $\omega \notin J_M(\Omega^+)$, then the problem (M_ω) is uniquely solvable.*

THEOREM 6.2 *If $\omega \in J_M(\Omega^+)$, then mixed type problem (M_ω) is solvable if and only if the orthogonality condition*

$$
\sum_{j=1}^{3}\langle f_j,\{U_j\}^+\rangle_S - \langle\{[\overline{\widetilde{T}U}]_4\}^+,\overline{f^{(D)}}\rangle_{S_D} + \langle f^{(N)},\{U_4\}^+\rangle_{S_N} = 0, \qquad U \in X^*_{M,\omega}(\Omega^+), \tag{18}
$$

holds, and the solution is defined modulo Jones modes $X_{M,\omega}(\Omega^+)$. Where the symbols $\langle\,,\,\rangle_S$, $\langle\,,\,\rangle_{S_D}$, $\langle\,,\,\rangle_{S_N}$ denote the duality between the spaces $H^{-1/2}(S)$ and $H^{1/2}(S)$, $\widetilde{H}^{-1/2}(S_D)$ and $H^{1/2}(S_D)$, $H^{-1/2}(S_N)$ and $\widetilde{H}^{1/2}(S_N)$ respectively.

REMARK 6.3 *Let $(f_1,f_2,f_3) = n\varphi$, where φ is scalar function and n is unit normal vector to S. Then necessary and sufficient condition (18), to be solvable the problem (M_ω), can be written*

$$
-\langle\{[\overline{\widetilde{T}U}]_4\}^+,\overline{f^{(D)}}\rangle_{S_D} + \langle f^{(N)},\{U_4\}^+\rangle_{S_N} = 0.
$$

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